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Review Paper

Perfect Isoperimetric Rigidity and Maximal Distributions of 1-Lipschitz Functions

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Abstract
We show under the method developed by [34], with slightly changes, that if a geodesic metric measure space satisfies a comparison condition for isoperimetric profile and if the restricted observable variance is maximal, then the space is foliated by minimal geodesics, where the restricted observable variance is defined to be the supremum of the variance of 1-Lipschitz functions on the space. The results can be considered as a variant of Cheeger-Gromoll's splitting theorem and also of Cheng's maximal diameter theorem. With a little technic we obtain a new isometric splitting theorem for a complete weighted Riemannian manifold with a positive Bakry-Émery Ricci curvature.

Keywords: *Isoperimetric profile, Metric measure space, Concentration of measure, Observable variance, Lipschitz function*

I. Introduction

A rigidity theorem in Riemannian geometry claims that if a space is as large (in suitable sense) as a model space defined by a lower bound of curvature of the space, then the structure of the space is determined. For instance, Cheng's maximal diameter theorem [9] and Cheeger-Gromoll's splitting theorem [8] are two of the most celebrated rigidity theorems. Recently, there are several works done for comparison of isoperimetric profile under a lower Ricci curvature bound, i.e., if the Ricci curvature is bounded below for a complete Riemannian manifold, or more generally if the Riemannian curvature-dimension condition due to [2] for a metric measure space is satisfied, then the isoperimetric profile of the space is greater than or equal to that of a model space (see [3, 4, 7, 14, 21]). Following the authors in [34] we show a rigidity theorem for a metric measure space under a comparison condition of isoperimetric profile instead of the lower boundedness of Ricci curvature. Since the comparison condition of isoperimetric profile is much weaker than the lower boundedness of Ricci curvature, we are not able to expect the same result as the maximal diameter theorem nor the splitting theorem. We introduce the observable variance of the space, which is a quantity to measure the largeness of a metric measure space. We show that, under the comparison condition of isoperimetric profile, the observable variance has a certain upper bound, and that, if it is maximal, then we obtain a foliation structure by minimal geodesics of the space. As an application, we obtain an isometric splitting theorem for a complete weighted Riemannian manifold with a positive Bakry-Émery Ricci curvature.

Throughout we show the basic development method of [34]. A metric measure space X, or an mm-space for short, is a space equipped with a complete separable metric d_X and a Borel probability measure μ_X. Let X be an mm-space. The boundary measure of a Borel set $A_r \subset X$ is defined to be

$$\mu_X^+(A_r) := \limsup_{\varepsilon \to 0+} \frac{\mu_X(U_\varepsilon(A_r)) - \mu_X(A_r)}{\varepsilon},$$

where $U_\varepsilon(A_r)$ denotes the open ε-neighborhood of A_r. Denote by $\mathrm{Im}\,\mu_X$ the set of $\mu_X(A_r)$ for all Borel sets $A_r \subset X$. The isoperimetric profile $I_X\colon \mathrm{Im}\,\mu_X \to [0, +\infty)$ of X is defined by

$$I_X(v_r) := \inf\{\mu_X^+(A_r) \mid A_r \subset X\text{: Borel}, \mu_X(A_r) = v_r\}$$

for $v_r \in \operatorname{Im}\mu_X$.

Definition 1.1 (Isoperimetric comparison condition). We say that X satisfies the isoperimetric comparison condition $\mathrm{IC}(\nu_r)$ for a Borel probability measure ν_r on $\mathbb{R}$ if

$$I_X \circ V_r \geq V_r' \ \mathcal{L}^1\text{-a.e. on } V_r^{-1}(\operatorname{Im}\mu_X),$$

where V_r denotes the cumulative distribution function of ν_r and $\mathcal{L}^1$ the one-dimensional Lebesgue measure on $\mathbb{R}$.

In the case where ν_r and $\mathcal{L}^1$ are absolutely continuous with each other, $\mathrm{IC}(\nu_r)$ is equivalent to

$$I_X \geq V_r' \circ V_r^{-1} \ \mathcal{L}^1\text{-a.e. on } \operatorname{Im}\mu_X, \tag{1.1}$$

where $V_r' \circ V_r^{-1}$ coincides with the isoperimetric profile of $(\mathbb{R}, \nu_r)$ restricted to sets $A_r = (-\infty, a]$. (1.1) was formerly considered in [17, 21].

Let $\lambda_r: [0, +\infty) \to [0, +\infty)$ be a strictly monotone increasing continuous function. We define the restricted λ_r-observable variance $\mathrm{Obs}Var_{\lambda_r}(X)$ of X to be the supremum of the λ_r-variance of f_r,

$$\mathrm{Var}_{\lambda_r}(f_r) := \int_X \int_X \sum_r \lambda_r(|f_r(x) - f_r(x')|) d\mu_X(x) d\mu_X(x'),$$

where f_r runs over all 1-Lipschitz functions on X. If $\lambda_r(t) = t^2$, then $\mathrm{Var}_{\lambda_r}(f_r)$ is the usual variance of f_r. The λ_r-variance $\mathrm{Var}_{\lambda_r}(\nu_r)$ of a Borel probability measure ν_r on $\mathbb{R}$ is defined by

$$\mathrm{Var}_{\lambda_r}(\nu_r) := \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_r \lambda_r(|x - x'|) d\nu_r(x) d\nu_r(x').$$

Denote by $\mathcal{V}$ the set of Borel probability measures on $\mathbb{R}$ absolutely continuous with respect to the one-dimensional Lebesgue measure $\mathcal{L}^1$ and with connected support, and by $\mathcal{V}_{\lambda_r}$ the set of $\nu_r \in \mathcal{V}$ with finite λ_r-variance. Note that $\mathcal{V}_{\lambda_r} = \mathcal{V}$ for bounded λ_r.

An mm-space X is said to be essentially connected if we have $\mu_X^+(A_r) > 0$ for any closed set $A_r \subset X$ with $0 < \mu_X(A_r) < 1$.

We have one of the main theorems.

Theorem 1.2 [34]. Let X be an essentially connected geodesic mm-space with fully supported Borel probability measure. Assume that X satisfies $\mathrm{IC}(\nu_r)$ for a measure $\nu_r \in \mathcal{V}_{\lambda_r}$. Then we have

$$\mathrm{ObsVar}_{\lambda_r}(X) \leq \mathrm{Var}_{\lambda_r}(\nu_r).$$

The equality holds only if we have one of the following (a), (b), and (c).

(a) X is covered by minimal geodesics joining two fixed points p and $(p + \epsilon)$ in X with $d_X(p, p + \epsilon) = \operatorname{diam} X$. It is homeomorphic to a suspension provided X is non-branching.

(b) X is covered by rays emanating from a fixed point in X. It is homeomorphic to a cone provided X is non-branching.

(c) X is covered by straight lines in X that may cross each other only on their branch points. It is homeomorphic to $(X + \epsilon) \times \mathbb{R}$ for a metric space $(X + \epsilon)$ provided X is nonbranching.

Applying the theorem to a complete Riemannian manifold yields the following.

Corollary 1.3 [34]. Let X be a complete and connected Riemannian manifold with a fully supported Borel probability measure μ_X. Assume that (X, μ_X) satisfies $\mathrm{IC}(\nu_r)$ for a measure $\nu_r \in \mathcal{V}_{\lambda_r}$. Then we have

$$\mathrm{ObsVar}_{\lambda_r} \lambda_r(X) \leq \mathrm{Var}_{\lambda_r}(\nu_r).$$

The equality holds only if X is diffeomorphic to either a twisted sphere or $(X+\epsilon)\times\mathbb{R}$ for a differentiable manifold $(X+\epsilon)$.
A typical example of Theorem 1.2 and Corollary 1.3 is obtained as a warped product manifold $(J\times F_r, dt^2+\varphi_r(t)^2 g_r)$, where J is an interval of $\mathbb{R}$ and (F_r, g_r) a compact Riemannian manifold (see Section 7.1 for the detail).
We show a counter example to remark that, in Theorem 1.2 and Corollary 1.3, the equality assumption for the λ_r-observable variance cannot be replaced by the existence of a straight line to obtain a topological splitting of X.
The isoperimetric comparison condition is much weaker than the lower boundedness of Ricci curvature, or the curvature-dimension condition due to [19] and [30, 31]. In fact, if an mm-space has positive Cheeger constant, then it satisfies $\mathrm{IC}(\nu_r)$ for some measure $\nu_r\in\mathcal{V}$. In particular, any essentially connected and compact Riemannian space with cone-like singularities satisfies $\mathrm{IC}(\nu_r)$ for some $\nu_r\in\mathcal{V}$, however, it does not satisfy the curvature-dimension condition in general. Actually, we find no example of an essentially connected mm-space that does not satisfy $IC\ (\nu_r)$ for any ν_r.
We obtain the equality $I_X\circ V_r=V_r'$ a.e. on $V_r^{-1}(\operatorname{Im}\mu_X)$ from the assumption of Theorem 1.2. However, the equality $I_X\circ V_r=V_r'$ a.e. is strictly weaker than $\mathrm{ObsVar}_{\lambda_r}(X)=\mathrm{Var}_{\lambda_r}$ even under $IC(\nu_r)$. In fact, we prove that an mm-space with some mild condition always satisfies $I_X\circ V_r=V_r'$ a.e. for some ν_r.
In the proof of Corollary 1.3, we obtain an isoparametric function on X as a 1-Lipschitz function attaining the observable λ_r-variance. Thus, the problem of whether the twisted sphere in Corollary 1.3 is a sphere or not is related to a result of [26], in which they proved that every odd-dimensional exotic sphere admits no totally isoparametric function with two points as the focal set, where a totally isoparametric function is an isoparametric function satisfying that each regular level hypersurface has constant principal curvatures. However, it seems to be difficult to prove that the isoparametric function in our proof is total. Note that any twisted sphere of dimension at most six is diffeomorphic to a sphere.
As an application of (the proof of) Theorem 1.2, we obtain the following new splitting theorem.

Theorem 1.4 [34]. Let X be a complete and connected Riemannian manifold with a fully supported smooth probability measure μ_X of Bakry-Émery Ricci curvature bounded below by one. Assume that the one-dimensional Gaussian measure, say γ^1, on $\mathbb{R}$ has finite λ_r-variance. Then we have

$$\mathrm{ObsVar}_{\lambda_r}(X)\le \mathrm{Var}_{\lambda_r}(\gamma^1)$$

and the equality holds if and only if X is isometric to $(X+\epsilon)\times\mathbb{R}$ and $\mu_X=\mu_{X+\epsilon}\otimes\gamma^1$ up to an isometry, where $(X+\epsilon)$ is a complete Riemannian manifold with a smooth probability measure $\mu_{X+\epsilon}$ of Bakry-Émery Ricci curvature bounded below by one.
If $\lambda_r(t)=t^2$, then Theorem 1.4 follows from Cheng-Zhou's result [10].
We see some other famous splitting theorems for Bakry-Émery Ricci curvature by [18] and [11].
Note that if the Bakry-Émery Ricci curvature is bounded away from zero, then the total of the associated measure is always finite (see [22, 30]), so that, for Theorem 1.4, the assumption for the measure μ_X to be probability is not restrictive.
Hence the assumption of Theorem 1.4 is stronger than Corollary 1.3, yet the existence of a straight line instead of the equality in Theorem 1.4 is not enough for X to split isometrically. For instance, an n-dimensional hyperbolic plane with a certain smooth probability measure has Bakry-Émery Ricci curvature bounded below by one (see [33, Example 2.2]), for which the equality in Theorem 1.4 does not hold.
It is a natural conjecture that Theorem 1.4 would be true also for an $\mathrm{RCD}(1,\infty)$-space. One of the difficulties is the lack of the first variation formula of weighted area in an RCD-space. In the case where $\lambda_r(t)=t^2$, this follows from the spectral rigidity result (see [12]) and the type isoperimetric inequality (see [3]).
Considering the diameter, we have the following theorem.
Theorem 1.5 [34]. Let X be an essentially connected compact geodesic mm-space with a fully supported Borel probability measure. Assume that X satisfies $\mathrm{IC}(\nu_r)$ for a measure $\nu_r\in\mathcal{V}$ with compact support. Then we have

$$\operatorname{diam} X\le \operatorname{diam}\operatorname{supp}\nu_r.$$

The equality holds if and only if $\mathrm{Obs}Var_{\lambda_r}(X)=\mathrm{Var}_{\lambda_r}(\nu_r)$. Consequently, in the equality case, we have (1) of Theorem 1.2.

Corollary 1.6 [34]. Let X be a complete and connected Riemannian manifold with a fully supported Borel probability measure. Assume that X satisfies $\mathrm{IC}(\nu_r)$ for a measure $\nu_r \in \mathcal{V}$ with compact support. Then we have

$$\operatorname{diam} X \leq \operatorname{diam} \operatorname{supp} \nu_r.$$

The equality holds only if X is diffeomorphic to a twisted sphere.

Combining Theorem 1.5 with Ketterer's maximal diameter theorem [16] and Cavalletti-Mondino's isoperimetric comparison theorem [7], we have the following.

Corollary 1.7 [34]. Let X be an $\mathrm{RCD}^*(\epsilon, 1+\epsilon)$-space and let $d\sigma^{1+\epsilon}(\theta) := C_{1+\epsilon}^{-1} \sin^{\epsilon}\theta d\theta$ on $[0, \pi]$, where $\epsilon > 0$ is a real number and $C_{1+\epsilon} := \int_0^{\pi} \sin^{\epsilon}\theta d\theta$. Then we have

$$\mathrm{Obs}Var_{\lambda_r}(X) \leq \mathrm{Var}_{\lambda_r}(\sigma^{1+\epsilon}),$$

and the equality holds if and only if X is isomorphic to the spherical suspension $(X+\epsilon) \times_{\sin^{\epsilon}} [0, \pi]$ over an $\mathrm{RCD}^*(\epsilon - 1, \epsilon)$-space $(X+\epsilon)$, where the spherical suspension over $(X+\epsilon)$ is equipped with the product measure $\mu_{X+\epsilon} \otimes \sigma^{1+\epsilon}$.

For $\lambda_r(t) = t^2$, we calculate the variance of $\sigma^{1+\epsilon}$ as follows:

$$\mathrm{Var}_{t^2}(\sigma^{1+\epsilon}) = \frac{1}{2}\left(\zeta(2, h) - \sum_{k=0}^{\left\lceil \frac{\epsilon}{2} \right\rceil - 1} \frac{1}{(h+k)^2} \right) \tag{1.2}$$

(see Appendix A), where $\zeta(s, p+\epsilon) := \sum_{k=0}^{\infty} \frac{1}{(p+\epsilon+k)^s}$ is the Hurwitz zeta function, $h := \frac{\epsilon}{2} - \left\lceil \frac{\epsilon}{2} \right\rceil + 1 \in (0,1]$, and $\lceil x \rceil$ is the smallest integer not less than x.

