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Review Paper

Fixed point theorems on $\mathcal{G}-$ metric spaces Via $C-$ Class Functions

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Abstract: *In this manuscript, we prove generalized fixed point theorems via $C-$ class functions on $\mathcal{G}-$ metric spaces. Further, we also provide some examples and corollaries to prove the existence and uniqueness of our results.*
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I. Introduction

Fixed point theory has an application in many fields such as physics, chemistry, biology and many areas of mathematics. The Banach contraction mapping principle [7] is the most pioneer result in the complete metric space. Banach contraction principle establishes that every mapping $\mathcal{R} : Y \to Y$, which is defined on complete metric space (Y, d) and satisfy the following condition for all $u, v \in Y$,

$$d\big(\mathcal{R}(u), \mathcal{R}(v)\big) \leq \lambda d(u, v),$$

where $0 < \lambda < 1$ is a constant, has a unique fixed point in Y.

The concept of a $\mathcal{G}-$ metric space was introduced by Mustafa and Sims [13] which is different from further spaces. After this appreciative work of Mustafa and Sims [13], many writers inspired to study the hurdles of the fixed point, common fixed point, common fuzzy fixed point by using different contractive conditions for mappings, see for examples ([1], [6], [8], [10], [11], [12]).

II. Preliminaries

Definition 2.1 [13] Let Y be a non-empty set and $\mathcal{G} : Y \times Y \times Y \to \mathbb{R}_0^+$ be a function such that for all $u, v, w, a \in Y$, satisfying the following properties

$(\mathcal{G}1)$ $\mathcal{G}(u, v, w) = 0$ if $u = v = w$;

$(\mathcal{G}2)$ $\mathcal{G}(u, u, v) > 0$ with $u \neq v$;

$(\mathcal{G}3)$ $\mathcal{G}(u, u, v) \leq \mathcal{G}(u, v, w)$ with $w \neq v$;

$(\mathcal{G}4)$ $\mathcal{G}(u, v, w) = \mathcal{G}(u, w, v) = \mathcal{G}(v, w, u) = \mathcal{G}(w, u, v) = \cdots$, (Symmetry in all three variables);

$(\mathcal{G}5)$ $\mathcal{G}(u,v,w) \leq \mathcal{G}(u,a,a) + \mathcal{G}(a,v,w)$, (Rectangle inequality).

Then, the function $\mathcal{G}$ is called a $\mathcal{G}-$ metric on Y, and the pair $(Y,\mathcal{G})$ is a $\mathcal{G}-$ metric space.

All these properties are satisfied when $\mathcal{G}(u,v,w)$ is the perimeter of the triangle with vertices at u,v and w in $\mathbb{R}^2$.

Example 2.2 [13] Let (Y,d) be a metric space. The mapping $\mathcal{G}_S : Y^3 \to \mathbb{R}_0^+$ defined by

$\mathcal{G}_S(u,v,w) = d(u,v) + d(v,w) + d(u,w)$, for all $u,v,w \in Y$,

is a $\mathcal{G}-$ metric and therefore $(Y,\mathcal{G}_S)$ is a $\mathcal{G}-$ metric space.

Definition 2.3 [13] The $\mathcal{G}-$ metric space $(Y,\mathcal{G})$ is called symmetric if $\mathcal{G}(u,u,v) = \mathcal{G}(v,v,u)$, for all $u,v \in Y$.

Proposition 2.4 [13] Let $(Y,\mathcal{G})$ be a $\mathcal{G}-$ metric space. Then for any $u,v,w,a \in Y$, it follows that:

(i) If $\mathcal{G}(u,v,w) = 0$, then $u = v = w$;
(ii) $\mathcal{G}(u,v,w) \leq \mathcal{G}(u,u,v) + \mathcal{G}(u,u,w)$;
(iii) $\mathcal{G}(u,v,v) \leq 2\mathcal{G}(v,u,u)$;
(iv) $\mathcal{G}(u,v,w) \leq \mathcal{G}(u,a,w) + \mathcal{G}(a,v,w)$;
(v) $\mathcal{G}(u,v,w) \leq \frac{2}{3}[\mathcal{G}(u,v,a) + \mathcal{G}(u,a,w) + \mathcal{G}(a,v,w)]$;
(vi) $\mathcal{G}(u,v,w) \leq \mathcal{G}(u,a,a) + \mathcal{G}(v,a,a) + \mathcal{G}(w,a,a)$.

