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Review Paper

Mixed Finite Element Methods for Second-Order Elliptic Eigenvalue Problems

Xiaomin Cai, Shan Nie, Qiuxia Tian
[1,2,3]*(School of Mathematical Sciences, Guizhou Normal University, Guiyang, China)*
Corresponding Author: Xiaomin Cai

ABSTRACT: *This paper focuses on the study of general elliptic eigenvalue problems and derives a priori error estimates for eigenvalues and eigenfunctions. First, based on the existence and uniqueness of solutions to the corresponding steady-state problem, a completely continuous operator T is defined, and an abstract error estimation expression is derived through deduction. On this basis, further derivations yield error estimates for eigenvalues and L^2-norm error estimates for eigenfunctions. Finally, the validity of the theoretical results is verified through numerical experiments on two-dimensional problems.*

KEYWORDS: *Eigenvalue Problems, A Priori Error, Mixed Finite Element, Raviart-Thomas Space*

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I. INTRODUCTION

Second-order elliptic eigenvalue problems play a significant role in scientific computing and engineering applications, with extensive use in structural vibration analysis, quantum mechanics, and electromagnetic field computations, among other fields. For instance, reference [1] explores extrapolation methods for eigenvalue problems, reference [2] investigates reconstruction algorithms for function values, and reference [3] proposes multilevel correction methods for eigenvalue problems. Over the years, the mixed finite element method, as an important branch of the finite element method, has gained increasing attention in practical applications. The Raviart-Thomas mixed finite element method, in particular, has garnered considerable attention due to its advantages in solving eigenvalue problems. The R-T mixed finite element method can also be applied to solve various eigenvalue problems, such as the Laplace eigenvalue problem [4], the Stokes eigenvalue problem [5] [6], and others.

This paper investigates the application of the Raviart-Thomas mixed finite element method to second-order elliptic eigenvalue problems. We systematically establish a priori error estimation theory and design corresponding numerical algorithms. The effectiveness and reliability of the proposed method are validated through numerical experiments.

1.1 Notations and Basic Preparation:

The following is a description of the notation that will be used in this article. For $s > 0$, we denote as $\|\cdot\|_{s,\Omega}$ the norms of the Sobolev space $H^s(\Omega)$ and $[H^s(\Omega)]^2$, with the convention $H^0(\Omega) = L^2(\Omega)$ and $[H^0(\Omega)]^2 = [L^2(\Omega)]^2$. In addition, we define the Hilbert space as follows

$$H(div,\Omega) = \left\{\boldsymbol{\tau} \in \left(L^2(\Omega)\right)^2 : div\boldsymbol{\tau} \in L^2(\Omega)\right\}$$

and the corresponding norm is given by:

$$\|\boldsymbol{\tau}\|^2_{H(div,\Omega)} = \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|div\boldsymbol{\tau}\|^2_{0,\Omega} \quad (1.1)$$

The Poincaré inequality: If Ω is a connected and bounded convex domain in one direction, then for any $v \in H^1(\Omega)$, we

$$\|v\|_{0,\Omega} \lesssim \|\nabla v\|_{0,\Omega} \quad (1.2)$$

Finally, the relation $a \lesssim b$ represents $a \leq Cb$, where C denotes a constant independent of h, the mesh size, and similarly, $a \gtrsim b$ represents $a \geq Cb$.

II. STANDARD MIXED FINITE ELEMENT APPROXIMATION

Consider the second-order elliptic eigenvalue problem: Find $\lambda \in R$, $u \in H_0^1(\Omega)$, such that

$$\begin{cases} -\nabla \cdot (c(x)\nabla u) = \lambda u, & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{cases} \tag{2.1}$$

where $c(x) \geq c_0 > 0$, $\Omega \subset R^2$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, and ∇ , $\nabla\cdot$ denote the gradient operator and the divergence operator, respectively.

Let $\boldsymbol{\sigma} = c(x) \cdot \nabla u$, then the problem (2.1) is equivalent to

$$\begin{cases} c(x)^{-1}\boldsymbol{\sigma} - \nabla u = 0, & in \quad \Omega \\ -div\boldsymbol{\sigma} = \lambda u, & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{cases} \tag{2.2}$$

let $\boldsymbol{H} = H(div, \Omega)$, $V = G = L^2(\Omega)$, from the equivalent form (2.2), the mixed variational form of the problem (2.1) is obtained as follows: Find $(\lambda, \boldsymbol{\sigma}, u) \in R \times \boldsymbol{H} \times V$, such that

$$\begin{cases} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H} \\ b(\boldsymbol{\sigma}, v) = -\lambda r(u, v), & \forall v \in V \end{cases} \tag{2.3}$$

where the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $r(\cdot,\cdot)$ are defined as follows:

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega c^{-1}\boldsymbol{\sigma\tau} dx, \quad b(\boldsymbol{\tau}, v) = \int_\Omega div\boldsymbol{\tau} \cdot v dx, \quad r(u, v) = \int_\Omega uv dx$$

and the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $r(\cdot,\cdot)$ have the following properties