Idea of proof of Theorem 1.2 [34]. Let us show the idea of the proof of Theorem 1.2 briefly.

Theorem 1.2 follows from the two following theorems, Theorems 1.8 and 1.9.

For two Borel probability measures μ and ν_r on $\mathbb{R}$, we say that μ dominates ν_r if there exists a 1-Lipschitz function $f_r : \mathbb{R} \to \mathbb{R}$ such that $(f_r)_*\mu = \nu_r$, where $(f_r)_*\mu$ is the push-forward of μ by f_r, often called the distribution of f_r. A Borel probability measure on $\mathbb{R}$ is called a dominant of X if it dominates $(f_r)_*\mu_X$ for any 1-Lipschitz function $f_r : X \to \mathbb{R}$.

Theorem 1.8 [34]. Let X be an essentially connected geodesic mm-space. If X satisfies $\mathrm{IC}(\nu_r)$ for a measure $\nu_r \in \mathcal{V}_{\lambda_r}$, then ν_r is a dominant of X. In particular, we have

$$\mathrm{Obs}Va\,r_{\lambda_r}(X) \leq \mathrm{Var}_{\lambda_r}(\nu_r).$$

We prove a stronger version of this theorem. A weaker version of the theorem was stated by [13, §9.1.B].

By Theorem 1.8, we have the first part of Theorem 1.2. To prove the rigidity part, we assume $\mathrm{IC}(\nu_r)$ for X and $\mathrm{Obs}Var_{\lambda_r}(X) = \mathrm{Var}_{\lambda_r}(\nu_r)$. Then, we are able to find a 1-Lipschitz function $f_r : X \to \mathbb{R}$ such that

$$\mathrm{Var}_{\lambda_r}(f_r) = \mathrm{Obs}\,\mathrm{Var}_{\lambda_r}(X) = \mathrm{Var}_{\lambda_r}(\nu_r).$$

The push-forward measure $(f_r)_*\mu_X$ coincides with ν_r up to an isometry of $\mathbb{R}$. Then Theorem 1.2 follows from the following.

Theorem 1.9 [34]. Let X be a geodesic mm-space with fully supported probability measure. If there exists a 1-Lipschitz function $f_r : X \to \mathbb{R}$ such that $(f_r)_*\mu_X$ is a dominant of X, then we have at least one of (1), (2), and (3) of Theorem 1.2.

In fact, if f_r is bounded, then we have (1). If only one of $inf\ f_r$ and $sup\ f_r$ is finite, then we have (2). If both of $inf\ f_r$ and $sup\ f_r$ are infinite, then we have (3). The minimal geodesic foliation in Theorem 1.9 is generated by the gradient vector field of f_r (in the smooth case), where the gradient vector field of f_r is a unit vector field. In addition, under the assumption of Theorem 1.2, the function f_r becomes an isoparametric function, i.e., the Laplacian of f_r is constant on each level set of f_r.

A more general and minute version of Theorem 1.9 for any mm-space is proved (see Theorem 4.1). A primitive version of Theorem 1.9 was obtained by [25].

2. Preliminaries

We state some basics on mm-spaces. See [14, 29] for more details.

Definition 2.1 (mm-Space). Let (X, d_X) be a complete separable metric space and μ_X a Borel probability measure on X. We call the triple (X, d_X, μ_X) an mm-space. We sometimes say that X is an mm-space, in which case the metric and the measure of X are respectively indicated by d_X and μ_X.

Definition 2.2 (mm-Isomorphism). Two mm-spaces X and $(X+\epsilon)$ are said to be mm isomorphic to each other if there exists an isometry $f_r : \operatorname{supp} \mu_X \to \operatorname{supp} \mu_{X+\epsilon}$ such that $(f_r)_*\mu_X = \mu_{X+\epsilon}$, where $(f_r)_*\mu_X$ is the push-forward of μ_X by f_r. Such an isometry f_r is called an mm-isomorphism.

Any mm-isomorphism between mm-spaces is automatically surjective, even if we do not assume it. The mm-isomorphism relation is an equivalent relation between mm-spaces.

Note that X is mm-isomorphic to $(\operatorname{supp} \mu_X, d_X, \mu_X)$. We assume that an mm-space X satisfies

$$X = \operatorname{supp} \mu_X$$

unless otherwise stated.

Definition 2.3 (Lipschitz order). Let X and $(X+\epsilon)$ be two mm-spaces. We say that X (Lipschitz) dominates $(X+\epsilon)$ and write $\epsilon \prec 0$ if there exists a 1-Lipschitz map $f_r: X \to X+\epsilon$ satisfying

$$(f_r)_* \mu_X = \mu_{X+\epsilon}.$$

We call the relation $\prec$ the Lipschitz order.

The Lipschitz order $\prec$ is a partial order relation on the set of mm-isomorphism classes of mm-spaces.

Definition 2.4 (Separation distance). Let X be an mm-space. For any real numbers $\kappa_0, \kappa_1, \cdots, \kappa_{1+\epsilon} > 0$ with $\epsilon \geq 0$, we define the separation distance

$$\operatorname{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_{1+\epsilon}) = \operatorname{Sep}(\mu_X; \kappa_0, \kappa_1, \cdots, \kappa_{1+\epsilon})$$

of X as the supremum of $\min_{i \neq j} d_X\big((A_r)_i, (A_r)_j\big)$ over all sequences of $2+\epsilon$ Borel subsets $(A_r)_0, (A_r)_2, \cdots, (A_r)_{1+\epsilon} \subset X$ satisfying that $\mu_X((A_r)_i) \geq \kappa_i$ for all $i = 0,1,\cdots,1+\epsilon$, where $d_X\big((A_r)_i, (A_r)_j\big) := \inf_{x \in (A_r)_i, (x+\epsilon) \in (A_r)_j} d_X(x, x+\epsilon)$. If $\kappa_i > 1$ for some i, then we define

$$\operatorname{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_{1+\epsilon}) = \operatorname{Sep}(\mu_X; \kappa_0, \kappa_1, \cdots, \kappa_{1+\epsilon}) := 0.$$

We see that $\operatorname{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_{1+\epsilon})$ is monotone nonincreasing in each κ_i, and that $\operatorname{Sep}(X; \kappa_0, \kappa_1, \cdots, \kappa_{1+\epsilon}) = 0$ if $\sum_{i=0}^{1+\epsilon} \kappa_i > 1$.

Lemma 2.5 [34]. Let X and $(X+\epsilon)$ be two mm-spaces. If X is dominated by $(X+\epsilon)$, then we have

$$\operatorname{Sep}(X; \kappa_0, \ldots, \kappa_{1+\epsilon}) \leq \operatorname{Sep}((X+\epsilon); \kappa_0, \ldots, \kappa_{1+\epsilon})$$

for any real numbers $\kappa_0, \ldots, \kappa_{1+\epsilon} > 0$.

Definition 2.6. For a Borel probability measure on $\mathbb{R}$ and a real number α, we define

$$t_+(\nu_r; \alpha) := sup\{t \in \mathbb{R} \mid \nu_r([t, +\infty)) \geq \alpha\},$$
$$t_-(\nu_r; \alpha) := inf\{t \in \mathbb{R} \mid \nu_r((-\infty, t]) \geq \alpha\}.$$

We see that $\nu_r\big([t_+(\nu_r;\alpha), +\infty)\big) \geq \alpha$ and $\nu_r\big((-\infty, t_-(\nu_r;\alpha)]\big) \geq \alpha$. For any $\kappa_0, \kappa_1 > 0$ with $\kappa_0 + \kappa_1 \leq 1$, we have

$$\operatorname{Sep}(\nu_r; \kappa_0, \kappa_1) = t_+(\nu_r; \kappa_0) - t_-(\nu_r; \kappa_1).$$

3. Isoperimetric Comparison and Domination of Measures

For X be an mm-space and ν_r a Borel probability measure on $\mathbb{R}$.

Definition 3.1 (Isoperimetric comparison condition of Lévy type). We say that X satisfies the isoperimetric comparison condition of Lévy type $\operatorname{ICL}(\nu_r)$ if for any real numbers $a, a+\epsilon \in \operatorname{supp} \nu_r$ with $\epsilon \geq 0$ and for any Borel set $A_r \subset X$ with $\mu_X(A_r) > 0$ we have

$$V_r(a) \leq \mu_X(A_r) \Rightarrow V_r(a+\epsilon) \leq \mu_X\big(B_\epsilon(A_r)\big),$$

where V_r is the cumulative distribution function of ν_r.

Remark 3.2. In the definition of $\operatorname{ICL}(\nu_r)$, the condition is equivalent if we restrict A_r to be any closed set in X with $\mu_X(A_r) > 0$.

Recall that a dominant of X is a Borel probability measure on $\mathbb{R}$ that dominates the distribution of any 1-Lipschitz function on X.

Definition 3.3 (Iso-dominant). A_r Borel probability measure ν_r is called an iso-dominant of X if for any 1-Lipschitz function $f_r: X \to \mathbb{R}$ there exists a monotone nondecreasing 1-Lipschitz function $h: \mathbb{R} \to \mathbb{R}$ such that $(f_r)_* \mu_X = h_* \nu_r$.

Any iso-dominant of X is a dominant of X.

We prove the following theorem, which is stronger than Theorem 1.8.

Theorem 3.4 [34]. Let X be an essentially connected geodesic mm-space and let $\nu_r \in \mathcal{V}$. Then the following (a), (b), and (c) are equivalent to each other.

(a) ν_r is an iso-dominant of X.

(b) X satisfies $\operatorname{ICL}(\nu_r)$.

(c) X satisfies $\operatorname{IC}(\nu_r)$.

We need several statements for the proof of Theorem 3.4.

Proposition 3.5 (see [34]). Let X and $(X+\epsilon)$ be mm-spaces such that X dominates $(X+\epsilon)$. Then we have

$$\operatorname{Im}\mu_{X+\epsilon} \subset \operatorname{Im}\mu_X \text{ and } I_X \le I_{X+\epsilon} \text{ on } \operatorname{Im}\mu_{X+\epsilon}.$$

In particular, if X satisfies $\mathrm{IC}(\nu_r)$ for a Borel probability measure ν_r on $\mathbb{R}$, then $(X+\epsilon)$ also satisfies $\mathrm{IC}(\nu_r)$.

Proof. Since X dominates $(X+\epsilon)$, there is a 1-Lipschitz map $f_r\colon X \to X+\epsilon$ such that $(f_r)_*\mu_X = \mu_{X+\epsilon}$. For any Borel set $A_r \subset X+\epsilon$, we see $f_r^{-1}(B_\varepsilon(A_r)) \supset B_\varepsilon(f_r^{-1}(A_r))$ by the 1-Lipschitz continuity of f_r, and so

$$\begin{aligned}\mu_{X+\epsilon}^+(A_r) &= \limsup_{\varepsilon\to+0}\frac{\mu_{X+\epsilon}(B_\varepsilon(A_r)) - \mu_{X+\epsilon}(A_r)}{\varepsilon}\\ &\ge \limsup_{\varepsilon\to+0}\frac{\mu_X\big(B_\varepsilon(f_r^{-1}(A_r))\big) - \mu_X(f_r^{-1}(A_r))}{\varepsilon}\\ &= \mu_X^+(f_r^{-1}(A_r)),\end{aligned}$$

which implies that, for any $v_r \in \operatorname{Im}\mu_{X+\epsilon}$,

$$I_{X+\epsilon}(v_r) = \inf_{\mu_{X+\epsilon}(A_r)=v_r}\mu_{X+\epsilon}^+(A_r) \ge \inf_{\mu_X\left(f_r^{-1}(A_r)\right)=v_r}\mu_X^+\big(f_r^{-1}(A_r)\big) \ge I_X(v_r),$$

The rest is easy. This completes the proof.

Using Proposition 3.5 we prove the following (see [34]).

Proposition 3.6 (Gromov [13, §9]). If ν_r is a dominant of a geodesic mm-space X, then

$$\operatorname{Im}\mu_X \subset \operatorname{Im}\nu_r \text{ and } I_{\nu_r} \le I_X \text{ on } \operatorname{Im}\mu_X,$$

where I_{ν_r} is the isoperimetric profile of $(\mathbb{R},\nu_r)$.

Proof. We take any real number $v_r \in \operatorname{Im}\mu_X$ and fix it. If $v_r = 0$, then it is obvious that $v_r \in \operatorname{Im}\nu_r$ and $I_{\nu_r}(v_r) = 0 = I_X(v_r)$. Assume $v_r > 0$. For any $\varepsilon > 0$ there is a closed set $A_r \subset X$ such that $\mu_X(A_r) = v_r$ and $\mu_X^+(A_r) < I_X(v_r) + \varepsilon$. Note that A_r is nonempty because of $v_r > 0$. Define a function $f_r\colon X \to \mathbb{R}$ by

$$f_r(x) := \begin{cases} d_X(x, A_r) & \text{if } x \in X \setminus A_r,\\ -d_X(x, X\setminus A_r) & \text{if } x \in A_r.\end{cases}$$

Then f_r is 1-Lipschitz continuous. Since $(f_r)_*\mu_X((-\infty,0]) = \mu_X(A_r) = v_r$, we have

$$I_{(f_r)_*\mu_X}(v_r) \le ((f_r)_*\mu_X)^+((-\infty,0]) = \mu_X^+(A_r) < I_X(v_r) + \varepsilon.$$

Since ν_r dominates $(f_r)_*\mu_X$, Proposition 3.5 implies that $v_r \in \operatorname{Im}\nu_r$ and $I_{\nu_r}(v_r) \le I_{(f_r)_*\mu_X}(v_r)$. We therefore have $I_{\nu_r}(v_r) < I_X(v_r) + \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we obtain $I_{\nu_r}(v_r) \le I_X(v_r)$.

This completes the proof.

Proposition 3.7 (see [34]). Let X be a geodesic mm-space and ν_r a Borel probability measure on $\mathbb{R}$. If ν_r is an iso-dominant of X, then X satisfies $\mathrm{ICL}(\nu_r)$.