Definition 2.5 [13] Let $(Y,\mathcal{G})$ be a $\mathcal{G}-$metric space and let $\{u_n\}$ be a sequence of points of Y. Then, the sequence $\{u_n\}$ is $\mathcal{G}-$ convergent to $u \in Y$ if $\mathcal{G}(u_m,u_n,u) \to 0$ as $m,n \to \infty$.

Proposition 2.6 [13] Let $(Y,\mathcal{G})$ be a $\mathcal{G}-$ metric space, therefore for a sequence $\{u_n\} \subseteq Y$ and a point $u \in Y$, the following are equivalent:

(i) $\{u_n\}$ is $\mathcal{G}-$ convergent to u.

(ii) $\mathcal{G}(u_n,u_n,u) \to 0$ as $n \to \infty$.

(iii) $\mathcal{G}(u_n,u,u) \to 0$ as $n \to \infty$.

Definition 2.7 [13] Let $(Y,\mathcal{G})$ be a $\mathcal{G}-$ metric space. A sequence $\{u_n\}$ is called $\mathcal{G}-$ Cauchy sequence, if for any $\varepsilon > 0$, there exists an $N_0 \in \mathbb{N}$ such that $\mathcal{G}(u_n,u_m,u_m) < \varepsilon$, for all $n,m \geq N_0$.

Definition 2.8 [13] If every $\mathcal{G}-$ Cauchy sequence in $(Y,\mathcal{G})$ is $\mathcal{G}-$ convergent in $(Y,\mathcal{G})$, then a $\mathcal{G}-$ metric space $(Y,\mathcal{G})$ is said to be $\mathcal{G}-$ complete.

Definition 2.9 [2] A mapping $\mathcal{F} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is called a $C-$ class function if it is continuous and satisfies the properties:

(i) $\mathcal{F}(r,t) \leq r$;
(ii) $\mathcal{F}(r,t) = r$ implies that either $r = 0$ or $t = 0$, for all $r,t \in \mathbb{R}$.

Also, for any $\mathcal{F}$, we obtain $\mathcal{F}(r,t) = 0$.

The class of all $C-$ class functions is denoted by $\mathcal{C}$. The upcoming example proves that $\mathcal{C}$ is non-empty.

Example 2.10 [2] Each of the functions $\mathcal{F} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ explained below are elements of $\mathcal{C}$.

(i) $\mathcal{F}(r,t) = r - t$;

(ii) $\mathcal{F}(r,t)=\frac{r}{(1+t)^s}, s\in(0,\infty)$;

(iii) $\mathcal{F}(r,t)=mr, 0<m<1$;

(iv) $\mathcal{F}(r,t)=r\beta_0(r)$ where $\beta_0:\mathbb{R}_0^+\to\mathbb{R}_0^+$ and is continuous;

(v) $\mathcal{F}(r,t)=\frac{r}{(1+r)^s}, s\in(0,\infty)$.

The above items (i) (iii) and (iv) are central results in [2]. Also see paper [4] and [9].

Definition 2.11 [2] Let $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ be a function which satisfies the properties:

(φ1) φ is continuous and non-decreasing function;

(φ2) $\varphi(t)=0$ if and only if $t=0$.

Then, φ is called an altering distance function.

Remark 2.12 The class of all altering distance functions is denoted by Φ.

Definition 2.13 [2] Let $\psi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ be a function which is also non-decreasing and continuous function such that $\psi(t)>0$ for $t>0$.

Then, ψ is called an ultra-altering distance function.

Remark 2.14 The class of all ultra-altering distance functions is denoted by Ψ_u.

Definition 2.15 Let $(Y,\mathcal{G})$ be a $\mathcal{G}-$metric space and $\mathcal{R}$ be a self-mapping defined on Y. Then $\mathcal{R}:Y\to Y$ is called a contraction if there exist a constant κ with $0\le\kappa<1$ such that for all $u,v,w\in Y$,

$\mathcal{G}(\mathcal{R}u,\mathcal{R}v,\mathcal{R}w)\le\kappa\mathcal{G}(u,v,w)$.

III. Main Results

In this manuscript, we prove fixed point theorems via $C-$ class functions on $\mathcal{G}-$ metric spaces. Further we also provide some examples and corollaries to prove the existence and uniqueness of our results.