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \lesssim \|\boldsymbol{\sigma}\|_{\boldsymbol{H}} \|\boldsymbol{\tau}\|_{\boldsymbol{H}}, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H} \tag{2.4}$$

$$a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \gtrsim \|\boldsymbol{\sigma}\|_{\boldsymbol{H}}^2, \quad \forall \boldsymbol{\sigma} \in \boldsymbol{H} \tag{2.5}$$

$$|b(\boldsymbol{\tau}, v)| \lesssim \|\boldsymbol{\tau}\|_{\boldsymbol{H}} \|v\|_V, \quad \forall \boldsymbol{\tau} \in \boldsymbol{H}, v \in V \tag{2.6}$$

$$|r(u, v)| \lesssim \|u\|_V \|v\|_V, \quad \forall u, v \in V \tag{2.7}$$

For the eigenvalue λ, there exists the Rayleigh quotient expression

$$\lambda = \frac{a(\sigma,\sigma)}{r(u,u)} \tag{2.8}$$

From [7] [10], the eigenvalue problem (2.3) has an eigenvalue sequence $\{\lambda_j\}$

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \qquad \lim_{k\to\infty} \lambda_k = \infty$$

and the associated eigenfunctions

$$(\boldsymbol{\sigma}_1, u_1), (\boldsymbol{\sigma}_2, u_2), \cdots, (\boldsymbol{\sigma}_k, u_k), \cdots$$

Let $\mathcal{T}_h = \{\kappa\}$ be a shape-regular mesh of Ω, where h_κ denotes the diameter of each element κ, and $h = \max_{\kappa\in\mathcal{T}_h} h_\kappa$. For any $\kappa \in \mathcal{T}_h$, we denote by $\mathcal{P}_k(\kappa)$ the space of polynomials defined on element κ, where $k \geq 0$. With these ingredients at hand, we define the Raviart-Thomas space as follows (see [8])

$$\boldsymbol{H}_h = \{\boldsymbol{\tau} \in H(div, \Omega): \boldsymbol{\tau}|_\kappa \in [\mathcal{P}_k(\kappa)]^2 \oplus \boldsymbol{x} \cdot \mathcal{P}_k(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \tag{2.9}$$

$$V_h = \{v \in L^2(\Omega): v|_\kappa \in \mathcal{P}_k(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \tag{2.10}$$

Then, according to the definitions of spaces $\boldsymbol{H}_h$ and V_h, we have

$$div\boldsymbol{H}_h = V_h \tag{2.11}$$

The mixed finite element approximation of problem (2.3) is: Find $(\lambda_h, \boldsymbol{\sigma}_h, u_h) \in R \times \boldsymbol{H}_h \times V_h$, such that

$$\begin{cases} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u_h) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H}_h \\ b(\boldsymbol{\sigma}_h, v) = -\lambda_h r(u_h, v), & \forall v \in V_h \end{cases} \tag{2.12}$$

for the eigenvalue λ_h, there exists the Rayleigh quotient expression

$$\lambda_h = \frac{a(\sigma_h,\sigma_h)}{r(u_h,u_h)} \tag{2.13}$$

From [7] [10], the eigenvalue problem (3.4) has eigenvalues as follow

$$0 \leq \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{k,h} \leq \cdots \leq \lambda_{N,h}, \lim_{k\to\infty} \lambda_k = \infty$$

and the associated eigenfunctions

$$(\boldsymbol{\sigma}_{1,h}, u_{1,h}), (\boldsymbol{\sigma}_{2,h}, u_{2,h}), \cdots, (\boldsymbol{\sigma}_{k,h}, u_{k,h}), \cdots, (\boldsymbol{\sigma}_{N,h}, u_{N,h})$$

For $f \in L^2(\Omega)$, consider the source problem corresponding to the eigenvalue problem (2.3) and its discrete mixed finite element form.

Find $(\boldsymbol{p}, q) \in \boldsymbol{H} \times V$, such that

$$\begin{cases} a(\boldsymbol{p}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, q) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H} \\ b(\boldsymbol{p}, v) = -(f, v), & \forall v \in V \end{cases} \tag{2.14}$$

Find $(\boldsymbol{p}_h, q_h) \in \boldsymbol{H}_h \times V_h$, such that

$$\begin{cases} a(\boldsymbol{p}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, q_h) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H}_h \\ b(\boldsymbol{p}_h, v) = -(f, v), & \forall v \in V_h \end{cases} \tag{2.15}$$

For the polygonal domain, it is known from [9] that the problem (2.14) has a unique solution, and the following regularity result holds: For $f \in V$, $\forall(\boldsymbol{p}, q) \in H^r(\Omega)^2 \times [H^{1+r}(\Omega) \cap H_0^{1+r}(\Omega)]$, such that

$$\|q\|_{1+r} + \|\boldsymbol{p}\|_r \lesssim \|f\|_0, \quad 1/2 < r \le 1 \tag{2.16}$$