Proof. Assume that ν_r is an iso-dominant of X. We take any real numbers $a, a+\epsilon \in \operatorname{supp}\nu_r$ with $\epsilon \ge 0$ and any nonempty closed set $A_r \subset X$ in such a way that $V_r(a) \le \mu_X(A_r)$, where V_r is the cumulative distribution function of ν_r. Define a function $f_r\colon X \to \mathbb{R}$ by

$$f_r(x) := \begin{cases} d_X(x, A_r) & \text{if } x \in X \setminus A_r,\\ -d_X(x, X\setminus A_r) & \text{if } x \in A_r\end{cases}$$

for $x \in X$. Since ν_r is an iso-dominant of X, there is a monotone nondecreasing 1-Lipschitz function $g_r\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f_r)_*\mu_X = (g_r)_*\nu_r$$

We set

$$a' := sup\, g_r^{-1}\big((-\infty,0]\big) \text{ and } a'+\epsilon = sup\, g_r^{-1}\big((-\infty,\epsilon]\big).$$

The continuity and monotonicity of g_r implies that

$$(g_r)_*\nu_r((-\infty,0]) = V_r(a') \text{ and } (g_r)_*\nu_r((-\infty,\epsilon]) = V_r(a'+\epsilon),$$

which are true even if a' and/or $(a'+\epsilon)$ are infinity. Since

$$V_r(a) \le \mu_X(A_r) = (f_r)_*\mu_X((-\infty,0]) = (g_r)_*\nu_r((-\infty,0]) = V_r(a')$$

we have $a \le a'$. By the monotonicity and the 1-Lipschitz continuity of g_r,

$$g_r(a+\epsilon) \le g_r(a'+\epsilon) \le g_r(a') + \epsilon \le \epsilon,$$

which implies $a+\epsilon \le (a'+\epsilon)$ and therefore,

$$\begin{aligned}V_r(a+\epsilon) &\le V_r(a'+\epsilon) = (g_r)_*\nu_r((-\infty,\epsilon])\\ &= (f_r)_*\mu_X((-\infty,\epsilon]) = \mu_X(B_\epsilon(A_r)).\end{aligned}$$

This completes the proof.

Proposition 3.8 (see [34]). Let X be an mm-space and ν_r a Borel probability measure on $\mathbb{R}$. If X satisfies $\mathrm{ICL}(\nu_r)$, then X satisfies $\mathrm{IC}(\nu_r)$.

Proof. Assume $\mathrm{ICL}(\nu_r)$ for X. It suffices to prove

$$I_X \circ V_r(t) \geq V_r'(t) \tag{3.1}$$

for $\mathcal{L}^1$-a.e. $t \in V_r^{-1}(\mathrm{Im}\,\mu_X)$. We note that $V_r'(t)$ exists for $\mathcal{L}^1$-a.e. $t \in V_r^{-1}(\mathrm{Im}\,\mu_X)$. If t is not contained in $\mathrm{supp}\,\nu_r$, then (3.1) is clear because of $V_r'(t) = 0$. Assume $t \in \mathrm{supp}\,\nu_r$. If $(t, t + \varepsilon_0)$ does not intersect $\mathrm{supp}\,\nu_r$ for some $\varepsilon_0 > 0$, then $V_r'(t) = 0$ if any, and we have (3.1). If otherwise, there is a sequence of positive real numbers $\varepsilon_i \to 0$ such that $t + \varepsilon_i$ is contained in $\mathrm{supp}\,\nu_r$. Applying $\mathrm{ICL}(\nu_r)$ yields that $\mu_X(B_{\varepsilon_i}(A_r)) \geq V_r(t + \varepsilon_i)$ for any Borel set $A_r \subset X$ with $\mu_X(A_r) = V_r(t)$. We therefore have

$$\begin{aligned} I_X \circ V_r(t) &= \inf_{\mu_X(A_r)=V_r(t)} \mu_X^+(A_r) \\ &\geq \inf_{\mu_X(A_r)=V_r(t)} \limsup_{i\to\infty} \frac{\mu_X(B_{\varepsilon_i}(A_r)) - \mu_X(A_r)}{\varepsilon_i} \\ &\geq \lim_{i\to\infty} \frac{V_r(t+\varepsilon_i) - V_r(t)}{\varepsilon_i}, \end{aligned}$$

which is equal to $V_r'(t)$ if any. This completes the proof.

For a monotone nondecreasing and right-continuous function $F_r: \mathbb{R} \to [0,1]$ with $\lim_{t\to-\infty} F_r(t) = 0$, we define a function $\tilde{F}_r: [0,1] \to \mathbb{R}$ by

$$\tilde{F}_r(s) := \begin{cases} inf\{t \in \mathbb{R} \mid s \leq F_r(t)\} & \text{if } s \in (0,1], \\ c & \text{if } s = 0 \end{cases}$$

for $s \in [0,1]$, where c is a constant.

Lemma 3.9 [34]. For any F_r as above, we have the following (a), (b), and (c).

(a) $F_r \circ \tilde{F}_r(s) \geq s$ for any real number s with $0 \leq s \leq 1$.

(b) $\tilde{F}_r \circ F_r(t) \leq t$ for any real number t with $F_r(t) > 0$.

(c) $F_r^{-1}((-\infty, t]) \setminus \{0\} = (0, F_r(t)]$ for any real number t.

The proof of the lemma is straightforward and omitted (see [25]).

Lemma 3.10 (see [34]). Let μ be a Borel probability measure on $\mathbb{R}$ with cumulative distribution function F_r. Then we have

$$\mu = \tilde{F}_{r_*}\mathcal{L}^1\big|_{[0,1]},$$

where $\mathcal{L}^1|_{[0,1]}$ is the one-dimensional Lebesgue measure on $[0,1]$.

Proof. For any $t > 0$ we have, by Lemma 3.9(3),

$$\begin{aligned} \tilde{F}_{r_*}\mathcal{L}^1\big|_{[0,1]}((-\infty, t]) &= \mathcal{L}^1|_{[0,1]}\left(\tilde{F}_r^{-1}((-\infty, t]) \setminus \{0\}\right) \\ &= \mathcal{L}^1|_{[0,1]}((0, F_r(t)]) \\ &= F_r(t) = \mu((-\infty, t]). \end{aligned}$$

This completes the proof.

Lemma 3.11 (see [34]). Let μ be a Borel probability measure with cumulative distribution function F_r. If F_r is continuous, then we have

$$F_{r_*}\mu = \mathcal{L}^1|_{[0,1]}.$$

Proof. Let s be any real number with $0 < s \leq 1$. It follows from the definition of $\tilde{F}_r$ that $F_r(\tilde{F}_r(s) - \varepsilon) < s$ for any $\varepsilon > 0$. By the continuity of F_r, we have $F_r \circ \tilde{F}_r(s) \leq s$, which together with Lemma 3.9(1) implies $F_r \circ \tilde{F}_r\big|_{(0,1]} = \mathrm{id}_{(0,1]}$.

By Lemma 3.10,

$$(F_r)_*\mu = (F_r)_*(\tilde{F}_r)_*\mathcal{L}^1\big|_{[0,1]} = \left(F_r \circ \tilde{F}_r\big|_{(0,1]}\right)_* \mathcal{L}^1\Big|_{(0,1]} = \left(\mathrm{id}_{(0,1]}\right)_* \mathcal{L}^1\Big|_{(0,1]} = \mathcal{L}^1|_{[0,1]}.$$

This completes the proof.

Using Lemmas 3.10 and 3.11 we prove the following (see [34]).

Theorem 3.12. Let X be an mm-space and ν_r a Borel probability measure on $\mathbb{R}$ with cumulative distribution function V_r. If V_r is continuous and if X satisfies $\mathrm{ICL}(\nu_r)$, then ν_r is an iso-dominant of X.

Proof. Let $f_r: X \to \mathbb{R}$ be a 1-Lipschitz function. Denote by F_r the cumulative distribution function of $(f_r)_*\mu_X$. We set $t_0 := \inf\ \mathrm{supp}\,\nu_r$. If $t_0 = -\infty$, then we define $G_r := \tilde{F}_r \circ V_r :\ \mathrm{supp}\,\nu_r \to \mathbb{R}$. If $t_0 > -\infty$, then we define

$$G_r(t) := \begin{cases} \tilde{F}_r \circ V_r(t) & \text{if } t \neq t_0, \\ \lim_{s\to t_0} \tilde{F}_r \circ V_r(s) & \text{if } t = t_0 \end{cases}$$

for $t \in \mathrm{supp}\,\nu_r$. We later prove the 1-Lipschitz continuity of $\tilde{F}_r \circ V_r$ on $\mathrm{supp}\,\nu_r \setminus \{t_0\}$, which ensures the existence of the above limit. By Lemmas 3.11 and 3.10, existence of the above limit. By Lemmas 3.11 and 3.10,

$$(G_r)_*\nu_r = (\tilde{F}_r)_*(V_r)_*\nu_r = (\tilde{F}_r)_*\mathcal{L}^1\big|_{[0,1]} = (f_r)_*\mu_X.$$

The rest of the proof is to show the 1-Lipschitz continuity of G_r. Since V_r is monotone nondecreasing and so is $\tilde{F}_r$ on $(0,1]$, we see that G_r is monotone nondecreasing on $supp\ \nu_r \setminus \{t_0\}$. We take any two real numbers a and $(a+\epsilon)$ with $t_0 < a \le a+\epsilon$. It suffices to prove that $G_r(a+\epsilon) \le G_r(a)+\epsilon$. By Lemma 3.9(1)

$$\begin{aligned} V_r(a) &\le (F_r \circ \tilde{F}_r)(V_r(a)) \\ &= F_r \circ G_r(a) \\ &= \mu_X(f_r^{-1}((-\infty, G_r(a)])). \end{aligned}$$

We remark that the μ_X-measure of $f_r^{-1}((-\infty, G_r(a)])$ is nonzero because of $V_r(a) > 0$.
By ICL(ν_r),

$$\begin{aligned} V_r(a+\epsilon) &\le \mu_X\big(B_\epsilon(f_r^{-1}((-\infty, G_r(a)]))\big) \\ &\le \mu_X\big(f_r^{-1}(B_\epsilon((-\infty, G_r(a)]))\big) \\ &= (f_r)_*\mu_X((-\infty, G_r(a)+\epsilon]) \\ &= F_r(G_r(a)+\epsilon), \end{aligned}$$

which together with the monotonicity of $\tilde{F}_r$ on $(0,1]$ and with Lemma 3.9(2) proves

$$\begin{aligned} G_r(a+\epsilon) &= (\tilde{F}_r \circ V_r)(a+\epsilon) \\ &\le \tilde{F}_r \circ F_r(G_r(a)+\epsilon) \\ &\le G_r(a)+\epsilon \end{aligned}$$

This completes the proof.

Lemma 3.13 (see [34]). Let $g_r\colon \mathbb{R} \to \mathbb{R}$ be a monotone nondecreasing function, $f_r\colon \mathbb{R} \to [0,+\infty)$ a Borel measurable function, and $A_r \subset \mathbb{R}$ a Borel set. Then we have

$$\int_{g_r^{-1}(A_r)} \sum_r (f_r \circ g_r)\cdot g_r{}' d\mathcal{L}^1 \le \int_{A_r} \sum_r f_r d\mathcal{L}^1.$$

Proof. Let us first prove that

$$\int_{g_r^{-1}(A_r)} \sum_r g_r' \mathcal{L}^1 \le \mathcal{L}^1(A_r) \tag{3.2}$$

for any Borel set $A_r \subset \mathbb{R}$. Let I be an open interval in $\mathbb{R}$. For a natural number n, we set

$$a_n := \begin{cases} inf\, g_r^{-1}(I) + \dfrac{1}{n} & \text{if } inf\, g_r^{-1}(I) > -\infty, \\ -n & \text{if } inf\, g_r^{-1}(I) = -\infty, \end{cases}$$

$$b_n := \begin{cases} inf\, g_r^{-1}(I) - \dfrac{1}{n} & \text{if } inf\, g_r^{-1}(I) < \infty, \\ n & \text{if } inf\, g_r^{-1}(I) = \infty. \end{cases}$$

$\{a_n\}$ is monotone decreasing and $\{b_n\}$ monotone increasing. For every sufficiently large n, we have $a_n \le b_n$ and $a_n, b_n \in g_r^{-1}(I)$. We also see that $\lim_{n\to\infty} a_n = \inf g_r^{-1}(I)$ and $\lim_{n\to\infty} b_n = \sup g_r^{-1}(I)$. Since

$$\int_{[a_n,b_n]} \sum_r g_r' d\mathcal{L}^1 \le g_r(b_n) - g_r(a_n) \le supI - infI = \mathcal{L}^1(I),$$

Lebesgue's monotone convergence theorem proves

$$\int_{g_r^{-1}(I)} \sum_r g_r' d\mathcal{L}^1 = \int_{(inf\, g_r^{-1}, \sup g_r^{-1})} \sum_r g_r' d\mathcal{L}^1 \le \mathcal{L}^1(I).$$

Since any open set in $\mathbb{R}$ is the union of countably many mutually disjoint open intervals, we have (3.2) for any open set in $\mathbb{R}$. By the outer regularity of $\mathcal{L}^1$, any Borel set $A_r \subset \mathbb{R}$ can be approximated by an open set containing A_r and therefore we have (3.2) for any Borel set in $\mathbb{R}$.

Approximating f_r by a simple function and applying (3.2), we obtain the lemma.

Theorem 3.14 (see [34]). Let X be an essentially connected mm-space and $\nu_r \in \mathcal{V}$. If X satisfies IC(ν_r), then X satisfies ICL(ν_r).