Theorem 3.1 Let $h:Y\to Y$ be a self-mapping defined on complete $\mathcal{G}-$ metric space $(Y,\mathcal{G})$ and

satisfy the following inequality for all $u,v,w\in Y$,

$$\varphi\big(\mathcal{G}(hu,hv,hw)\big)\le\mathcal{F}\Big(\varphi\big(\Theta_0(u,v,w)\big),\psi\big(\Theta_0(u,v,w)\big)\Big), \qquad (1)$$

where

$$\Theta_0(u,v,w)=k_1\mathcal{G}(u,v,w)+k_2\mathcal{G}(u,u,hu)+k_3\mathcal{G}(w,w,hw)+$$
$$k_4[\mathcal{G}(w,w,hu)+\mathcal{G}(u,u,hw)]+k_5\left(\frac{\mathcal{G}(w,w,hw)}{(1+\mathcal{G}(u,v,w))}\right)$$

and $k_1,k_2,k_3,k_4,k_5>0$ are non-negative reals with $k_1+k_2+k_3+2k_4+k_5<1, \varphi\in\Phi$,

$\psi\in\Psi_u$ *and* $\mathcal{F}\in\mathcal{C}$. Then, h has a unique fixed point in Y, that is, $hl=l$.

Proof. Let $u_0\in Y$ be any arbitrary point.

Consider that $u_{2n+1}=hu_{2n}$ for $n=0,1,2,...$

Now, we have to show that $\{u_n\}$ is a $\mathcal{G}-$ Cauchy sequence in $(Y,\mathcal{G})$. But for this, firstly we will prove that

$lim_{n\to\infty}\mathcal{G}(u_{n+1},u_{n+1},u_n)=0.$

Now, putting $u=v=u_{2n}, w=u_{2n-1}$ in equation (1) and using property $(\mathcal{G}1)$, $(\mathcal{G}5)$ and Definition 2.3, we get

$$\varphi\big(\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})\big)=\varphi\big(\mathcal{G}(hu_{2n},hu_{2n},hu_{2n-1})\big)$$

$$\leq \mathcal{F}\Big(\varphi\big(\Theta_0(u_{2n},u_{2n},u_{2n-1})\big),\psi\big(\Theta_0(u_{2n},u_{2n},u_{2n-1})\big)\Big), \qquad (2)$$

where

$$\Theta_0(u_{2n},u_{2n},u_{2n-1})$$

$$=k_1\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+k_2\mathcal{G}(u_{2n},u_{2n},hu_{2n})+k_3\mathcal{G}(u_{2n-1},u_{2n-1},hu_{2n-1})$$

$$+k_4[\mathcal{G}(u_{2n-1},u_{2n-1},hu_{2n})+\mathcal{G}(u_{2n},u_{2n},hu_{2n-1})]+k_5\left(\frac{\mathcal{G}(u_{2n-1},u_{2n-1},hu_{2n-1})}{(1+\mathcal{G}(u_{2n},u_{2n},u_{2n-1}))}\right)$$

$$=k_1\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+k_2\mathcal{G}(u_{2n},u_{2n},u_{2n+1})+k_3\mathcal{G}(u_{2n-1},u_{2n-1},u_{2n})$$

$$+k_4[\mathcal{G}(u_{2n-1},u_{2n-1},u_{2n+1})+\mathcal{G}(u_{2n},u_{2n},u_{2n})]+k_5\left(\frac{\mathcal{G}(u_{2n-1},u_{2n-1},u_{2n})}{(1+\mathcal{G}(u_{2n},u_{2n},u_{2n-1}))}\right)$$

$$\leq k_1\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+k_2\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})+k_3\mathcal{G}(u_{2n},u_{2n},u_{2n-1})$$

$$+k_4[\mathcal{G}(u_{2n-1},u_{2n-1},u_{2n})+\mathcal{G}(u_{2n},u_{2n},u_{2n+1})]+k_5\left(\frac{\mathcal{G}(u_{2n},u_{2n},u_{2n-1})}{(1+\mathcal{G}(u_{2n},u_{2n},u_{2n-1}))}\right)$$

$$=k_1\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+k_2\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})+k_3\mathcal{G}(u_{2n},u_{2n},u_{2n-1})$$

$$+k_4[\mathcal{G}(u_{2n-1},u_{2n-1},u_{2n})+\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})]+k_5\left(\frac{\mathcal{G}(u_{2n},u_{2n},u_{2n-1})}{(1+\mathcal{G}(u_{2n},u_{2n},u_{2n-1}))}\right)$$

$$\leq k_1\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+k_2\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})+k_3\mathcal{G}(u_{2n},u_{2n},u_{2n-1})$$

$$+k_4[\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})]+k_5\mathcal{G}(u_{2n},u_{2n},u_{2n-1})$$

$$=(k_1+k_3+k_4+k_5)\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+(k_2+k_4)\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n}). \qquad (3)$$