Assume that the mixed finite element spaces $\boldsymbol{H}_h \subset \boldsymbol{H}$ and $V_h \subset V$ satisfy the inf-sup condition, i.e., there exists a constant $\beta > 0$ such that

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{H}_h} \frac{b(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{\boldsymbol{H}}} \ge \beta \|v\|_0, \quad \forall v \in V_h \tag{2.17}$$

Then (2.15) also exists a unique solution $(\boldsymbol{p}_h, q_h) \in \boldsymbol{H}_h \times V_h$ and the following error estimate is valid (see [8] [11])

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\boldsymbol{H}} + \|q - q_h\|_0 \lesssim \inf_{\boldsymbol{\tau}_h \in \boldsymbol{H}_h} \|\boldsymbol{p} - \boldsymbol{\tau}\|_{\boldsymbol{H}} + \inf_{v_h \in V_h} \|q - v\|_0 \tag{2.18}$$

By setting $G = L^2(\Omega)$ and $\boldsymbol{W} = L^2(\Omega) \times L^2(\Omega)$, a linear bounded operator can be defined as follows:

$$\boldsymbol{S}: G \to \boldsymbol{H} \quad T: G \to G$$
$$\boldsymbol{S}_h: G \to \boldsymbol{H} \quad T_h: G \to G$$

Thus, the eigenvalue problems (2.3) and (2.12) have equivalent operator forms, respectively.

$$\begin{cases} \lambda T u = u \\ \boldsymbol{S}(\lambda u) = \boldsymbol{\sigma} \end{cases} \tag{2.19}$$

$$\begin{cases} \lambda_h T_h u_h = u_h \\ \boldsymbol{S}_h(\lambda_h u_h) = \boldsymbol{\sigma}_h \end{cases} \tag{2.20}$$

In this way, finding the eigenpairs $(\lambda, \boldsymbol{\sigma}, u)$ of (2.3) can be reduced to finding the eigenpairs (λ^{-1}, u) of T and $\boldsymbol{\sigma} = \boldsymbol{S}(\lambda u)$; finding the eigenpairs $(\lambda_h, \boldsymbol{\sigma}_h, u_h)$ of (2.12) can be reduced to finding the eigenpairs (λ_h^{-1}, u_h) of T_h and $\boldsymbol{\sigma}_h = \boldsymbol{S}_h(\lambda_h u_h)$.

Lemma 2.1. T and T_h are self-adjoint operators.

Proof. For $f \in L^2(\Omega)$,(2.14) can be written as

$$\begin{cases} a(\boldsymbol{S}f, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, Tf) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H} \\ b(\boldsymbol{S}f, v) = -(f, v), & \forall v \in V \end{cases} \tag{2.21}$$

Similarly, for $g \in L^2(\Omega)$, (2.14) also holds

$$\begin{cases} a(\boldsymbol{S}g, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, Tg) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H} \\ b(\boldsymbol{S}g, v) = -(g, v), & \forall v \in V \end{cases} \tag{2.22}$$

By taking $\boldsymbol{\tau} = \boldsymbol{S}g$,$v = Tg$ in (2.18), we obtain

$$\begin{cases} a(\boldsymbol{S}f, \boldsymbol{S}g) + b(\boldsymbol{S}g, Tf) = 0 \\ \quad b(\boldsymbol{S}f, Tg) = -(f, Tg) \end{cases} \tag{2.23}$$

by taking $\boldsymbol{\tau} = \boldsymbol{S}f$,$v = Tf$ in (2.19), we obtain

$$\begin{cases} a(\boldsymbol{S}g, \boldsymbol{S}f) + b(\boldsymbol{S}f, Tg) = 0 \\ \quad b(\boldsymbol{S}g, Tf) = -(g, Tf) \end{cases} \tag{2.24}$$

From (2.23) and (2.24), we have

$$(f, Tg) = -b(\boldsymbol{S}f, Tg) = a(\boldsymbol{S}g, \boldsymbol{S}f) = -b(\boldsymbol{S}g, Tf) = (g, Tf) \tag{2.25}$$

Therefore, T is self-adjoint. Similarly, it can be shown that T_h is also self-adjoint.

Theorem 2.1. For the operators T and T_h defined above, as $h \to 0$, $\| T - T_h \|_0 \to 0$.

Proof. It has been proven in [12]

$$\|u - u_h\|_0 \lesssim \begin{cases} h\|u\|_2 & k \ge 0 \\ h^2\|u\|_2 & k \ge 1 \end{cases} \tag{2.26}$$

From the regularity estimate $\|u\|_2 \lesssim \|f\|_0$, it follows that:

$$\begin{cases} \|Tf - T_h f\|_0 \lesssim h\|f\|_0 & h = 0 \\ \|Tf - T_h f\|_0 \lesssim h^2\|f\|_0 & h \ge 1 \end{cases} \tag{2.27}$$

Therefore, as $h \to 0$, $\|T - T_h\|_0 \to 0$.