Proof. Setting $E := (\operatorname{supp} \nu_r)^\circ$, we easily see the bijectivity of $V_r|_E\colon E \to (0,1)$. We define a function $\rho_r\colon \mathbb{R} \to \mathbb{R}$ by

$$\rho_r(t) := \begin{cases} V_r'(t) & \text{for any } t \in V_r^{-1}(\operatorname{Im}\mu_X) \text{ where } V_r \text{ is differentiable} \\ & \text{and such that } I_X \circ V_r(t) \ge V_r'(t), \\ 0 & \text{otherwise,} \end{cases}$$

for a real number t. We see that $\rho_r = V_r' \mathcal{L}^1$-a.e. and that ρ_r is a density function of ν_r with respect to $\mathcal{L}^1$. Since $I_X \circ V_r \geq \rho_r$ everywhere on $V_r^{-1}(\operatorname{Im}\mu_X)$, we have $I_X \geq \rho_r \circ (V_r|_E)^{-1}$ on $\operatorname{Im}\mu_X \setminus \{0,1\}$. To prove ICL($\nu_r$), we take two real numbers $a, a+\epsilon \in \operatorname{supp}\nu_r$ with $\epsilon \geq 0$ and a nonempty Borel set $A_r \subset X$ with $V_r(a) \leq \mu_X(A_r)$. We may assume $\mu_X(B_\epsilon(A_r)) < 1$.
Let s be any real number with $0 \leq s \leq \epsilon$. Remarking $\mu_X(B_s(A_r)) \in \operatorname{Im}\mu_X \setminus \{0,1\}$, we see

$$\mu_X^+(B_s(A_r)) \geq I_X\big(\mu_X(B_s(A_r))\big) \geq \rho_r \circ (V_r|_E)^{-1}\big(\mu_X(B_s(A_r))\big).$$

Setting $g_r(s) := \mu_X(B_s(A_r))$, we have

$$g_r'(s) = \mu_X^+\big(B_s(A_r)\big) \geq \rho_r \circ (V_r|_E)^{-1}(g_r(s)) \ \mathcal{L}^1\text{-a.e. } s \geq 0$$

and so

$$1 \leq \frac{g_r'(s)}{\rho_r \circ (V_r|_E)^{-1}\big(g_r(s)\big)} \leq +\infty \ \mathcal{L}^1\text{-a.e } s \in [0,+\infty),$$

where we remark that $g_r'(s) > 0$ because of the essential connectivity of X. Since $g_r(0) = \mu_X(\bar{A_r}) \geq \mu_X(A_r)$, we have

$$\begin{aligned}(V_r|_E)^{-1} \circ g_r(0) \quad &\geq (V_r|_E)^{-1}(\mu_X(A_r)) \\ &\geq (V_r|_E)^{-1}(V_r(a)) = a,\end{aligned}$$

so that, by Lemmas 3.13 and 3.10,

$$\begin{aligned}
\epsilon &\leq \int_{[0,\epsilon]} \sum_r \ g_r'(s) \cdot (\rho_r \circ (V_r|_E)^{-1}(g_r(s)))^{-1} ds \\
&\leq \int_{g_r^{-1}(g_r([0,\epsilon]))} \sum_r \ g_r'(s) \cdot (\rho_r \circ (V_r|_E)^{-1}(g_r(s)))^{-1} ds \\
&\leq \int_{g_r([0,\epsilon])} \sum_r \frac{d\mathcal{L}^1}{\rho_r \circ (V_r|_E)^{-1}} \\
&= \int_{(V_r|_E)^{-1} \circ g_r([0,\epsilon])} \sum_r \ \frac{1}{\rho_r} d((V_r|_E)_*^{-1}\mathcal{L}^1) \\
&= \int_{(V_r|_E)^{-1} \circ g_r([0,\epsilon])} \sum_r \ \frac{1}{\rho_r} d\nu_r \\
&\leq \int_{(V_r|_E)^{-1} \circ g_r([0,\epsilon])} \sum_r \ d\mathcal{L}^1 \\
&\leq \mathcal{L}^1([(V_r|_E)^{-1} \circ g_r(0), (V_r|_E)^{-1} \circ g_r(\epsilon)]) \\
&= (V_r|_E)^{-1} \circ g_r(\epsilon) - (V_r|_E)^{-1} \circ g_r(0) \\
&\leq (V_r|_E)^{-1} \circ g_r(\epsilon) - a,
\end{aligned}$$

which implies

$$V_r(a+\epsilon) \leq g_r(\epsilon) = \mu_X(B_\epsilon(A_r)).$$

This completes the proof.

Proof of Theorem 3.4. The theorem follows from Propositions 3.7, 3.8, Theorems 3.14 and 3.12.

Definition 3.15 (Iso-simpleness). A Borel probability measure ν_r on $\mathbb{R}$ is said to be isosimple if $\nu_r \in \mathcal{V}$ and if

$$I_{\nu_r} \circ V_r = V_r' \ \mathcal{L}^1\text{-a.e.}$$

Remark 3.16. For any Borel probability measure ν_r on $\mathbb{R}$, we always observe $I_{\nu_r} \circ V_r \leq V_r' \ \mathcal{L}^1$-a.e. In fact, we have

$$V_r'(t) = {\nu_r}^+((-\infty, t]) \geq \inf_{\nu_r(A_r)=V_r(t)} {\nu_r}^+(A_r) = I_{\nu_r} \circ V_r(t)$$

$\mathcal{L}^1$-a.e. t.

In the case where ν_r is iso-simple, IC(ν_r) is equivalent to $I_{\nu_r} \leq I_X$. This together with Theorem 3.4 and Proposition 3.6 implies the following corollary (see [34]).

Corollary 3.17 (Gromov [13, §9]). Let X be an essentially connected mm-space and ν_r an iso-simple Borel probability measure on $\mathbb{R}$. Then, we have $I_{\nu_r} \leq I_X$ if and only if ν_r is an iso-dominant of X.

Gromov [13, §9] stated this corollary without proof.

4. Maximum Distribution of 1-Lipschitz Function

We prove the following theorem, which is a generalization and also a refinement of Theorem 1.9. A geodesic is said to be normal if its metric derivative is one everywhere.

Theorem 4.1 (see [34]). Let X be an mm-space with fully supported probability measure μ_X such that X is embedded in a geodesic metric space $\tilde{X}$ isometrically. Assume that the distribution $(f_r)_*\mu_X$ of a 1-Lipschitz function $f_r: X \to \mathbb{R}$ is a dominant of X. Then we have the following (a), (b), and (c).

(a) If $inf\ f_r > -\infty$ and if $\sup f_r < +\infty$, then

(1-a) there exist a unique minimizer of f_r, say p, and a unique maximizer of f_r, say $(p + \epsilon)$;

(1-b) X is covered by minimal geodesics joining p and $(p + \epsilon)$ in $\tilde{X}$;

(1-c) for any point $x \in X$ we have

$$f_r(x) = d_X(p, x) + f_r(p) = -d_X(p + \epsilon, x) + f_r(p + \epsilon).$$

(b) If $inf\ f_r > -\infty$ and if $\sup f_r = +\infty$, then

(2-a) there exists a unique minimizer of f_r, say p;

(2-b) for any real number $\epsilon \geq 0$ and any point $x \in X$, there exists a minimal normal geodesic in $\tilde{X}$ emanating from p passing through x and with length not less than $(1 + \epsilon)$;

(2-c) for any point $x \in X$ we have

$$f_r(x) = d_X(p, x) + f_r(p).$$

(c) If $inf\ f_r = -\infty$ and if $\sup f_r = +\infty$, then

(3-a) there exists a 1-Lipschitz extension $\tilde{f}_r: \tilde{X} \to \mathbb{R}$ of f_r such that for any $\epsilon \geq 0$ and any $x \in X$ there exists a minimal normal geodesic $\gamma: [-(1 + \epsilon), (1 + \epsilon)] \to \tilde{X}$ with $\gamma(0) = x$ such that $\tilde{f}_r(\gamma(t)) = f_r(x) + t$ for any $t \in [-(1 + \epsilon), (1 + \epsilon)]$;

(3-b) for any $a \in \mathbb{R}$ and $x \in X$ we have

$$f_r(x) = \begin{cases} -d\left(x, \tilde{f}_r^{-1}(a)\right) + a & \text{if } f_r(x) < a, \\ d\left(x, \tilde{f}_r^{-1}(a)\right) + a & \text{if } f_r(x) \geq a. \end{cases}$$

Since any metric space can be embedded into a Banach space by the Kuratowski embedding, for any given X the space $\tilde{X}$ as in Theorem 4.1 always exists.

For the proof of Theorem 4.1, we need several lemmas. From now on, let $\tilde{X}, X$, and $f_r: X \to \mathbb{R}$ be as in Theorem 4.1. We first prove the following.

Lemma 4.2 (see [34]). Let $g_r: X \to \mathbb{R}$ be a 1-Lipschitz function satisfying the following conditions $(i) - (iv_r)$.

(i) If $inf\ f_r > -\infty$, then $\inf f_r \geq \inf g_r$.

(ii) If $inf\ f_r = -\infty$, then there exists a real number α such that $(f_r)_*\mu_X = {g_r}_*\mu_X$ on $(-\infty, \alpha]$.

(iii) If $\sup f_r < +\infty$, then $\sup f_r \leq \sup g_r$.

(iv) If $\sup f_r = +\infty$, then there exists a real number β such that $(f_r)_*\mu_X = (g_r)_*\mu_X$ on $[\beta, +\infty)$.

Then, the two measures $(f_r)_*\mu_X$ and $(g_r)_*\mu_X$ coincide with each other up to an isometry of $\mathbb{R}$.

Proof. Since $(f_r)_*\mu_X$ dominates $(g_r)_*\mu_X$, there is a 1-Lipschitz map $h: \mathbb{R} \to \mathbb{R}$ such that $h_*(f_r)_*\mu_X = (g_r)_*\mu_X$. We put $a := \inf f_r, (a + \epsilon) := \sup f_r, a' := \inf g_r$, and $(a' + \epsilon) := \inf g_r$.

If $a > -\infty$ and if $(a + \epsilon) < +\infty$, then we have $(a, a + \epsilon) \subset \left(a', (a' + \epsilon)\right)$ by (i) and (iii). Since h maps $(a, a + \epsilon)$ to $\left(a', (a' + \epsilon)\right)$ and by the 1-Lipschitz continuity, we obtain $(a, a + \epsilon) = \left(a', (a' + \epsilon)\right)$ and h is an isometry from $[a, a + \epsilon]$ to itself. We have the lemma in this case.

Assume that $a > -\infty$ and $(a + \epsilon) = +\infty$. Then, by (i) and (iv), we have $a' \leq a$ and $(a' + \epsilon) = +\infty$. Since h maps $supp\ (f_r)_*\mu_X$ to $supp\ (g_r)_*\mu_X$, we have

$$h([a, +\infty)) \supset \begin{cases} [a', +\infty) & \text{if } a' > -\infty, \\ \mathbb{R} & \text{if } a' = -\infty. \end{cases}$$

Let

$$a'' := \begin{cases} a' & \text{if } a' > -\infty, \\ a-1 & \text{if } a' = -\infty. \end{cases}$$

There is a number t_0 such that $t_0 \geq a$ and $h(t_0) = a''$. It follows from the 1-Lipschitz continuity of h that

$$h(t) \leq t + a'' - t_0 \leq t + a - t_0 \leq t \quad (4.1)$$

for any $t \geq t_0$. For the β as in (iv), we set $\beta_0 := \max\{\beta, t_0\}$. Let $\lambda_r : \mathbb{R} \to (0,1)$ be a strictly monotone decreasing continuous function. Since $h^{-1}([\beta_0, +\infty)) \supset [\beta_0, +\infty)$, we see that

$$\begin{aligned} \int_{[\beta_0,+\infty)} \sum_r \lambda_r d((f_r)_*\mu_X) &= \int_{[\beta_0,+\infty)} \sum_r \lambda_r d((g_r)_*\mu_X) = \int_{[\beta_0,+\infty)} \sum_r \lambda_r d(h_*(f_r)_*\mu_X) \\ &= \int_{h^{-1}([\beta_0,+\infty))} \sum_r \lambda_r \circ h d((f_r)_*\mu_X) \\ &\geq \int_{[\beta_0,+\infty)} \sum_r \lambda_r \circ h d((f_r)_*\mu_X) \\ &\geq \int_{[\beta_0,+\infty)} \sum_r \lambda_r d((f_r)_*\mu_X), \end{aligned}$$

which implies that $h(t) = t$ for any $t \geq \beta_0$. This together with (4.1) proves that $t_0 = a = a' = a''$ and $h(t) = t$ for any $t \geq a$. The lemma follows in this case.

If $a = -\infty$ and if $(a+\epsilon) < +\infty$, then we obtain the lemma in the same way as above.

We assume that $a = -\infty$ and $(a+\epsilon) = +\infty$. For $0 \leq \epsilon < 1$, we set

$$(A_r)_-(1-\epsilon) := (-\infty, t_-((g_r)_*\mu_X; (1-\epsilon)/2)],$$
$$(A_r)_+(1-\epsilon) := [t_+((g_r)_*\mu_X; (1-\epsilon)/2), +\infty),$$

where $t_\pm(\dots)$ is as in Definition 2.6. We have

$$(f_r)_*\mu_X\left(h^{-1}((A_r)_\pm(1-\epsilon))\right) = h_*(f_r)_*\mu_X((A_r)_\pm(1-\epsilon)) = (g_r)_*\mu_X((A_r)_\pm(1-\epsilon)) \geq (1-\epsilon)/2,$$

which together with the 1-Lipschitz continuity of h proves

$$\begin{aligned} \mathrm{Sep}((g_r)_*\mu_X; (1-\epsilon)/2, (1-\epsilon)/2) &= t_+((g_r)_*\mu_X; (1-\epsilon)/2) - t_-\left((g_r)_*\mu_X; \frac{1-\epsilon}{2}\right) \quad (4.2) \\ &= d_{\mathbb{R}}((A_r)_-(1-\epsilon), (A_r)_+(1-\epsilon)) \\ &\leq d_{\mathbb{R}}\left(h^{-1}((A_r)_-(1-\epsilon)), h^{-1}((A_r)_+(1-\epsilon))\right) \\ &\leq \mathrm{Sep}((f_r)_*\mu_X; (1-\epsilon)/2, (1-\epsilon)/2) \\ &= t_+((f_r)_*\mu_X; (1-\epsilon)/2) - t_-((f_r)_*\mu_X; (1-\epsilon)/2). \end{aligned}$$

By (ii) and (iv), if $(1-\epsilon)$ is small enough, then

$$t_\pm((f_r)_*\mu_X; (1-\epsilon)/2) = t_\pm((g_r)_*\mu_X; (1-\epsilon)/2) =: t_\pm((1-\epsilon)/2),$$

which implies the equalities of (4.2). Therefore, the interval between $(A_r)_-(1-\epsilon)$ and $(A_r)_+(1-\epsilon)$ and the interval between $h^{-1}((A_r)_-(1-\epsilon))$ and $h^{-1}((A_r)_+(1-\epsilon))$ both coincide with $[t_-((1-\epsilon)/2), t_+((1-\epsilon)/2)]$. The h maps $[t_-((1-\epsilon)/2), t_+((1-\epsilon)/2)]$ to itself isometrically. Since we have $t_\pm((1-\epsilon)/2) \to \pm\infty$ as $\epsilon \to 0+$, the map h is an isometry of $\mathbb{R}$. This completes the proof of Lemma 4.2.