Putting the value of $\Theta_0(u_{2n},u_{2n},u_{2n-1})$ from equation (3) in equation (2) and also using the property of $\mathcal{F}$, we have

$$\varphi\big(\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})\big)$$

$$\leq \mathcal{F}\begin{pmatrix}\varphi\big((k_1+k_3+k_4+k_5)\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+(k_2+k_4)\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})\big),\\ \psi\big((k_1+k_3+k_4+k_5)\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+(k_2+k_4)\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})\big)\end{pmatrix}$$

$$\leq \varphi\big((k_1+k_3+k_4+k_5)\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+(k_2+k_4)\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})\big). \qquad (4)$$

As $\varphi\in\Phi$, then using the property of φ, we get

$$\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})\leq (k_1+k_3+k_4+k_5)\mathcal{G}(u_{2n},u_{2n},u_{2n-1})+$$

$$(k_2+k_4)\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n}).$$

That is, $\mathcal{G}(u_{2n+1},u_{2n+1},u_{2n})\leq\left(\frac{k_1+k_3+k_4+k_5}{1-k_2-k_4}\right)\mathcal{G}(u_{2n},u_{2n},u_{2n-1})=\alpha\mathcal{G}(u_{2n},u_{2n},u_{2n-1}),$ (5)

where $\alpha=\left(\frac{k_1+k_3+k_4+k_5}{1-k_2-k_4}\right)<1$,

because $k_1+k_2+k_3+2k_4+k_5<1$.

Therefore,

$$\mathcal{G}(u_{n+1},u_{n+1},u_n)\le\alpha\mathcal{G}(u_n,u_n,u_{n-1}), \tag{6}$$

for $n=0,1,2,\ldots$

Now, consider $d_n=\mathcal{G}(u_{n+1},u_{n+1},u_n)$ and $d_{n-1}=\mathcal{G}(u_n,u_n,u_{n-1})$.

Hence, from equation (6), we obtain

$$d_n\le\alpha d_{n-1}\le\alpha^2 d_{n-2}\le\cdots\le\alpha^n d_0. \tag{7}$$

As $0\le\alpha<1$, then taking the limit as $n\to\infty$, we obtain

$$lim_{n\to\infty}\mathcal{G}(u_{n+1},u_{n+1},u_n)=0. \tag{8}$$

Next, we will prove that $\{u_n\}$ is a $\mathcal{G}-$ Cauchy sequence in $(Y,\mathcal{G})$.

We assume that $m>n$, for all $n,m\in\mathbb{N}$ and also using the property $(\mathcal{G}5)$, Definition 2.3 and using equation (7), we get

$$\begin{aligned}\mathcal{G}(u_n,u_n,u_m)&\le\mathcal{G}(u_n,u_n,u_{n+1})+\mathcal{G}(u_{n+1},u_{n+1},u_m)\\&\le\mathcal{G}(u_n,u_n,u_{n+1})+\mathcal{G}(u_{n+1},u_{n+1},u_{n+2})+\mathcal{G}(u_{n+2},u_{n+2},u_m)\\&\le\mathcal{G}(u_n,u_n,u_{n+1})+\mathcal{G}(u_{n+1},u_{n+1},u_{n+2})+\mathcal{G}(u_{n+2},u_{n+2},u_{n+3})+\cdots\\&\quad+\mathcal{G}(u_{m-1},u_{m-1},u_m)\\&\le(\alpha^n+\alpha^{n+1}+\alpha^{n+2}+\cdots+\alpha^{m-1})\mathcal{G}(u_0,u_0,u_1)\\&=(\alpha^n+\alpha^{n+1}+\alpha^{n+2}+\cdots+\alpha^{m-1})d_0\\&=\left(\textstyle\sum_{p=n}^{m-1}\alpha^p\right)d_0.\end{aligned} \tag{9}$$

Letting $n,m\to\infty$, we get $\mathcal{G}(u_n,u_n,u_m)\to0$, as $0\le\alpha<1$.