III. A PRIORI ERROR ESTIMATE

Let $(\lambda,\boldsymbol{\sigma},u)$ be an eigenpair of (2.3), and $(\lambda_h,\boldsymbol{\sigma}_h,u_h)$ be an eigenpair of (2.12). Suppose $(\lambda_h,\boldsymbol{\sigma}_h,u_h)$ approximates $(\lambda,\boldsymbol{\sigma},u)$, and let V_λ denote the eigenspace of (2.3) corresponding to λ. Then, the following estimates hold.

Lemma 3.1. Let the multiplicity of λ be m, for $l = 1,2,\dots,m$, we have the following estimate

$$\left|\lambda-\lambda_{l,h}\right| \lesssim \left\|(\boldsymbol{S}-\boldsymbol{S}_h)|_{V_\lambda}\right\|_0^2 + \left\|(\boldsymbol{S}-\boldsymbol{S}_h)|_{V_\lambda}\right\|_0 \left\|(T-T_h)|_{V_\lambda}\right\|_0 + \left\|(T-T_h)|_{V_\lambda}\right\|_0^2 \tag{3.1}$$

Proof. Since the multiplicity of eigenvalues of a self-adjoint operator is equal to the dimension of its eigenspace, let $\varphi_1,\varphi_2,\dots,\varphi_m$ be an orthogonal basis for V_λ. From Theorem 3 in [7] and the fact that the steepness of a self-adjoint operator $\alpha = 1$, we obtain the following

$$|\lambda^{-1}-\lambda_{l,h}^{-1}| \lesssim \sum_{i,j=1}^m \left|\left((T-T_h)\varphi_i,\varphi_j\right)\right| + \left\|(T-T_h)|_{V_\lambda}\right\|_0^2 \tag{3.2}$$

For $f,g \in L^2(\Omega)$, we estimate $|((T-T_h)g,f)|$. Express (2.14) and (2.15) in operator form.

$$\begin{cases} a(\boldsymbol{S}f,\boldsymbol{\tau}) + b(\boldsymbol{\tau},Tf) = 0 & \forall \boldsymbol{\tau}\in \boldsymbol{H} \\ b(\boldsymbol{S}f,v) = -(f,v) & \forall v \in V \end{cases} \tag{3.3}$$

$$\begin{cases} a(\boldsymbol{S}_h f,\tau) + b(\boldsymbol{\tau},T_h f) = 0 & \forall \boldsymbol{\tau}\in \boldsymbol{H}_h \\ b(\boldsymbol{S}_h f,v) = -(f,v) & \forall v \in V_h \end{cases} \tag{3.4}$$

From the two equations in (3.3), we obtain the following

$$(f,v) = -a(\boldsymbol{S}f,\boldsymbol{\tau}) - b(\boldsymbol{\tau},Tf) - b(\boldsymbol{S}f,v) \qquad \forall(\boldsymbol{\tau},v)\in \boldsymbol{H}\times\boldsymbol{V} \tag{3.5}$$

For $g\in L^2(\Omega)$, let $\boldsymbol{\tau}\in(\boldsymbol{S}-\boldsymbol{S}_h)g$,$v\in(T-T_h)g$. Then, the following holds

$$(f,(T-T_h)g) = -a(\boldsymbol{S}f,(\boldsymbol{S}-\boldsymbol{S}_h)g) - b\left((\boldsymbol{S}-\boldsymbol{S}_h)g,Tf\right) - b(\boldsymbol{S}f,(T-T_h)g) \tag{3.6}$$

Replacing f with $g\in L^2(\Omega)$, we derive from (3.3) and (3.4) that

$$\begin{cases} a\left((\boldsymbol{S}-\boldsymbol{S}_h)g,\boldsymbol{\tau}\right) + b(\boldsymbol{\tau},(T-T_h)g) = 0 \\ b\left((\boldsymbol{S}-\boldsymbol{S}_h)g,v\right) = 0 \end{cases} \tag{3.7}$$

Adding the two equations in (3.7), we have

$$a\left((\boldsymbol{S}-\boldsymbol{S}_h)g,\boldsymbol{\tau}\right) + b(\boldsymbol{\tau},(T-T_h)g) + b\left((\boldsymbol{S}-\boldsymbol{S}_h)g,v\right) = 0 \tag{3.8}$$

Due to the symmetry of $a(\cdot,\cdot)$, adding (2.23) and (2.24) gives

$$(f,(T-T_h)g) = a\left((\boldsymbol{S}-\boldsymbol{S}_h)g,\boldsymbol{\tau}-\boldsymbol{S}f\right) + b\left((\boldsymbol{S}-\boldsymbol{S}_h)g,v-Tf\right) + b(\boldsymbol{\tau}-\boldsymbol{S}f,(T-T_h)g) \tag{3.9}$$