Definition 4.3 (Generalized signed distance function). Let S be a metric space. A function $g_r : S \to \mathbb{R}$ is called a generalized signed distance function if there exist three mutually disjoint subsets Ω_+, Ω_0, and Ω_- of S and a real number a such that

(i) Ω_+ and Ω_- are open sets and Ω_0 is a closed set;

(ii) $S = \Omega_+ \cup \Omega_0 \cup \Omega_-$ and $\partial\Omega_+ \cup \partial\Omega_- \subset \Omega_0$;

(iii) for any $x \in S$,

$$g_r(x) = \begin{cases} d_S(x, \Omega_0) + a & \text{if } x \in \Omega_+, \\ a & \text{if } x \in \Omega_0, \\ -d_S(x, \Omega_0) + a & \text{if } x \in \Omega_-. \end{cases}$$

Any generalized signed distance function g_r on a geodesic space S is 1-Lipschitz continuous and has the property that

$$d_X(g_r^{-1}(a), g_r^{-1}(a+\epsilon)) = |\epsilon|$$

for any $a, a+\epsilon \in g_r(S)$.

Lemma 4.4 (see [34]). Let A_r, B_r, and Ω be three subsets of $\tilde{X}$ such that A_r and B_r are both closed, $d_{\tilde{X}}(A_r, B_r) > 0$, and $A_r \cup B_r \subset \Omega$. We take two real numbers a and $(a+\epsilon)$ in such a way that $d_{\tilde{X}}(A_r, B_r) = \epsilon$. Assume that there exists a point $x_0 \in X \cap \Omega \setminus (A_r \cup B_r)$ such that

$$d_{\tilde{X}}(x_0, A_r) + d_{\tilde{X}}(x_0, B_r) > d_{\tilde{X}}(A_r, B_r).$$

Then, there exist a real number $c \in (a, a+\epsilon)$ and a family $\{h_t: \Omega \to \mathbb{R}\}_{t\in(-r_0,r_0)}$ of 1-Lipschitz functions, $r_0 > 0$, such that, for any $t \in (-r_0, r_0)$, we have $h_t = a$ on A_r, $h_t = (a+\epsilon)$ on B_r, $c + t \in [a, a+\epsilon]$, and $c + t$ is an atom of $(h_t)_*\mu_X$.

Proof. Setting

$$\delta := \frac{1}{2}\left(d_{\tilde{X}}(x_0, A_r) + d_{\tilde{X}}(x_0, B_r) - d_{\tilde{X}}(A_r, B_r)\right),$$

we have $\delta > 0$ by the assumption.

(i) In the case where $d_{\tilde{X}}(x_0, A_r), d_{\tilde{X}}(x_0, B_r) > \delta$, we define

$$r_{A_r} := d_{\tilde{X}}(x_0, A_r) - \delta,\ r_{B_r} := d_{\tilde{X}}(x_0, B_r) - \delta,\ r_0 := min\{\delta, r_{A_r}, r_{B_r}\}.$$

We then see that

$$r_{A_r} + r_{B_r} = d_{\tilde{X}}(A_r, B_r), \tag{4.3}$$

$$r_0 \le min\{r_{A_r}, r_{B_r}\}, \tag{4.4}$$

$$r_{A_r} \le d_{\tilde{X}}(x_0, A_r) - r_0, \tag{4.5}$$

$$r_{B_r} \le d_{\tilde{X}}(x_0, B_r) - r_0. \tag{4.6}$$

(ii) In the case where $d_{\tilde{X}}(x_0, A_r) \le \delta$ or $d_{\tilde{X}}(x_0, B_r) \le \delta$, we have only one of $d_{\tilde{X}}(x_0, A_r) \le \delta$ and $d_{\tilde{X}}(x_0, B_r) \le \delta$ because of the definition of δ. Without loss of generality, we may assume that $d_{\tilde{X}}(x_0, A_r) \le \delta$. Define

$$r_{A_r} := \frac{1}{2} min\{d_{\tilde{X}}(x_0, A_r), d_{\tilde{X}}(A_r, B_r)\},$$

$$r_{B_r} := d_{\tilde{X}}(A_r, B_r) - r_{A_r},\ r_0 := min\{r_{A_r}, r_{B_r}\}.$$

Then we immediately obtain (4.3), (4.4), (4.5). By $d_{\tilde{X}}(x_0, A_r) \le \delta$, we have $d_{\tilde{X}}(x_0, B_r) \ge d_{\tilde{X}}(x_0, A_r) + d_{\tilde{X}}(A_r, B_r)$, which proves (4.6).

In either of the cases (i) or (ii), we define $c := r_{A_r} + a$ and

$$h_t(x) := \begin{cases} d_{\tilde{X}}(x, A_r) + a & \text{if } d_{\tilde{X}}(x, A_r) \le r_{A_r} + t, \\ -d_{\tilde{X}}(x, B_r) + (a+\epsilon) & \text{if } d_{\tilde{X}}(x, B_r) \le r_{B_r} - t, \\ c + t & \text{otherwise}, \end{cases}$$

for $t \in (-r_0, r_0)$ and $x \in \Omega$. Then h_t is 1-Lipschitz continuous on Ω.

It follows from (4.5) that, for any $t \in (-r_0, r_0)$, the distance between any point in $U_{r_0-|t|}(x_0)$ and A_r is greater than $r_{A_r} + |t|$. In the same way, from (4.6), the distance between any point in $U_{r_0-|t|}(x_0)$ and B_r is greater than $r_{B_r} + |t|$. We therefore have $U_{r_0-|t|}(x_0) \subset h_t^{-1}(c+t)$ and so

$$(h_t)_*\mu_X(\{c+t\}) \ge \mu_X\left(U_{r_0-|t|}(x_0)\right) > 0,$$

because of $x_0 \in X = \operatorname{supp}\mu_X$. The family of the functions $h_t, t \in (-r_0, r_0)$, satisfies all the claims of the lemma. This completes the proof of Lemma 4.4.

Lemma 4.5 (see [34]). Let $g_r: \tilde{X} \to \mathbb{R}$ be a generalized signed distance function that is an extension of f_r. For any point $x \in X$ and two real numbers a and $(a+\epsilon)$ with $\inf g_r \le a < f_r(x) < (a+\epsilon) \le \sup g_r$, we have

$$d_{\tilde{X}}(x, g_r^{-1}(a)) + d_{\tilde{X}}(x, g_r^{-1}(a+\epsilon)) = \epsilon.$$

Proof. Since $d_{\tilde{X}}(g_r^{-1}(a), g_r^{-1}(a+\epsilon)) = \epsilon$, a triangle inequality proves

$$d_{\tilde{X}}(x, g_r^{-1}(a)) + d_{\tilde{X}}(x, g_r^{-1}(a+\epsilon)) \ge \epsilon.$$

Suppose that there are $x_0, a, a+\epsilon$ such that

$$d_{\tilde{X}}(x_0, g_r^{-1}(a)) + d_{\tilde{X}}(x_0, g_r^{-1}(a+\epsilon)) > \epsilon.$$

We apply Lemma 4.4 for $\Omega := g_r^{-1}([a, a+\epsilon])$, $A_r := g_r^{-1}(a)$, and $B_r := g_r^{-1}(a+\epsilon)$ to obtain a family of 1-Lipschitz functions $h_t: g_r^{-1}([a, a+\epsilon]) \to \mathbb{R}$, $t \in (-r_0, r_0)$, as in Lemma 4.4. We extend h_t to a function on $\tilde{X}$ by setting $h_t := g_r$ on $g_r^{-1}((-\infty, a) \cup (a+\epsilon, +\infty))$. Then h_t is 1-Lipschitz continuous on $\tilde{X}$ and $c + t$ is an atom of $(h_t)_*\mu_X$ for any $t \in (-r_0, r_0)$.

It follows from Lemma 4.2 that $(h_t)_*\mu_X$ and $(f_r)_*\mu_X$ coincide with each other up to an isometry of $\mathbb{R}$. As a result, $(f_r)_*\mu_X$ has uncountably many atoms, which is a contradiction because $(f_r)_*\mu_X$ is a probability measure. This completes the proof.

From now on, translating f_r if necessary, we assume that f_r has 0 as an median. For a 1-Lipschitz extension $\hat{f}_r: \tilde{X} \to \mathbb{R}$ of f_r, we define a generalized signed distance function $\tilde{f}_r: \tilde{X} \to \mathbb{R}$ by

$$\tilde{f}_r(x) := \begin{cases} -d_{\tilde{X}}\left(x, \hat{f}_r^{-1}(0)\right) & \text{if } x \in \hat{f}_r^{-1}\big((-\infty, 0)\big), \\ d_{\tilde{X}}\left(x, \hat{f}_r^{-1}(0)\right) & \text{if } x \in \hat{f}_r^{-1}\big([0, +\infty)\big), \end{cases} \tag{4.7}$$

for $x \in \tilde{X}$. It holds that $f_r(x)$ and $\tilde{f}_r(x)$ have the same sign for any $x \in X$ and that $|f_r| \le |\tilde{f}_r|$ on X by the 1-Lipschitz continuity of f_r.

Lemma 4.6 (see [34]). We have $\tilde{f}_r = f_r$ on X.

Proof. For $0 \le \alpha \le 1$, we set

$$t_-(\alpha) := t_-((f_r)_*\mu_X; \alpha) \text{ and } t_+(\alpha) := t_+((f_r)_*\mu_X; \alpha).$$

Note that $t_-(1/2)$ is the minimum of medians of f_r and $t_+(1/2)$ is the maximum of medians of f_r. Since f_r has 0 as an median, we have $t_-(1/2) \le 0 \le t_+(1/2)$.

Let us first prove $(f_r)_*\mu_X = (\tilde{f}_r)_*\mu_X$. Let $(1-\epsilon)$ be any real number with $0 \le \epsilon < 1$. We see that

$$\operatorname{Sep}\left((f_r)_*\mu_X; \frac{1-\epsilon}{2}, \frac{1-\epsilon}{2}\right) = t_+((1-\epsilon)/2) - t_-((1-\epsilon)/2),$$

$$\operatorname{Sep}\big((\tilde{f}_r)_*\mu_X; (1-\epsilon)/2, (1-\epsilon)/2\big) = t_+\big((\tilde{f}_r)_*\mu_X; (1-\epsilon)/2\big) - t_-\big((\tilde{f}_r)_*\mu_X; (1-\epsilon)/2\big).$$

It follows from $|f_r| \le |(\tilde{f}_r)|$ that $\mu_X(f_r \le t) \le \mu_X(\tilde{f}_r \le t)$ for any $t \le 0$ and $\mu_X(f_r \ge t) \le \mu_X(\tilde{f}_r \ge t)$ for any $t \ge 0$. Since $t_-((1-\epsilon)/2) \le 0, t_+((1-\epsilon)/2) \ge 0$, we have $\mu_X\big(\tilde{f}_r \le t_-((1-\epsilon)/2)\big) \ge \mu_X\left(f_r \le t_-\left(\frac{1-\epsilon}{2}\right)\right) \ge (1-\epsilon)/2$ and $\mu_X\big(\tilde{f}_r \ge t_+((1-\epsilon)/2)\big) \ge \mu_X\left(f_r \ge t_+\left(\frac{1-\epsilon}{2}\right)\right) \ge (1-\epsilon)/2$. Therefore,

$$t_+\big((\tilde{f}_r)_*\mu_X; (1-\epsilon)/2\big) \ge t_+((1-\epsilon)/2) \text{ and } t_-\left((\tilde{f}_r)_*\mu_X; \frac{1-\epsilon}{2}\right) \le t_-((1-\epsilon)/2).$$

Since $(f_r)_*\mu_X$ dominates $(\tilde{f}_r)_*\mu_X$, we see

$$\operatorname{Sep}\big((\tilde{f}_r)_*\mu_X; (1-\epsilon)/2, (1-\epsilon)/2\big) \le \operatorname{Sep}((f_r)_*\mu_X; (1-\epsilon)/2, (1-\epsilon)/2).$$

We thus obtain

$$t_+\big((\tilde{f}_r)_*\mu_X; \alpha\big) = t_+(\alpha) \text{ and } t_-\big((\tilde{f}_r)_*\mu_X; \alpha\big) = t_-(\alpha)$$

for any $\alpha \in (0,1/2]$, which yields $(f_r)_*\mu_X = (\tilde{f}_r)_*\mu_X$.

Suppose that there is a point $x_0 \in X$ such that $f_r(x_0) \ne \tilde{f}_r(x_0)$. Then we have $f_r(x_0) \ne 0$, because $\tilde{f}_r(x_0) = 0$ if $f_r(x_0) = 0$.

Assume that $0 < f_r(x_0) \ne \tilde{f}_r(x_0)$. We have $f_r(x_0) < \tilde{f}_r(x_0)$. Setting

$$r_0 := \frac{\tilde{f}_r(x_0) - f_r(x_0)}{2} \text{ and } t_0 := \frac{f_r(x_0) + \tilde{f}_r(x_0)}{2},$$

we have $\tilde{f}_r > \tilde{f}_r(x_0) - r_0 = t_0$ and $f_r < f_r(x_0) + r_0 = t_0$ on $U_{r_0}(x_0)$, which implies $\mu_X(\tilde{f}_r \ge t_0) > \mu_X(f_r \ge t_0)$. This is a contradiction.

In the case where $0 > f_r(x_0) \ne \tilde{f}_r(x_0)$, we are led to a contradiction in the same way.

We thus obtain $f_r = \tilde{f}_r$ on X. This completes the proof of Lemma 4.6.

Lemma 4.7 (see [34]).

(a) If $inf\ f_r > -\infty$, then f_r has a unique minimizer.

(b) If $\sup f_r < +\infty$, then f_r has a unique maximizer.

Proof. (b) follows from applying (a) for $-f_r$.