Therefore, $\{u_n\}$ is a $\mathcal{G}-$ Cauchy sequence in Y. Also, $(Y,\mathcal{G})$ is $\mathcal{G}-$ complete, then there exists $l\in Y$ such that $lim_{n\to\infty}u_n=l$.

Now, we will prove that l is a fixed point of h.

Putting $u=v=u_{2n}$ and $w=l$ in equation (1), we obtain

$$\begin{aligned}\varphi\big(\mathcal{G}(u_{2n+1},u_{2n+1},hl)\big)&=\varphi\big(\mathcal{G}(hu_{2n},hu_{2n},hl)\big)\\&\le\mathcal{F}\Big(\varphi\big(\Theta_0(u_{2n},u_{2n},l)\big),\psi\big(\Theta_0(u_{2n},u_{2n},l)\big)\Big),\end{aligned} \tag{10}$$

where

$\Theta_0(u_{2n},u_{2n},l)$

$= k_1\mathcal{G}(u_{2n}, u_{2n}, l) + k_2\mathcal{G}(u_{2n}, u_{2n}, hu_{2n}) + k_3\mathcal{G}(l, l, hl)$

$+k_4[\mathcal{G}(l, l, hu_{2n}) + \mathcal{G}(u_{2n}, u_{2n}, hl)] + k_5\left(\frac{\mathcal{G}(l,l,hl)}{(1+\mathcal{G}(u_{2n},u_{2n},l))}\right)$

$= k_1\mathcal{G}(u_{2n}, u_{2n}, l) + k_2\mathcal{G}(u_{2n}, u_{2n}, u_{2n+1}) + k_3\mathcal{G}(l, l, hl)$

$+k_4[\mathcal{G}(l, l, u_{2n+1}) + \mathcal{G}(u_{2n}, u_{2n}, hl)] + k_5\left(\frac{\mathcal{G}(l,l,hl)}{(1+\mathcal{G}(u_{2n},u_{2n},l))}\right).$ (11)

Taking $n \to \infty$ in the above equation (11) and using the property $(\mathcal{G}1)$, we obtain

$\Theta_0(u_{2n}, u_{2n}, l) = (k_3 + k_4 + k_5)\mathcal{G}(l, l, hl).$ (12)

Using the property of $\mathcal{F}$ and also using equation (12) in equation (10), we get

$$\varphi\big(\mathcal{G}(u_{2n+1}, u_{2n+1}, hl)\big) = \varphi\big(\mathcal{G}(hu_{2n}, hu_{2n}, hl)\big)$$

$$\leq \mathcal{F}\Big(\varphi\big((k_3 + k_4 + k_5)\mathcal{G}(l, l, hl)\big), \psi\big((k_3 + k_4 + k_5)\mathcal{G}(l, l, hl)\big)\Big)$$

$$\leq \varphi\big((k_3 + k_4 + k_5)\mathcal{G}(l, l, hl)\big). \quad (13)$$

Again, taking $n \to \infty$ in equation (13), we get

$\varphi\big(\mathcal{G}(l, l, hl)\big) \leq \varphi\big((k_3 + k_4 + k_5)\mathcal{G}(l, l, hl)\big).$ (14)

As $\varphi \in \Phi$, then using the property of φ in equation (14), we conclude that

$$\mathcal{G}(l, l, hl) \leq (k_3 + k_4 + k_5)\mathcal{G}(l, l, hl)$$

$$\leq (k_1 + k_2 + k_3 + 2k_4 + k_5)\mathcal{G}(l, l, hl)$$

$$< \mathcal{G}(l, l, hl),$$

as $(k_1 + k_2 + k_3 + 2k_4 + k_5) < 1$,

a contradiction.

Therefore, $\mathcal{G}(l, l, hl) = 0$.

In other words, $hl = l$.

This proves that l is a fixed point of h.

Uniqueness: Let l' be another fixed point of h such that $hl' = l'$ with $l' \neq l$.