From equations (2.4), (2.6), and (2.7), it follows that for all $\forall\boldsymbol{\tau}\in\boldsymbol{H}_h$, $v\in V_h$, we have

$$\begin{aligned} |(f,(T-T_h)g)| \lesssim{} & \|(\boldsymbol{S}-\boldsymbol{S}_h)g\|_0\|\boldsymbol{\tau}-\boldsymbol{S}f\|_0 \\ & +\|(\boldsymbol{S}-\boldsymbol{S}_h)g\|_0\|v-Tf\|_0 \\ & +\|\boldsymbol{\tau}-\boldsymbol{S}f\|_0\|(T-T_h)g\|_0 \end{aligned} \tag{3.10}$$

In (3.10), taking $\boldsymbol{\tau}=\boldsymbol{S}_h f$,$v=T_h f$, we have

$$\begin{aligned} |(f,(T-T_h)g)| \lesssim{} & \|(\boldsymbol{S}-\boldsymbol{S}_h)g\|_0\|(\boldsymbol{S}-\boldsymbol{S}_h)f\|_0 \\ & +\|(\boldsymbol{S}-\boldsymbol{S}_h)g\|_0\|(T-T_h)f\|_0 \\ & +\|(\boldsymbol{S}-\boldsymbol{S}_h)f\|_0\|(T-T_h)g\|_0 \end{aligned} \tag{3.11}$$

In (3.11), replacing g with φ_i and f with φ_j, we obtain

$$\left|\left((T-T_h)\varphi_i,\varphi_j\right)\right| \lesssim \left\|(\boldsymbol{S}-\boldsymbol{S}_h)|_{V_\lambda}\right\|_0^2 + 2\left\|(\boldsymbol{S}-\boldsymbol{S}_h)|_{V_\lambda}\right\|_0\left\|(T-T_h)|_{V_\lambda}\right\|_0 \tag{3.12}$$

Substituting (3.12) into (3.2) yields (3.1).

Lemma 3.2. The following estimate holds

$$\|u-u_h\|_G \lesssim \left\|(T-T_h)|_{V_\lambda}\right\|_0 \tag{3.13}$$

$$|\lambda-\lambda_h| \lesssim \left\|(T-T_h)|_{V_\lambda}\right\|_0 \tag{3.14}$$

Proof. Equation (3.13) can be immediately derived from Theorem 7.4 in [7] and the preceding Lemma 2.1. Equation (3.14) can be derived from Theorem 7.3 in [7] and the preceding Theorem 2.1 and Lemma 2.1.

In (2.14) and (2.15), taking $f=\lambda u$, it follows from the definitions of T, $\boldsymbol{S}$, T_h, and $\boldsymbol{S}_h$ that $T(\lambda u)$,and $\boldsymbol{S}(\lambda u)$ are solutions to (2.14), while $T_h(\lambda u)$, and $\boldsymbol{S}_h(\lambda u)$ are solutions to (2.15).

Lemma 3.3. The following estimate holds for the eigenfunction $\boldsymbol{\sigma}_h$.

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 \lesssim \left\|(T-T_h)|_{V_\lambda}\right\|_0 + \|(\boldsymbol{S}_h-\boldsymbol{S})(\lambda u)\|_0 \tag{3.15}$$

Proof. Due to the triangle inequality

$$\|\boldsymbol{S}(\lambda u)-\boldsymbol{S}_h(\lambda_h u_h)\|_0 \le \|\boldsymbol{S}(\lambda u)-\boldsymbol{S}_h(\lambda u)\|_0 + \|\boldsymbol{S}_h(\lambda u)-\boldsymbol{S}_h(\lambda_h u_h)\|_0 \tag{3.16}$$

Thus, it suffices to prove

$$\|\boldsymbol{S}_h(\lambda u) - \boldsymbol{S}_h(\lambda_h u_h)\|_0 \lesssim \left\|(T - T_h)|_{V_\lambda}\right\|_0 \tag{3.17}$$

In (2.15), taking $f = \lambda u$ and expressing it in operator form

$$\begin{cases} a(\boldsymbol{S}_h(\lambda u), \boldsymbol{\tau}) + b\big(\boldsymbol{\tau}, T_h(\lambda u)\big) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_h \\ b(\boldsymbol{S}_h(\lambda u), v) = -(\lambda u, v) & \forall v \in V_h \end{cases} \tag{3.18}$$

Expressing (2.12) in operator form and taking the difference with (3.18) yields

$$\begin{cases} a(\boldsymbol{S}_h(\lambda_h u_h - \lambda u), \boldsymbol{\tau}) + b\big(\boldsymbol{\tau}, T_h(\lambda_h u_h - \lambda u)\big) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_h \\ b(\boldsymbol{S}_h(\lambda_h u_h - \lambda u), v) = -(\lambda_h u_h - \lambda u, v) & \forall v \in V_h \end{cases} \tag{3.19}$$