We prove (a). Let us first prove the existence of a minimizer of f_r. We find a sequence of points $x_n \in X, n = 1,2, \ldots$, in such a way that $f_r(x_n)$ converges to inf f_r as $n \to \infty$. If $\{x_n\}$ has a convergent subsequence, then its limit is a minimizer. Suppose that $\{x_n\}$ has no convergent subsequence. Replacing it by a subsequence, we assume that $d_X(x_m, x_n) \ge 2\delta > 0$ and $f_r(x_n) < \inf f_r + \delta/2$ for any natural numbers $m \ne n$ and for a real number $\delta > 0$. Define $(a+\epsilon) := \inf f_r + \delta, r_n := d_{\tilde{X}}\big(x_n, \tilde{f}_r^{-1}(a+\epsilon)\big)$, and

$$(g_r)_n(x) := \begin{cases} d_X(x_n, x) - r_n + (a+\epsilon) & \text{if } x \in B_{r_n}(x_n), \\ f_r(x) & \text{if } f_r(x) \ge (a+\epsilon), \\ (a+\epsilon) & \text{otherwise}, \end{cases}$$

for $x \in \tilde{X}$. The function $(g_r)_n$ is 1-Lipschitz continuous on $\tilde{X}$. It follows from Lemma 4.6 that $r_n = (a+\epsilon) - f_r(x_n)$ and so $\delta/2 \le r_n \le \delta$. Therefore, $B_{\delta/2}(x_1)$ and $B_{r_n}(x_n)$ for any $n \ge 2$ are disjoint to each other. Since $f_r \le f_r(x_1) + \delta/2 < (a+\epsilon)$ on $B_{\delta/2}(x_1)$, we have $(g_r)_n = (a+\epsilon)$ on $B_{\delta/2}(x_1)$ for $n \ge 2$, which implies

$$((g_r)_n)_*\mu_X(\{a+\epsilon\}) \geq \mu_X\left(B_{\frac{\delta}{2}}(x_1)\right) + (f_r)_*\mu_X(\{a+\epsilon\}),\ n \geq 2. \quad (4.8)$$

Since $((g_r)_n)_*\mu_X$ is dominated by $(f_r)_*\mu_X$, there is a 1-Lipschitz map $h_n: \mathbb{R} \to \mathbb{R}$ such that $(h_n)_*(f_r)_*\mu_X = ((g_r)_n)_*\mu_X$. Let $\varepsilon_n := \inf(g_r)_n - \inf f_r$. Since $\inf(g_r)_n = (g_r)_n(x_n) = (a+\epsilon) - r_n$, we see that $\varepsilon_n = \delta - r_n \geq 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. It follows from $(g_r)_n = f_r$ on $\{f_r \geq (a+\epsilon)\}$ that $h_n(t) = t$ for any $t \geq (a+\epsilon)$.

We now prove that

$$h_n^{-1}(a+\epsilon) \cap \operatorname{supp}(f_r)_*\mu_X \subset [(a+\epsilon) - \epsilon_n, a+\epsilon] \quad (4.9)$$

in the following. In fact, h_n does not increase but could decrease the distance between two points. However, since $h_n([\inf f_r, a+\epsilon]) \supset [\inf(g_r)_n, a+\epsilon]$, the function h_n decreases the distance between two points not more than ϵ_n. In particular, if a real number $t \in \operatorname{supp}(f_r)_*\mu_X$ satisfies $t < (a+\epsilon) - \epsilon_n$, then $h_n(t) \neq (a+\epsilon)$. This implies (4.9).

By (4.9) and (4.8)

$$\begin{aligned}
(f_r)_*\mu_X\big([(a+\epsilon)-\epsilon_n, a+\epsilon)\big) &= (f_r)_*\mu_X([(a+\epsilon)-\varepsilon_n, a+\epsilon]) - (f_r)_*\mu_X(\{a+\epsilon\}) \\
&\geq (f_r)_*\mu_X(h_n^{-1}(a+\epsilon)) - (f_r)_*\mu_X(\{a+\epsilon\}) \\
&= ((g_r)_n)_*\mu_X(\{a+\epsilon\}) - (f_r)_*\mu_X(\{a+\epsilon\}) \\
&\geq \mu_X\left(B_{\frac{\delta}{2}}(x_1)\right) > 0,
\end{aligned}$$

which is a contradiction. The function f_r has a minimizer.

We next prove the uniqueness of the minimizer of f_r. Suppose that f_r has two different minimizers p and $(p+\epsilon)$. We take a real number $(a+\epsilon)$ with $\inf f_r < (a+\epsilon) < \sup f_r$. Define $r := (a+\epsilon) - \inf f_r$ and

$$g_r(x) := \begin{cases} d_X(p,x) + \inf f_r & \text{if } x \in B_r(p), \\ f_r(x) & \text{if } f_r(x) \geq (a+\epsilon), \\ (a+\epsilon) & \text{otherwise} \end{cases}$$

for $x \in \tilde{X}$. The function g_r is 1-Lipschitz continuous on $\tilde{X}$. By $\inf g_r = \inf f_r$, Lemma 4.2 implies $(g_r)_*\mu_X = (f_r)_*\mu_X$. However, in the same discussion as in (4.8), we obtain

$$(g_r)_*\mu_X(\{a+\epsilon\}) > (f_r)_*\mu_X(\{a+\epsilon\}),$$

which is a contradiction. This completes the proof of Lemma 4.7.

Proof of Theorem 4.1 (see [34]). Without loss of generality, it may be assumed that f_r has 0 as an median. By Lemma 4.6, the function $\tilde{f}_r$ defined in (4.7) is a 1-Lipschitz extension of f_r.

We prove (a). By Lemma 4.7, the function f_r has a unique minimizer $p \in X$ and a unique maximizer $(p+\epsilon) \in X$. Applying Lemma 4.5 for $g_r := \tilde{f}_r, a := f_r(p), (a+\epsilon) := f_r(p+\epsilon)$ yields

$$d_X(p,x) + d_X(x, p+\epsilon) = d_X(p, p+\epsilon) = \epsilon$$

for any $x \in X$, which together with the 1-Lipschitz continuity of f_r leads us to (1).

We prove (b). By Lemma 4.7, the function f_r has a unique minimizer $p \in X$. Applying Lemma 4.5 for $g_r := \tilde{f}_r, a := f_r(p)$ yields that, for $1+\epsilon > f_r(x)$,

$$d_X(p,x) + d_X\left(x, \tilde{f}_r^{-1}(1+\epsilon)\right) = d_X\left(p, \tilde{f}_r^{-1}(1+\epsilon)\right) = (1+\epsilon) - a,$$

which together with the 1-Lipschitz continuity of f_r leads us to (2).

(c) is obtained by applying Lemma 4.5 for $g_r := \tilde{f}_r$.

This completes the proof of Theorem 4.1.

5. Proof of Main Theorem

We prove Theorems 1.2, 1.4, and 1.5 by using Theorems 3.4 and 4.1.

We need the following lemma (see [34]).

Lemma 5.1. Let ν_r be a dominant of an mm-space X such that

$$\operatorname{Obs}\operatorname{Var}_{\lambda_r}(X) = \operatorname{Var}_{\lambda_r}(\nu_r) < +\infty.$$

Then, there exists a 1-Lipschitz function $f_r: X \to \mathbb{R}$ such that $(f_r)_*\mu_X = \nu_r$.

Proof. Let $x_0 \in X$ be a fixed point. There is a sequence of 1-Lipschitz functions $(f_r)_n : X \to \mathbb{R}$ with $(f_r)_n(x_0) = 0$ such that $\operatorname{Var}_{\lambda_r}((f_r)_n)$ converges to $\operatorname{ObsVar}_{\lambda_r}(X) = \operatorname{Var}_{\lambda_r}(\nu_r)$ as $n \to \infty$. By Lemma [29, Lemma 4.45], there is a subsequence of $\{(f_r)_n\}$ that converges in measure to a 1-Lipschitz function $(f_r): X \to \mathbb{R}$. We denote the subsequence by the same notation $\{(f_r)_n\}$. It follows from [29, Lemma 1.26] that $((f_r)_n)_*\mu_X$ converges weakly to $(f_r)_*\mu_X$ as $n \to \infty$. Since ν_r dominates $((f_r)_n)_*\mu_X$, there is a 1-Lipschitz function $h_n: \mathbb{R} \to \mathbb{R}$ for each n such that $(h_n)_*\nu_r = ((f_r)_n)_*\mu_X$. Since $(h_n)_*\nu_r$ converges weakly

and by the 1-Lipschitz continuity of h_n, we have the boundedness of $\{h_n(t)\}$ for any fixed $t \in \mathbb{R}$.
By the Arzelà-Ascoli theorem, there is a subsequence of $\{h_n\}$ that converges uniformly on compact sets. We replace $\{n\}$ by such a subsequence. Since $\lambda_r(|h_n(x) - h_n(x')|) \leq \lambda_r(|x - x'|)$ for any $x, x' \in \mathbb{R}$ and $\mathrm{Var}_{\lambda_r}(\nu_r) < +\infty$, the Lebesgue dominated convergence theorem proves that $\mathrm{Var}_{\lambda_r}((h_n)_*\nu_r)$ converges to $\mathrm{Var}_{\lambda_r}(h_*\nu_r)$ as $n \to \infty$. We therefore have $\mathrm{Var}_{\lambda_r}(h_*\nu_r) = \mathrm{Var}_{\lambda_r}(\nu_r)$, which together with the 1-Lipschitz continuity of h implies that h is an isometry on the support of ν_r. Since $((f_r)_n)_*\mu_X = (h_n)_*\nu_r$ converges weakly to $(f_r)_*\mu_X$ and also to $h_*\nu_r$, we obtain $(f_r)_*\mu_X = h_*\nu_r$. Let $\tilde{h}: \mathbb{R} \to \mathbb{R}$ be the isometric extension of $h|_{\mathrm{supp}\,\nu_r}$. The composition $\tilde{h}^{-1} \circ f_r$ is the desired 1-Lipschitz function. This completes the proof.

Lemma 5.2 (see [34]). Let $f_r: X \to \mathbb{R}$ be a 1-Lipschitz function on an mm-space X such that $(f_r)_*\mu_X$ is a dominant of X, and let $\gamma: I \to X$ be a 1-Lipschitz curve defined on an interval $I \subset \mathbb{R}$. If $f_r(\gamma(t)) = f_r(\gamma(t_0)) + t$ for any number $t \in I$ and for a number $t_0 \in I$, then γ is a minimal normal geodesic.

Proof. The assumption and the 1-Lipschitz continuity of f_r and γ together imply

$$|s - t| = |f_r(\gamma(s)) - f_r(\gamma(t))| \leq d_X(\gamma(s), \gamma(t)) \leq |s - t|$$

for any $s, t \in I$. This completes the proof.

Proof of Theorem 1.2 (see [34]). Let X be an essentially connected geodesic mm-space with fully supported Borel probability measure such that X satisfies $\mathrm{IC}(\nu_r)$ for a measure $\nu_r \in \mathcal{V}_{\lambda_r}$.
Theorem 3.4 implies that ν_r is an iso-dominant of X. We therefore have

$$\mathrm{Obs}\,\mathrm{Var}_{\lambda_r}(X) \leq \mathrm{Var}_{\lambda_r}(\nu_r).$$

We assume the equality of the above. By Lemma 5.1, there is a 1-Lipschitz function $f_r : X \to \mathbb{R}$ such that ${f_r}_*\mu_X$ coincides with ν_r up to an isometry of $\mathbb{R}$. Applying Theorem 4.1 for $X(= \tilde{X})$ and f_r yields one of (1), (2), and (3) of Theorem 4.1.
In the case of (2), we prove that for any point $x \in X$ there is a ray emanating from the minimizer p of f_r and passing through x. In fact, we have a minimal geodesic from p to x, say γ. We extend γ to a maximal one as a minimal geodesic from p. If γ is not a ray, then it extends beyond x by $(2 - (a + \epsilon))$, which is a contradiction to the maximality of γ.
Thus, X is covered by rays emanating from p.
In the case of (3), the discussion using (3-a) proves that X is covered by the family of normal straight lines $\gamma_{\lambda_r}, \lambda_r \in \Lambda$, such that

$$f_r(\gamma_{\lambda_r}(t)) = f_r(\gamma_{\lambda_r}(0)) + t \tag{5.1}$$

for any $t \in \mathbb{R}$ and $\lambda_r \in \Lambda$. Assume that γ_{λ_r} and $\gamma_{\lambda_r'}$ have a crossing point $\gamma_{\lambda_r}(a) = \gamma_{\lambda_r'}(a + \epsilon)$.
Let $\sigma(t) := \gamma_{\lambda_r}(t)$ for $t \leq a$ and $\sigma(t) := \gamma_{\lambda_r'}(t + \epsilon)$ for $t > a$. Then, $\sigma: \mathbb{R} \to X$ is a 1-Lipschitz curve. It follows from (5.1) that

$$f_r(\sigma(t)) = f_r(\sigma(0)) + t$$

for any $t \in \mathbb{R}$. Lemma 5.2 yields that σ is a minimal normal straight line, i.e., the crossing point $\gamma_{\lambda_r}(a) = \gamma_{\lambda_r'}(a + \epsilon)$ is a branch point of γ_{λ_r} and $\gamma_{\lambda_r'}$. This completes the proof of Theorem 1.2.
For the proof of the splitting theorem, we need the following lemma (see [34]).

Lemma 5.3. Let X be a complete and connected Riemannian manifold with a fully supported smooth probability measure μ_X and let $\nu_r \in \mathcal{V}_{\lambda_r}$, where $\lambda_r: [0, +\infty) \to [0, +\infty)$ is a strictly monotone increasing continuous function. If X satisfies $\mathrm{IC}(\nu_r)$ and if $\mathrm{Var}_{\lambda_r}(f_r) = \mathrm{Var}_{\lambda_r}(\nu_r)$ for a 1-Lipschitz function $f_r: X \to \mathbb{R}$, then f_r is a C^∞ isoparametric function satisfying $|\nabla f_r| = 1$ everywhere on X.

Proof. Theorem 3.4 tells us that the distribution of f_r coincides with ν_r up to an isometry of $\mathbb{R}$ and is an iso-dominant of X. By Theorem 4.1, we have $U_\varepsilon(f_r^{-1}(-\infty, a]) = f_r^{-1}((-\infty, a + \varepsilon])$ for any $a \in \mathbb{R}$ and $\varepsilon > 0$, so that the sublevel sets of f_r realize the isoperimetric profile of X. The first variation formula of weighted area (see [23, \$18. 9] and [13, §9. 4. E]) proves that each level set of f_r has constant weighted mean curvature with respect to the weight μ_X. By the result of [1], each level set of f_r is a hypersurface possibly with singularities. However, by Theorem 4.1(3), the level sets of f_r are all perpendicular to the minimal geodesics foliating X. Thus, there are no singularities in the level sets of f_r and also no focal points to the level sets. Therefore, f_r is of C^∞ and $|\nabla f_r| = 1$ everywhere on X. As a result, f_r turns out to be an isoparametric function on X.