Now, using equation (1) for $u = v = l$ and $w = l'$ and also using the property $(\mathcal{G}1)$ and Definition 2.3, we obtain

$$\varphi\big(\mathcal{G}(l, l, l')\big) = \varphi\big(\mathcal{G}(hl, hl, hl')\big)$$

$$\leq \mathcal{F}\Big(\varphi\big(\Theta_0(l, l, l')\big), \psi\big(\Theta_0(l, l, l')\big)\Big), \quad (15)$$

where

$\Theta_0(l, l, l')$

$= k_1\mathcal{G}(l,l,l') + k_2\mathcal{G}(l,l,hl) + k_3\mathcal{G}(l',l',hl') + k_4[\mathcal{G}(l',l',hl) + \mathcal{G}(l,l,hl')] + k_5\left(\frac{\mathcal{G}(l',l',hl')}{(1+\mathcal{G}(l,l,l'))}\right) =$ $k_1\mathcal{G}(l,l,l') + k_2\mathcal{G}(l,l,l) + k_3\mathcal{G}(l',l',l') + k_4[\mathcal{G}(l',l',l) + \mathcal{G}(l,l,l')] + k_5\left(\frac{\mathcal{G}(l',l',l')}{(1+\mathcal{G}(l,l,l'))}\right) = k_1\mathcal{G}(l,l,l') + 0 + 0 +$ $k_4[2\mathcal{G}(l,l,hl')] + 0$

$= (k_1 + 2k_4)\mathcal{G}(l,l,l').$ (16)

Putting the value of $\Theta_0(l,l,l')$ from equation (16) in equation (15) and also using the property of $\mathcal{F}$, we get

$\varphi(\mathcal{G}(l,l,l')) = \varphi(\mathcal{G}(hl,hl,hl'))$

$\leq \mathcal{F}\left(\varphi((k_1 + 2k_4)\mathcal{G}(l,l,l')), \psi((k_1 + 2k_4)\mathcal{G}(l,l,l'))\right),$

$\leq \varphi((k_1 + 2k_4)\mathcal{G}(l,l,l')).$ (17)

As $\varphi \in \Phi$, then again using the property of φ in equation (17), we obtain

$\mathcal{G}(l,l,l') \leq (k_1 + 2k_4)\mathcal{G}(l,l,l')$

$\leq (k_1 + k_2 + k_3 + 2k_4 + k_5)\mathcal{G}(l,l,l')$

$< \mathcal{G}(l,l,l'),$ (18)

as $(k_1 + k_2 + k_3 + 2k_4 + k_5) < 1,$

again, we get a contradiction.

Hence, $\mathcal{G}(l,l,l') = 0$, that is, $l = l'$.

Therefore, l is a fixed point of h in Y.

Corollary 3.2 Let $h : Y \to Y$ be a self-mapping defined on complete $\mathcal{G}-$metric space $(Y,\mathcal{G})$ and satisfy the following inequality for all $u,v,w \in Y$,

$\varphi(\mathcal{G}(hu,hv,hw)) \leq \varphi(\Theta_0(u,v,w)) - \psi(\Theta_0(u,v,w)),$ (19)

where

$\Theta_0(u,v,w) = k_1\mathcal{G}(u,v,w) + k_2\mathcal{G}(u,u,hu) + k_3\mathcal{G}(w,w,hw) +$

$k_4[\mathcal{G}(w,w,hu) + \mathcal{G}(u,u,hw)] + k_5\left(\frac{\mathcal{G}(w,w,hw)}{(1+\mathcal{G}(u,v,w))}\right)$

and $k_1,k_2,k_3,k_4,k_5 > 0$ are non-negative reals with $k_1 + k_2 + k_3 + 2k_4 + k_5 < 1, \varphi \in \Phi,$

$\psi \in \Psi_u \text{ and } \mathcal{F} \in \mathcal{C}.$

Then, h has a unique fixed point in Y, that is, $hl = l$.

Proof. If we take $\mathcal{F}(r,t) = r - t$ in Theorem 3.1, then we get the required result.

Corollary 3.3 Let $h : Y \to Y$ be a self-mapping defined on compete $\mathcal{G}-$ metric space $(Y,\mathcal{G})$ which satisfy the following inequality for all $u,v,w \in Y$,

$\mathcal{G}(hu,hv,hw) \leq \kappa\,\mathcal{G}(u,v,w),$ (20)

where $\kappa \in [0,1)$ is a constant.

Then, h has a unique fixed point in Y, that is, $hl = l$.

Proof. If we consider $\mathcal{F}(r,t) = \mathcal{m}r$ for some $\mathcal{m}$ such that $0 < \mathcal{m} < 1, \varphi(t) = t$, for all $t \geq 0$ and taking $k_1 = \kappa$, where $\kappa \in [0,1)$ and also $k_2 = k_3 = k_4 = k_5 = 0$ in Theorem 3.1, then we get the required result (with $\mathcal{m}\kappa \to \kappa$).