In (3.19), taking $\boldsymbol{\tau} = \boldsymbol{S}_h(\lambda_h u_h - \lambda u)$ and $v = T_h(\lambda_h u_h - \lambda u)$, we obtain

$$\begin{cases} a\big(\boldsymbol{S}_h(\lambda_h u_h - \lambda u), \boldsymbol{S}_h(\lambda_h u_h - \lambda u)\big) + b\big(\boldsymbol{S}_h(\lambda_h u_h - \lambda u), T_h(\lambda_h u_h - \lambda u)\big) = 0 \\ \qquad b\big(\boldsymbol{S}_h(\lambda_h u_h - \lambda u), T_h(\lambda_h u_h - \lambda u)\big) = -\big(\lambda_h u_h - \lambda u, T_h(\lambda_h u_h - \lambda u)\big) \end{cases} \tag{3.20}$$

Adding the two equations in (3.20) gives

$$a\big(\boldsymbol{S}_h(\lambda_h u_h - \lambda u), \boldsymbol{S}_h(\lambda_h u_h - \lambda u)\big) = \big(\lambda_h u_h - \lambda u, T_h(\lambda_h u_h - \lambda u)\big) \tag{3.21}$$

From (3.21) and (2.5), we obtain

$$\begin{aligned} \|\boldsymbol{S}_h(\lambda_h u_h - \lambda u)\|_0^2 &\lesssim \|\lambda_h u_h - \lambda u\|_0 \|T_h(\lambda_h u_h - \lambda u)\|_0 \\ &\lesssim \|\lambda_h u_h - \lambda u\|_0^2 \end{aligned} \tag{3.22}$$

Consequently, we have

$$\|\boldsymbol{S}_h(\lambda_h u_h - \lambda u)\|_0 \lesssim \|\lambda_h u_h - \lambda u\|_0 \tag{3.23}$$

From (3.13), (3.14), and (3.23), we obtain.

$$\|\boldsymbol{S}_h(\lambda_h u_h - \lambda u)\|_0 \lesssim \left\|(T - T_{\square})|_{V_\lambda}\right\|_0 \tag{3.24}$$

This is precisely equation (3.15).

Theorem 3.1. Let $(\lambda, \boldsymbol{\sigma}, u)$ and $(\lambda_h, \boldsymbol{\sigma}_h, u_h)$ be the solutions to the eigenvalue problem (2.3) and (3.4), respectively, for $u \in H^{m+1}(\Omega)$, $1 \le m \le k+1$, such that the following priori error estimates hold

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u - u_h\|_0 \lesssim h^{k+1}, \ k \ge 0 \tag{3.25}$$

$$|\lambda - \lambda_h| \lesssim h^{2k+2}, \ k \ge 0 \tag{3.26}$$

Proof. From Theorem 4 in [12], we obtain

$$\|(\boldsymbol{S} - \boldsymbol{S}_h)g\| \lesssim h^{k+1}\|Tg\|_{k+1} \tag{3.27}$$

$$\|(T - T_h)g\|_0 \lesssim \begin{cases} h^{k+2}\|Tg\|_{k+2} & k \ge 1 \\ h\|Tg\|_2 & k = 0 \end{cases} \tag{3.28}$$

$$\|(\boldsymbol{S} - \boldsymbol{S}_h)g\|_0 \lesssim h^k \|Tg\|_{k+2} \tag{3.29}$$

If $g \in V_\lambda$, then $Tg = \lambda_l^{-1} g$, and thus we have

$$\left\|(\boldsymbol{S} - \boldsymbol{S}_h)|_{V_\lambda}\right\|_0 \lesssim h^{k+1} \tag{3.30}$$

$$\left\|(T - T_h)|_{V_\lambda}\right\|_0 \lesssim h^{k+1} \tag{3.31}$$

From (3.1), (3.13), (3.15), (3.30) and (3.31), and noting that $G = V = L^2(\Omega)$, we obtain (3.25) and (3.26).

IV. NUMERICAL EXAMPLE

In this section, some numerical experiments will be reported to demonstrate the effectiveness of the method. For problem (2.1), we consider three cases with

$$c(x) = 1,\ c(x) = \frac{1}{1+(x-1/2)^2}, \text{ and } c(x) = \frac{1}{1+x^2y^2}.$$

The corresponding numerical results are shown in the tables and figures. Additionally, the numerical examples in this paper were computed using MATLAB 2020b under the iFEM software package (see [13]).

In the experiment, we consider three test domains: the L-shaped domain $\Omega_L = (-1,1)^2 \backslash (0,1) \times -1,0)$, the crack structure domain $\Omega_{SL} = (-1,1)^2 \backslash \{0 \le x \le 1, y = 0\}$ and the square domain Ω_S with vertices at $(0,-1), (1,0), (0,1), (-1,0)$. Since the exact eigenvalues are unknown, we select nine sufficiently accurate approximate values as the reference for the numerical test. These reference eigenvalues are obtained as accurately as possible through adaptive computations. From Tables 1 to 3, we can observe that the algorithm achieves optimal convergence rates.