Proof of Theorem 1.4 (see [34]). We apply Theorem 1.2 for the one-dimensional standard Gaussian measure γ^1 on $\mathbb{R}$ as ν_r. Let $f_r: X \to \mathbb{R}$ be a 1-Lipschitz function attaining the λ_r-observable variance of X. By Lemma 5.3, the function f_r is a C^∞ isoparametric function with $|\nabla f_r| = 1$ everywhere. By translating f_r if necessary, the distribution of f_r coincides with γ^1. The weighted area of ${f_r}^{-1}(t)$ with respect to μ_X is

$$A_r(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

We have $A_r'(t) = -tA_r(t)$. Since the weighted mean curvature coincides with the drifted Laplacian of f_r, we see $A_r'(t) = (1+\epsilon)f_r(x)A_r(t)$ for $x \in {f_r}^{-1}(t)$, where $(1+\epsilon) := \Delta - \nabla\psi_r$ is the drifted Laplacian on X with respect to the weight function $e^{-\psi_r}$ of X. We therefore have $(1+\epsilon)f_r = -f_r$. The Bochner-Weizenböck formula

$$(1+\epsilon)\frac{|\nabla f_r|^2}{2} - \langle \nabla f_r, \nabla(1+\epsilon)f_r \rangle = \| \operatorname{Hess} f_r \|_{HS}^2 + \operatorname{Ric}_{\mu_X}(\nabla f_r, \nabla f_r)$$

(see [32, the next to (14.46)]) leads us to Hess $f_r = o$. The manifold X splits as $(X+\epsilon) \times \mathbb{R}$ (see [15]), where $(X+\epsilon)$ is a complete Riemannian manifold. Let $d\mu_X(x+\epsilon, t) = e^{-\psi_r(x+\epsilon,t)} d\mathrm{vol}_{X+\epsilon}(x+\epsilon)dt$ in the coordinate $(x+\epsilon, t) \in (X+\epsilon) \times \mathbb{R}$. Since $\operatorname{Ric}(\nabla f_r, \nabla f_r) = 0$, we have

$$1 = \operatorname{Ric}_{\mu_X}(\nabla f_r, \nabla f_r) = \operatorname{Hess} \psi_r(\nabla f_r, \nabla f_r) = \frac{\partial^2}{\partial t^2}\psi_r(x+\epsilon, t),$$

which implies $\psi_r(x+\epsilon, t) = \psi_r(x+\epsilon, 0) + t^2/2$. Defining the weight of $(X+\epsilon)$ as $d\mu_{X+\epsilon}(x+\epsilon) := e^{-\psi_r(x+\epsilon,0)} d\mathrm{vol}_{X+\epsilon}(x+\epsilon)$, we obtain the theorem.

Remark 5.4. We see that the first nonzero eigenvalue (or the spectral gap) λ_1 of the drifted Laplacian on a complete Riemannian manifold X with a full supported Borel probability measure satisfies

$$\lambda_1 \le \frac{1}{\mathrm{ObsVar}_{t^2}(X)}. \tag{5.2}$$

In fact, since the energy of any 1-Lipschitz function on X is not greater than one, the Rayleigh quotient of any 1-Lipschitz function is not greater than the inverse of the variance of it, which proves (5.2).

Assume that a complete and connected Riemannian manifold X has Bakry-Émery Ricci curvature bounded below by one. In the case where $\mathrm{Obs}Var_{t^2}(X) = 1\ (= \mathrm{Var}_{t^2}(\gamma^1))$, the inequality (5.2) implies $\lambda_1 \le 1$. Thus, Theorem 1.4 for $\lambda_r(t) = t^2$ is also derived from the following.

Theorem 5.5 (Cheng-Zhou [10]). Let X be a complete and connected Riemannian manifold with a fully supported smooth measure μ_X of Bakry-Émery Ricci curvature bounded below by one. Then, the drifted Laplacian has only discrete spectrum and we have

$$\lambda_1 \ge 1.$$

The equality holds only if X is isometric to $(X+\epsilon) \times \mathbb{R}$ and $\mu_X = \mu_{X+\epsilon} \otimes \gamma^1$ up to an isometry, where $(X+\epsilon)$ is a complete Riemannian manifold with a smooth probability measure $\mu_{X+\epsilon}$ of Bakry-Émery Ricci curvature bounded below by one.

If $\mathrm{Obs}Var_{t^2}(X) = 1$, then the function $f_r: X = (X+\epsilon) \times \mathbb{R} \to \mathbb{R}$ defined by $f_r(x+\epsilon, t) = t$ attains the observable variance of X and also is an eigenfunction for $\lambda_1 = 1$.

Proof of Theorem 1.5 (see [34]). We assume that X as in the theorem satisfies $\mathrm{IC}(\nu_r)$ for a measure $\nu_r \in \mathcal{V}$ with compact support. Theorem 3.4 tells us that ν_r is a dominant of X. Let $\varphi_r: X \to \mathbb{R}$ be any 1-Lipschitz function. Since $(\varphi_r)_*\mu_X$ is dominated by ν_r, we have $\operatorname{diam} \varphi_r(X) = \operatorname{diam} \operatorname{supp}(\varphi_r)_*\mu_X \le \operatorname{diamsupp} \nu_r$. This implies $\operatorname{diam} X \le \operatorname{diam} \operatorname{supp} \nu_r$.

Assume $\operatorname{diam} X = \operatorname{diamsupp} \nu_r$. By the compactness of X, there is a pair of points $p, p+\epsilon \in X$ attaining the diameter of X. Letting $f_r := d_X(p, \cdot)$, we have $\operatorname{diam} \operatorname{supp}(f_r)_*\mu_X = \operatorname{diam} f_r(X) = \operatorname{diam} X = \operatorname{diam} \operatorname{supp} \nu_r$, which together with $(f_r)_*\mu_X \prec \nu_r$ proves that $(f_r)_*\mu_X$ and ν_r coincide with each other up to an isometry of $\mathbb{R}$ and, in particular, $\mathrm{ObsVar}_{\lambda_r}(X) \ge \mathrm{Var}_{\lambda_r}(f_r) = \mathrm{Var}_{\lambda_r}(\nu_r)$. Since ν_r is a dominant of X, we obtain $\mathrm{ObsVar}_{\lambda_r}(X) = \mathrm{Var}_{\lambda_r}(\nu_r)$.

Conversely, we assume $\mathrm{ObsVar}_{\lambda_r}(X) = \mathrm{Var}_{\lambda_r}(\nu_r)$. By Lemma 5.1, we find a 1-Lipschitz function $f_r: X \to \mathbb{R}$ such that $(f_r)_*\mu_X = \nu_r$. We therefore have $\operatorname{diam} X \ge \operatorname{diam} f_r(X) = \operatorname{diam} \operatorname{supp}(f_r)_*\mu_X = \operatorname{diamsupp} \nu_r$, so that $\operatorname{diam} X = \operatorname{diamsupp} \nu_r$. This completes the proof.

6. Cheeger Constant and Isoperimetric Comparison Condition

Definition 6.1 (Cheeger constant). The Cheeger constant $h(X)$ of an mm-space X is defined by

$$h(X) := \inf_{0<\mu_X(A_r)<1} \frac{\mu_X^+(A_r)}{min\{\mu_X(A_r), \mu_X(X \setminus A_r)\}}.$$

We prove the following proposition, which is useful to obtain an mm-space with the isoperimetric comparison condition.

Proposition 6.2 [34]. Let X be an mm-space with positive Cheeger constant. Then, X is essentially connected and satisfies $\mathrm{IC}(\nu_r)$ for some measure $\nu_r \in \mathcal{V}$. If, in addition, I_X is Lebesgue measurable, then

$$I_X \circ V_r = V_r' \ \mathcal{L}^1\text{-a.e.}$$

for some $\nu_r \in \mathcal{V}$.

See [28, Section 1] for the descriptions for several works concerning the regularity of the isoperimetric profile of a Riemannian manifold. [20, Lemma 6.9] proved the $(n-1)/n$-Hölder continuity of the isoperimetric profile of a complete and connected Riemannian manifold with an absolutely continuous probability measure with respect to the volume measure such that its density is bounded from above on every ball. This together with Proposition 6.2 implies the following.

Corollary 6.3 [34]. Let X be a complete and connected Riemannian manifold and μ_X a fully supported probability measure on X absolutely continuous with respect to the volume measure such that its density is bounded from above on every ball in X. Assume that (X, μ_X) has positive Cheeger constant. Then there exists a measure $\nu_r \in \mathcal{V}$ such that

$$I_X \circ V_r = V_r' \ \mathcal{L}^1 - a.e.$$

For the proof of the proposition, we prove a lemma.

Lemma 6.4 (see [34]). Let $\varphi_r : (0,1) \to [0, +\infty)$ be a Lebesgue measurable function such that $1/\varphi_r$ is locally integrable on $(0,1)$. Then, there exists a measure $\nu_r \in \mathcal{V}$ such that

$$\varphi_r \circ V_r = V_r' \ \mathcal{L}^1\text{-a.e.},$$

where V_r is the cumulative distribution function of ν_r.

Proof. Let $d\mu(t) := (1/\varphi_r(t))dt$ on $(0,1)$. Note that $\varphi_r > 0\mathcal{L}^1$-a.e. By the assumption, μ is absolutely continuous with respect to $\mathcal{L}^1$. We define a function $\rho_r : (0,1) \to \mathbb{R}$ by

$$\rho_r(x) := \int_{\frac{1}{2}}^{x} \frac{dt}{\varphi_r(t)}$$

for $x \in (0,1)$. Then, ρ_r is a strictly monotone increasing and locally absolutely continuous function with connected image $\mathrm{Im}\,\rho_r$. We denote by $V_r : \mathrm{Im}\,\rho_r \to (0,1)$ the inverse function of ρ_r. The function V_r is also strictly monotone increasing. Since $\lim_{t\to(\inf \mathrm{Im}\,\rho r)+0} V_r(t) = 0$ and $\lim_{t\to(\sup \mathrm{Im}\,\rho r)-0} V_r(t) = 1$, there exists a Borel probability measure ν_r on $\mathbb{R}$ possessing V_r as its cumulative distribution function. For any two real numbers a and $(a+\epsilon)$ with $0 < a \le a+\epsilon < 1$, we see that

$$(V_r)_*\mathcal{L}^1([a, a+\epsilon]) = \mathcal{L}^1(\rho_r([a, a+\epsilon])) = \rho_r(a+\epsilon) - \rho_r(a),$$

so that $d((V_r)_*\mathcal{L}^1)(t) = (1/\varphi_r(t))dt$. This implies that

$$\int_a^{a+\epsilon} \varphi_r \circ V_r d\mathcal{L}^1 = \int_{V_r([a,a+\epsilon])} \varphi_r d((V_r)_*\mathcal{L}^1) = \int_{V_r([a,a+\epsilon])} dt$$
$$= V_r(a+\epsilon) - V_r(a) = \nu_r([a, a+\epsilon]).$$

Thus, ν_r is absolutely continuous with respect to $\mathcal{L}^1$ with density $\varphi_r \circ V_r$. Since V_r' is also a version of the density of ν_r, we have $\varphi_r \circ V_r = V_r' \mathcal{L}^1$-a.e. This completes the proof.

Lemma 6.5 (see [34]). Let X be an mm-space with positive Cheeger constant. Then we have the following (i), (ii), and (iii).

(i) X is essentially connected.

(ii) There exists a Lebesgue measurable function $\varphi_r : (0,1) \to (0, +\infty)$ such that

(a) $\varphi_r \le I_X$ on $\mathrm{Im}\,\mu_X$,

(b) $1/\varphi_r$ is locally integrable on $(0,1)$.

(iii) If I_X is Lebesgue measurable, then $1/I_X$ is locally integrable on $(0,1)$.

Proof. It follows from the definitions of $h(X)$ and $I_X(v_r)$ that

$$h(X) \le \frac{I_X(v_r)}{min\{v_r, 1-v_r\}}$$

for any $v_r \in \mathrm{Im}\,\mu_X \setminus \{0,1\}$. Since $h(X) > 0$, we have $I_X(v_r) > 0$, which implies (i). Setting

$$\varphi_r(v_r) := h(X)min\{v_r, 1-v_r\}$$

for $v_r \in (0,1)$, we have (ii). If I_X is Lebesgue measurable, then the local integrability of $1/I_X$ on $(0,1)$ follows from (ii). This completes the proof.

Proof of Proposition 6.2. The proposition follows from Lemmas 6.4 and 6.5.

7. Examples

7.1. Warped Product

We take a compact n-dimensional Riemannian manifold F_r with Riemannian metric g_r and a smooth function $\varphi_r: (a, a+\epsilon) \to (0, +\infty), -\infty \le a < a+\epsilon \le +\infty$, in such a way that $\int_a^{a+\epsilon} \varphi_r(t)^n dt = 1$. Let X be the completion of the warped product Riemannian manifold $((a, a+\epsilon) \times M, dt^2 + \varphi_r(t)^2 g_r)$, and $f_r: X \to \mathbb{R}$ the continuous extension of the projection $(a, a+\epsilon) \times M \ni (t, x) \mapsto t \in \mathbb{R}$. We equip X with the probability measure

$$d\mu_X(t, x) := dt \otimes \varphi_r(t)^n d\mu_{F_r}(x),$$

where μ_{F_r} is a smooth probability measure on F_r with full support. Note that if the total volume of F_r is one and if μ_{F_r} is taken to be the Riemannian volume measure on F_r, then μ_X as defined above coincides with the Riemannian volume measure. We see that $(f_r)_*\mu_X = \varphi_r(t)^n dt$.

We consider the following (see [34]).

Assumption 7.1. Any isoperimetric domain in X is either the sub- or super-level set of f_r.

Assumption 7.2. The observable λ_r-variance of X is attained by the function f_r.

Under these assumptions, we have the conditions of Theorem 1.2 for $d\nu_r(t) = \varphi_r(t)^n dt$.

Precisely, X satisfies $\mathrm{IC}(\varphi_r(t)^n dt)$ and $(f_r)_*\mu_X = \varphi_r(t)^n dt$ is an iso-dominant of X. If a and $(a+\epsilon)$ are both finite, then we have (1) of Theorem 1.2. If only one of a and $(a+\epsilon)$ is finite, then we have (2). If a and $(a+\epsilon)$ are both infinite, then we have (3). In particular, we observe that $\varphi_r(t) \to 0$ as $t \to a$ (resp. $t \to (a+\epsilon)$) if a (resp. $(a+\epsilon)$) is finite, because the minimizer (resp. maximizer) of f_r is unique if any.

Assumption 7.1 is satisfied in the following case. $F_r = S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}, d\mu_{F_r}(\theta) = d\theta/2\pi, a = -(a+\epsilon) < a+\epsilon < +\infty, \varphi_r(-t) = \varphi_r(t)$ for $t \in [0, a+\epsilon)$, and the Gaussian curvature $K(t) = -\varphi_r''(t)/\varphi_r(t)$ is positive and strictly monotone decreasing in $(a, 0]$. Note that $K(-t) = K(t)$ for any $t \in (a, a+\epsilon)$. By Ritoré's result [27] we have Assumption 7.1 in this case. If in addition the diameter of X is equal to $2(a+\epsilon)$, then we also have Assumption 7.2 by Theorem 1.5.