Corollary 3.4 Let $h : Y \to Y$ be a self-mapping defined on complete $\mathcal{G}-$ metric space $(Y, \mathcal{G})$ which satisfy the following inequality for all $u, v, w \in Y$,

$$\mathcal{G}(hu,hv,hw) \leq k_1\mathcal{G}(u,v,w) + k_2\mathcal{G}(u,u,hu) + k_3\mathcal{G}(w,w,hw) + k_4[\mathcal{G}(w,w,hu) + \mathcal{G}(u,u,hw)] + k_5\left(\frac{\mathcal{G}(w,w,hw)}{(1+\mathcal{G}(u,v,w))}\right), \tag{21}$$

where $k_1, k_2, k_3, k_4, k_5 > 0$ are non-negative reals with $k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$.

Then, h has a unique fixed point in Y, that is, $hl = l$.

Proof. If we consider $\mathcal{F}(r,t) = \mathcal{m}r$ for some $0 < \mathcal{m} < 1, \varphi(t) = t$, for all $t \geq 0$ in Theorem 3.1, then we get the required result (with $\mathcal{m}k_1 \to k_1, \mathcal{m}k_2 \to k_2, \mathcal{m}k_3 \to k_3, \mathcal{m}k_4 \to k_4, \mathcal{m}k_5 \to k_5$).

Example 3.5 Let $Y = [0,2]$ and $h : Y \to Y$ be a mapping defined as $h(u) = \frac{u}{3}$, for all $u \in Y$.

Also, a mapping $\mathcal{G} : Y^3 \to [0,\infty)$ be defined by

$$\mathcal{G}(u,v,w) = \begin{cases} 0, & u = v = w, \\ \max\{u,v,w\}, & \text{otherwise}, \end{cases}$$

for all $u, v, w \in Y$, is a $\mathcal{G}-$ metric space on Y.

Case 1: If we consider $u = v = w$, then both equations (21) and (20) are truly hold.

Case 2: (a) If we consider $u > v > w$, for all $u, v, w \in Y$, then

$$\mathcal{G}(hu,hv,hw) = \max\left\{\frac{u}{3},\frac{v}{3},\frac{w}{3}\right\} = \frac{u}{3},$$

$$\mathcal{G}(u,v,w) = \max\{u,v,w\} = u,$$

$$\mathcal{G}(u,u,hu) = \max\left\{u,u,\frac{u}{3}\right\} = u,$$

$$\mathcal{G}(w,w,hw) = \max\left\{w,w,\frac{w}{3}\right\} = w,$$

$$\mathcal{G}(w,w,hu) = \max\left\{w,w,\frac{u}{3}\right\} = \frac{u}{3},$$

$$\mathcal{G}(u,u,hw) = \max\left\{u,u,\frac{w}{3}\right\} = u.$$

Using equation (21) of Corollary 3.4, we obtain

$$\frac{u}{3} \leq k_1u + k_2u + k_3w + \frac{4}{3}k_4u + k_5\frac{w}{1+u}.$$

Now, consider $u = 2, v = 1, w = \frac{2}{3}$, then we get

$\frac{2}{3} \leq 2k_1 + 2k_2 + \frac{2}{3}k_3 + \frac{8}{3}k_4 + \frac{2}{9}k_5.$

That is, $6 \leq 18k_1 + 18k_2 + 6k_3 + 24k_4 + 2k_5.$ (22)

The above equation (22) is valid for:

(i) $k_1 = \frac{2}{4}, k_2 = \frac{2}{5}$ and $k_3 = k_4 = k_5 = 0;$

(ii) $k_1 = \frac{1}{3}, k_3 = \frac{1}{4}, k_4 = \frac{1}{5}$ and $k_2 = k_5 = 0;$

(iii) $k_2 = \frac{2}{7}, k_3 = \frac{3}{7}$ and $k_1 = k_4 = k_5 = 0,$

with $k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$.

Hence, all the required conditions of Corollary 3.4 are satisfied.

Therefore, h has a unique fixed point in Y by applying Corollary 3.4.

Evidently, $0 \in Y$ is the unique fixed point of h in this case.

(b) Now assume equation (20) of Corollary 3.3, we obtain

$\frac{u}{3} \leq \kappa u,$

or $\kappa \geq \frac{1}{3}$.

If we consider $0 < \kappa < 1$, then all the required conditions of Corollary 3.3 are satisfied and $0 \in Y$ is the unique fixed point of h in this case also.

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