Table 1: When $c(x) = 1$,the numerical solution for the eigenvalues for regions Ω_L,Ω_S, Ω_{SL}.

$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_L	9.6397238440	1/8	992	9.4920831004	0.14764	1.3721
		1/16	3904	9.5826845073	0.05704	1.3577
		1/32	15488	9.6174666168	0.02226	1.3486
		1/64	61696	9.6309840196	0.00874	1.3429
		1/128	246272	9.6362784156	0.00345	
$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_S	9.8696044011	1/8	336	9.9112971978	0.04169	1.9848
		1/16	1312	9.8801376559	0.01053	1.9963
		1/32	5184	9.8722445283	0.00264	1.9991
		1/64	20608	9.8702648574	0.00066	1.9998
		1/128	82176	9.8697695417	0.00017	
$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_{SL}	8.3713297112	1/8	1320	8.0463118980	0.32502	0.9766
		1/16	5200	8.2061613471	0.16517	0.9886
		1/32	20640	8.2880920691	0.08324	0.9944
		1/64	82240	8.3295494719	0.04178	0.9972
		1/128	328320	8.3503995143	0.02093	

Table 2: When $c(x) = \frac{1}{1+(x-1/2)^2}$, the numerical solution for the eigenvalues for regions Ω_L,Ω_S, Ω_{SL}.

$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_L	5.3470894509	1/8	992	5.2845072785	0.06258	1.4048
		1/16	3904	5.3234532993	0.02364	1.3834
		1/32	15488	5.3380295990	0.00906	1.3792
		1/64	61696	5.3436066085	0.00348	1.3998
		1/128	246272	5.3457695017	0.00132	
$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_S	7.1106875689	1/8	336	7.1330615920	0.02237	1.9352
		1/16	1312	7.1165381849	0.00585	1.9377
		1/32	5184	7.1122148112	0.00153	1.8115
		1/64	20608	7.1111226589	0.0004	1.4311
		1/128	82176	7.1108489255	0.00016	
$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_{SL}	4.7612704287	1/8	1320	4.6140761432	0.14719	0.9893
		1/16	5200	4.6871233652	0.07415	0.9976
		1/32	20640	4.7241363336	0.03713	1.0046
		1/64	82240	4.7427627754	0.01851	1.0138
		1/128	328320	4.7521049773	0.00917	

Table 3: When $c(x) = \frac{1}{1+x^2y^2}$, the numerical solution for the eigenvalues for regions Ω_L,Ω_S, Ω_{SL}.

$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_L	9.0569153630	1/8	992	8.9155212051	0.14139	1.4001
		1/16	3904	9.0033419260	0.05357	1.3804
		1/32	15488	9.0363370039	0.02058	1.3723
		1/64	61696	9.0489666936	0.00795	1.3809
		1/128	246272	9.0538631875	0.00305	
$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_S	9.7122874395	1/8	336	9.7539955778	0.04171	1.9755
		1/16	1312	9.7228932340	0.01061	1.9761
		1/32	5184	9.7149831817	0.00270	1.9251
		1/64	20608	9.7129972694	0.00071	1.7378
		1/128	82176	9.7125002670	0.00021	
$domain$	ref	h	dof	λ_1	$Error$	$rate$
Ω_{SL}	7.8686199409	1/8	1320	7.5639009183	0.30472	0.9926
		1/16	5200	7.7154741472	0.15315	0.9991

	1/32	20640	7.7919991445	0.07662	1.0041
	1/64	82240	7.8304187125	0.03820	1.0109
	1/128	328320	7.8496636620	0.01896	

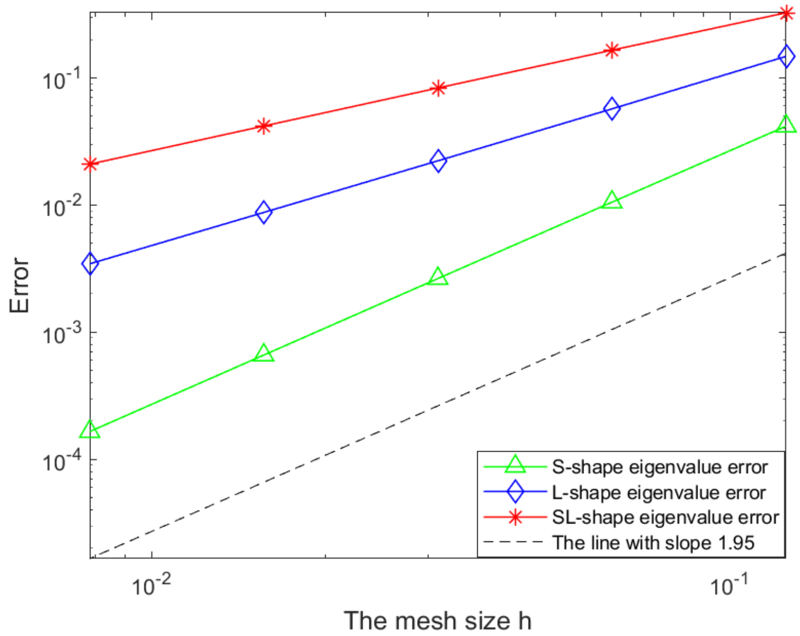

Figure 1: When $c(x) = 1$, the error curve of the first eigenvalue in the regions Ω_L,Ω_S, Ω_{SL}.