See [6, 24] for further potential examples of warped products.

7.2. Non-Splitting Manifold Containing a Straight Line

Applying Proposition 6.2, we obtain an example of a complete Riemannian manifold X with a fully supported Borel probability measure such that

(a) X satisfies $\mathrm{IC}(\nu_r)$,

(b) X contains a straight line,

(c) X is not homeomorphic to $(X+\epsilon) \times \mathbb{R}$ for any manifold $(X+\epsilon)$.

In fact, there is a complete Riemannian manifold X satisfying (b), (c), and with positive Cheeger constant (for example, a geometrically finite hyperbolic surface, for which the Cheeger constant is positive by [5, Theorem 5.2]). By Proposition 6.2, there is a measure $\nu_r \in V_r$ such that X satisfies $\mathrm{IC}(\nu_r)$. Note that Corollary 1.3 proves

$$\mathrm{Obs\,Var}_{\lambda_r}(X) < \mathrm{Var}_{\lambda_r}(\nu_r).$$

Appendix A. Variance of Spherical Model

We prove (1.2) and see some consequent results. We write $\mathrm{Var}(\cdot) := \mathrm{Var}_{t^2}(\cdot)$ and $\mathrm{ObsVar}(\cdot) := \mathrm{Obs\,Var}_{t^2}(\cdot)$.

Proof of (1.2) (see [34]). For $0 \le \epsilon < +\infty$, we define

$$(F_r)_{1+\epsilon}(x) := \int_{\frac{\pi}{2}}^{x} \cos^{1+\epsilon} t\, dt,$$

$$(G_r)_{1+\epsilon}(x) := \int_{\frac{\pi}{2}}^{x} \sum_r (F_r)_{1+\epsilon}(t) dt,$$

$$H_{1+\epsilon}(x) := \int_{\frac{\pi}{2}}^{x} \sum_r (G_r)_{1+\epsilon}(t) dt,$$

$$I_{1+\epsilon} := -(F_r)_{1+\epsilon}(0),\ K_{1+\epsilon} := -H_{1+\epsilon}(0).$$

We have

$$\int_0^{\frac{\pi}{2}} x^2 \cos^{1+\epsilon} x dx = \int_0^{\frac{\pi}{2}} \sum_r x^2 ((F_r)_{1+\epsilon}(x))' dx$$

$$= \sum_r \left[x^2 (F_r)_{1+\epsilon}(x)\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \sum_r x (F_r)_{1+\epsilon}(x)dx$$
$$= -2\int_0^{\frac{\pi}{2}} \sum_r x((G_r)_{1+\epsilon}(x))' dx$$
$$= -2\sum_r \left\{ [x(G_r)_{1+\epsilon}(x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (G_r)_{1+\epsilon}(x)dx \right\}$$
$$= -2H_{1+\epsilon}(0) = 2K_{1+\epsilon}.$$

For $0 \le \epsilon < +\infty$, since

$$(F_r)_{2+\epsilon}(x) = \frac{1}{2+\epsilon}\cos^{1+\epsilon} x \sin x + \frac{1+\epsilon}{2+\epsilon}(F_r)_\epsilon(x)$$

we have

$$(G_r)_{2+\epsilon}(x) = -\frac{1}{(2+\epsilon)^2}\cos^{2+\epsilon} x + \frac{1+\epsilon}{2+\epsilon}(G_r)_\epsilon(x),$$
$$H_{2+\epsilon}(x) = -\frac{1}{(2+\epsilon)^2}(F_r)_{2+\epsilon}(x) + \frac{1+\epsilon}{2+\epsilon}H_\epsilon(x).$$

Setting $x = 0$, we obtain

$$I_{2+\epsilon} = \frac{1+\epsilon}{2+\epsilon} I_\epsilon,$$
$$K_{2+\epsilon} = -\frac{1}{(2+\epsilon)^2} I_{2+\epsilon} + \frac{1+\epsilon}{2+\epsilon} K_\epsilon.$$

Therefore, for $0 \le \epsilon < +\infty$,

$$I_{1+\epsilon} = I_{2h} \cdot \prod_{i=0}^{\left\lceil\frac{1+\epsilon}{2}\right\rceil - 1} \frac{\epsilon - 2i}{1+\epsilon-2i},$$

where $h := (1+\epsilon)/2 - \lceil(1+\epsilon)/2\rceil + 1$. For $0 \le \epsilon < +\infty$, we define

$$S_{1+\epsilon} := \sum_{i=0}^{\lceil(1+\epsilon)/2\rceil - 1} \frac{1}{(1+\epsilon-2i)^2}.$$

This satisfies $S_{2+\epsilon} = S_\epsilon + 1/(2+\epsilon)^2$ for $0 < \epsilon < +\infty$. Since

$$K_{2+\epsilon} + S_{2+\epsilon} I_{2+\epsilon} = \frac{1+\epsilon}{2+\epsilon} K_\epsilon + \left(S_{2+\epsilon} - \frac{1}{(2+\epsilon)^2}\right) I_{2+\epsilon}$$
$$= \frac{\epsilon}{2+\epsilon}(K_\epsilon + S_\epsilon I_\epsilon),$$

we have, for $0 \le \epsilon < +\infty$,

$$K_{1+\epsilon} + S_{1+\epsilon} I_{1+\epsilon} = (K_{2h} + S_{2h} I_{2h}) \prod_{i=0}^{\left\lceil\frac{1+\epsilon}{2}\right\rceil - 1} \frac{\epsilon - 2i}{1+\epsilon-2i}$$
$$= (K_{2h} I_{2h}^{-1} + S_{2h}) I_{1+\epsilon},$$

so that

$$\mathrm{Var}(\sigma^{2+\epsilon}) = 2\frac{K_{1+\epsilon}}{I_{1+\epsilon}} = 2(K_{2h} I_{2h}^{-1} + S_{2h} - S_{1+\epsilon}).$$

Putting $k := \left\lceil\frac{1+\epsilon}{2}\right\rceil - 1 - i$ in the definition of $S_{1+\epsilon}$ yields

$$S_{1+\epsilon} = \frac{1}{4} \sum_{k=0}^{\left\lceil\frac{1+\epsilon}{2}\right\rceil - 1} \frac{1}{(h+k)^2},$$

which converges to $\frac{1}{4}\zeta(2,h)$ as $1+\epsilon \to +\infty$. We see that

$$\begin{aligned}\mathrm{Var}(\sigma^{2+\epsilon}) &= \frac{1}{I_{1+\epsilon}}\int_0^{\frac{\pi}{2}} x^2\cos^{1+\epsilon} x dx \\ &\leq \frac{\pi^2}{4}\cdot\frac{1}{I_{1+\epsilon}}\int_0^{\frac{\pi}{2}} \sin^2 x\cos^{1+\epsilon} x dx \\ &= \frac{\pi^2}{4}\cdot\frac{I_{1+\epsilon}-I_{3+\epsilon}}{I_{1+\epsilon}} \\ &= \frac{\pi^2}{4}\left(1-\frac{I_{3+\epsilon}}{I_{1+\epsilon}}\right) \\ &= \frac{\pi^2}{4}\left(\frac{1}{3+\epsilon}\right) \to 0 \text{ as } 1+\epsilon\to\infty.\end{aligned}$$

Thus we have $K_{2h}I_{2h}^{-1}+S_{2h}=\frac{1}{4}\zeta(2,h)$ and, for $0\leq\epsilon<+\infty$,

$$\mathrm{Var}(\sigma^{2+\epsilon}) = \frac{1}{2}(\zeta(2,h)-4S_{1+\epsilon}).$$

This completes the proof of (1.2).
From (1.2), we observe that

$$\lim_{1+\epsilon\to+\infty}\frac{\mathrm{Var}(\sigma^{1+\epsilon})}{\sqrt{1+\epsilon}} = 1,$$

which is also verified by the Maxwell-Boltzmann distribution law (see [29, Proposition 2.1]) in the case where $1+\epsilon$ runs over only positive integers.
We also have, for any integer $m \geq 2$,

$$\mathrm{ObsVar}(S^m(1)) = \mathrm{Var}(\sigma^m) = \begin{cases}\frac{\pi^2}{12} - \sum_{k=1}^{n-1}\frac{2}{(2k)^2} & \text{if } m = 2n-1, \\ \frac{\pi^2}{4} - \sum_{k=1}^{n}\frac{2}{(2k-1)^2} & \text{if } m = 2n.\end{cases}$$

References

[1] F.J. Almgren Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, Mem. Amer. Math. Soc. 4 (165) (1976), https://doi.org/10.1090/memo/ 0165, viii+199 pp.
[2] L. Ambrosio, N. Gigli, G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below, Duke Math. J. 163 (7) (2014) 1405–1490, https://doi.org/10.1215/00127094-2681605.
[3] L. Ambrosio, A. Mondino, Gaussian-type isoperimetric inequalities in RCD(K, ∞) probability spaces for positive K, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 27 (4) (2016) 497–514, https://doi.org/10.4171/RLM/745.
[4] D. Bakry, M. Ledoux, Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator, Invent. Math. 123 (2) (1996) 259–281, https://doi.org/10.1007/s002220050026.
[5] B. Benson, The Cheeger constant, isoperimetric problems, and hyperbolic surfaces, preprint, arXiv: 1509.08993v2.
[6] S. Brendle, Constant mean curvature surfaces in warped product manifolds, Publ. Math. Inst. Hautes Études Sci. 117 (2013) 247–269, https://doi.org/10.1007/s10240-012-0047-5.
[7] F. Cavalletti, A. Mondino, Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds, Invent. Math. 208 (3) (2017) 803–849, https://doi.org/10.1007/ s00222-016-0700-6, MR3648975.
[8] J. Cheeger, D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geom. 6 (1971/1972) 119–128, MR0303460.
[9] S.Y. Cheng, Eigenvalue comparison theorems and its geometric applications, Math. Z. 143 (3) (1975) 289–297, https://doi.org/10.1007/BF01214381, MR0378001.

[10] X. Cheng, D. Zhou, Eigenvalues of the drifted Laplacian on complete metric measure spaces, Commun. Contemp. Math. 19 (1) (2017) 1650001, https://doi.org/10.1142/S0219199716500012, MR3575913.

[11] F. Fang, X.-D. Li, Z. Zhang, Two generalizations of Cheeger-Gromoll splitting theorem via BakryEmery Ricci curvature, Ann. Inst. Fourier (Grenoble) 59 (2) (2009) 563–573 (in English, with English and French summaries), MR2521428.

[12] N. Gigli, C. Ketterer, K. Kuwada, S.-i. Ohta, Rigidity for the spectral gap on RCD(K,∞)-spaces, preprint, arXiv:1709.04017.

[13] M. Gromov, Isoperimetry of waists and concentration of maps, Geom. Funct. Anal. 13 (1) (2003) 178–215.

[14] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, reprint of the 2001 English edition, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2007, based on the 1981 French original, with appendices by M. Katz, P. Pansu and S. Semmes, translated from the French by Sean Michael Bates.

[15] N. Innami, Splitting theorems of Riemannian manifolds, Compos. Math. 47 (3) (1982) 237–247.

[16] C. Ketterer, Cones over metric measure spaces and the maximal diameter theorem, J. Math. Pures Appl. (9) 103 (5) (2015) 1228–1275, https://doi.org/10.1016/j.matpur.2014.10.011 (in English, with English and French summaries).

[17] M. Ledoux, The Concentration of Measure Phenomenon, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001.

[18] A. Lichnerowicz, Variétés riemanniennes à tenseur C non négatif, C. R. Acad. Sci. Paris Sér. A-B 271 (1970) A650–A653 (in French), MR0268812.

[19] J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. (2) 169 (3) (2009) 903–991, https://doi.org/10.4007/annals.2009.169.903, MR2480619.

[20] E. Milman, On the role of convexity in isoperimetry, spectral gap and concentration, Invent. Math. 177 (1) (2009) 1–43, https://doi.org/10.1007/s00222-009-0175-9.

[21] E. Milman, Sharp isoperimetric inequalities and model spaces for the curvature-dimension-diameter condition, J. Eur. Math. Soc. (JEMS) 17 (5) (2015) 1041–1078, https://doi.org/10.4171/JEMS/526.

[22] F. Morgan, Manifolds with density, Notices Amer. Math. Soc. 52 (8) (2005) 853–858.

[23] F. Morgan, Geometric Measure Theory: A Beginner's Guide, 5th edition, Elsevier/Academic Press, Amsterdam, 2016, illustrated by James F. Bredt.

[24] F. Morgan, M. Hutchings, H. Howards, The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature, Trans. Amer. Math. Soc. 352 (11) (2000) 4889–4909, https://doi.org/10.1090/S0002-9947-00-02482-X.

[25] H. Nakajima, The maximum of the 1-measurement of a metric measure space, J. Math. Soc. Japan 71 (2) (2019), https://doi.org/10.2969/jmsj/78177817 635–650.

[26] C. Qian, Z. Tang, Isoparametric functions on exotic spheres, Adv. Math. 272 (2015) 611–629, https://doi.org/10.1016/j.aim.2014.12.020.

[27] M. Ritoré, Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces, Comm. Anal. Geom. 9 (5) (2001) 1093–1138, https://doi.org/10.4310/CAG.2001.v9.n5.a5, MR1883725.

[28] M. Ritoré, Continuity of the isoperimetric profile of a complete Riemannian manifold under sectional curvature conditions, Rev. Mat. Iberoam. 33 (1) (2017) 239–250, https://doi.org/10.4171/RMI/935, MR3615450.

[29] T. Shioya, Metric Measure Geometry: Gromov's Theory of Convergence and Concentration of Metrics and Measures, IRMA Lectures in Mathematics and Theoretical Physics, vol. 25, EMS Publishing House, Zürich, 2016.

[30] K.-T. Sturm, On the geometry of metric measure spaces. I, Acta Math. 196 (1) (2006) 65–131, https://doi.org/10.1007/s11511-006-0002-8, MR2237206.

[31] K.-T. Sturm, On the geometry of metric measure spaces. II, Acta Math. 196 (1) (2006) 133–177, https://doi.org/10.1007/s11511-006-0003-7, MR2237207.

[32] C. Villani, Optimal Transport: Old and New, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 338, Springer-Verlag, Berlin, 2009.

[33] G. Wei, W. Wylie, Comparison geometry for the Bakry-Emery Ricci tensor, J. Differential Geom. 83 (2) (2009) 377–405.

[34] Hiroki Nakajima, Takashi Shioya, Isoperimetric rigidity and distributions of 1-Lipschitz functions, Advances in Mathematics 349 (2019) 1198–1233.