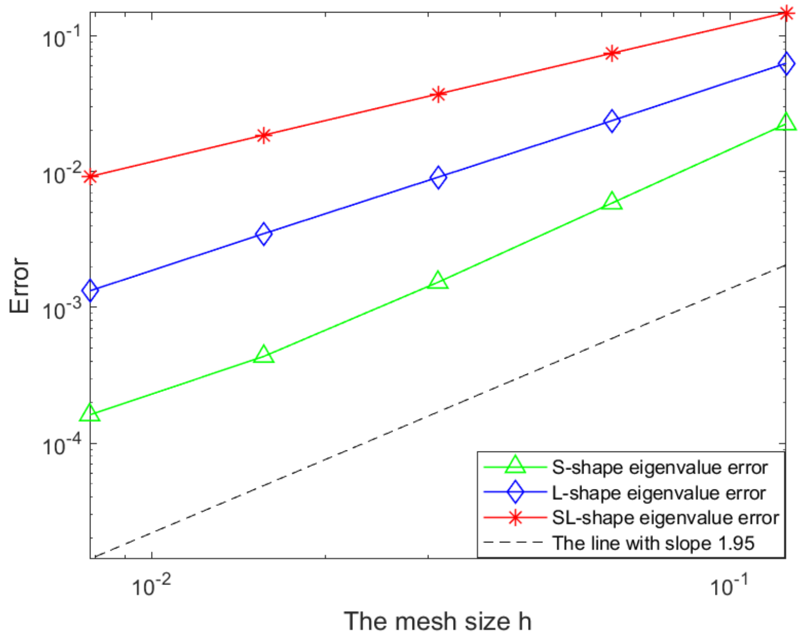

Figure 2: When $c(x) = \frac{1}{1+(x-1/2)^2}$, the error curve of the first eigenvalue in the regions Ω_L,Ω_S, Ω_{SL}.

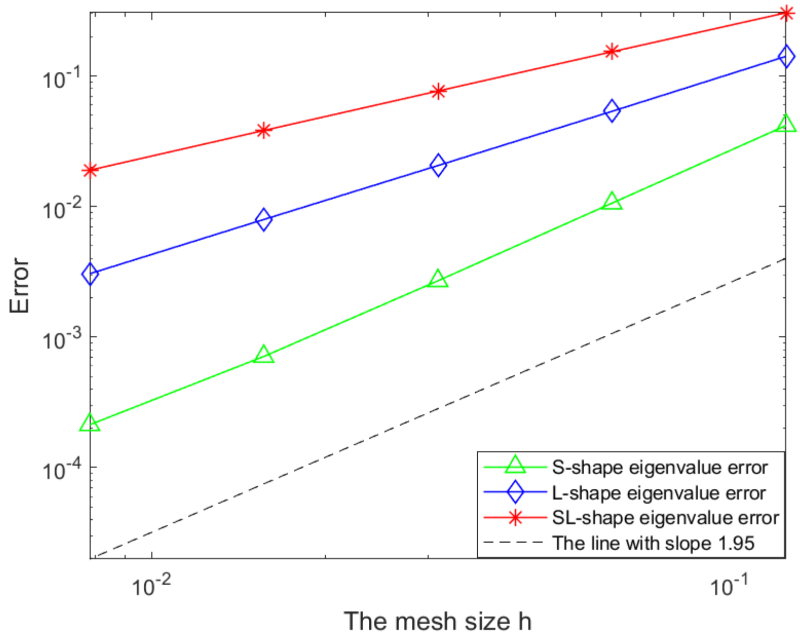

Figure 3: When $c(x) = \frac{1}{1+x^2y^2}$, the error curve of the first eigenvalue in the regions Ω_L,Ω_S, Ω_{SL}.

V. CONCLUSION

This paper presents the Raviart-Thomas mixed finite element method for solving second-order elliptic eigenvalue problems. In order to derive the a priori error estimates, it is crucial to define the operator T, S, T_h, S_h, and investigate its complete continuity. In this paper, we conducted numerical experiments on three test domains Ω_L,Ω_S, Ω_{SL}. the results of these experiments demonstrate that our method is capable of achieving the optimal convergence order for eigenvalues and obtaining the optimal order error estimates for eigenfunctions. The numerical experiments confirm the effectiveness of the proposed algorithm.

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