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Review Paper

Difference of Weighted Composition Operators with Series of Symbols

Fatima Yousief (1) and Shawgy Hussein(2)
(1) *Sudan University of Science and Technology, Sudan.*
(2) *Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.*

Abstract
Following the stream of the authors in [32] we introduce complete characterizations in terms of Carleson measures for bounded/compact differences of weighted composition operators with series of symbols acting on the standard weighted Bergman spaces over the unit disk. We allow the weight functions to be non-holomorphic and unbounded. We obtain a compactness characterization for differences of unweighted composition operators acting on the Hardy spaces in terms of Carleson measures and show that compact differences of composition operators with univalent symbols on the Hardy and weighted Bergman spaces are exactly the same. We also show that an earlier characterization due to Acharyya and Wu for compact differences of weighted composition operators with
bounded holomorphic weights does not extend to the case of non-holomorphic weights. Some extended explicit examples are shown.

Keywords: *Difference, Weighted composition operator, weighted Bergman space, Hardy space*

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I. INTRODUCTION

For $\mathbf{D}$ be the unit disk in the complex plane $\mathbf{C}$. Denote by $\mathcal{S}(\mathbf{D})$ the set of all holomorphic self-maps of $\mathbf{D}$. Given the sequence $\varphi_j \in \mathcal{S}(\mathbf{D})$, the composition operator $C_{\sum_j \varphi_j}$ with symbol $\sum_j \varphi_j$ is defined by

$$C_{\sum_j \varphi_j} f_j := f_j \circ \sum_j \varphi_j \, \psi_j$$

for functions f_j holomorphic on $\mathbf{D}$. A wide study on the theory of composition operators has been established on various settings, see [8] and [27] for various aspects on the theory of composition operators acting on classical holomorphic function spaces.

A weighted composition operator is a composition operator followed by a multiplication operator. Hence, for a Borel function u on $\mathbf{D}$, the weighted composition operator $C_{\sum_j \varphi_j, u}$ with symbol $\sum_j \varphi_j$ and weight u is defined by

$$C_{\sum_j \varphi_j, u} f_j := u\left(f_j \circ \sum_j \varphi_j\right)$$

for functions f_j holomorphic on $\mathbf{D}$. These operators appear naturally in studying operator theory on classical holomorphic functions spaces. For example, isometries on Hardy or Bergman spaces are weighted composition operators; see [12, 16, 17]. Also, Brennan's Conjecture, an important conjecture in univalent function theory, is closely connected with weighted composition operators; see [21]. The boundedness and compactness of weighted composition operators with holomorphic weights are characterized on various settings; see [7, 9, 10, 24, 25].

Suppose we are familiar by the study of path-connected components in the space of composition operators, the study on differences, or more generally linear combinations, of unweighted composition operators has been of growing interest; see [4]. In similar way, one may study differences of weighted composition operators. In fact such operators also appear naturally, when the setting of differences of unweighted composition operators is changed from a holomorphic function space into another; (see, [1]) first obtained a compactness characterization for differences of weighted composition operators acting from a weighted Bergman space into another. However, their weights are restricted to a class of holomorphic functions satisfying certain growth rates. For example, for operators acting from a weighted Bergman space into itself, their weights are restricted to the class of bounded holomorphic functions so that individual operators are automatically bounded. Wang, Yao and Chen also [30] obtained some partial results concerning compactness of those operators acting on the weighted Bergman spaces. Their weights are also restricted to the class of bounded holomorphic functions. Following the authors in [32] we obtain complete characterizations in terms of Carleson measures for bounded/compact differences of weighted composition operators with series of symbols acting on the weighted Bergman spaces. The significance of our characterizations lies in allowing the weights to be non-holomorphic and unbounded. In particular, individual operators are no longer guaranteed to be bounded.

Now given $\epsilon > 0$, we denote by $A_{\epsilon-1}$ the normalized weighted measure defined by

$$dA_{\epsilon-1}(z) \coloneqq (\epsilon)(1-|z|^2)^{\epsilon-1}dA(z), \quad z \in \mathbf{D}$$

where A denotes the area measure on $\mathbf{D}$ normalized to have the total mass 1.

For $0 \le \epsilon < \infty$, the $(\epsilon-1)$-weighted Bergman space $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is the space of all holomorphic functions f_j on $\mathbf{D}$ for which the "norm"

$$\|f_j\|_{A^{1+\epsilon}_{\epsilon-1}} \coloneqq \left\{\int_{\mathbf{D}} \sum_j |f_j|^{1+\epsilon} dA_{\epsilon-1}\right\}^{1/1+\epsilon}$$

is finite. The space $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is a closed subspace of $L^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \coloneqq L^{1+\epsilon}(\mathbf{D}, A_{\epsilon-1})$, the standard Lebesgue space associated with the measure $A_{\epsilon-1}$. So, it is a Banach space for $0 \le \epsilon < \infty$ and a complete metric space for $0 < \epsilon < 1$ with respect to the translationin-variant metric $(f_j, g_j) \mapsto \|f_j - g_j\|^{1+\epsilon}_{A^{1+\epsilon}_{\epsilon-1}}$. We fix the parameters $\epsilon > 0$ and $0 < \epsilon < \infty$ throughout the sequel.

To begin with we keep the notations of [32]. We reserve symbol functions $\varphi_j, \psi_j \in \mathcal{S}(\mathbf{D})$ and weights u, v to be considered throughout the sequel. We put

$$\rho(z) \coloneqq d\left(\varphi_j(z), \psi_j(z)\right), \quad z \in \mathbf{D}$$

where d denotes the pseudohyperbolic distance on $\mathbf{D}$; see Section 2.2. Given a positive Borel measure μ on $\mathbf{D}$ and $\varphi_j \in \mathcal{S}(\mathbf{D})$, we denote by $\mu \circ \varphi_j^{-1}$ the pullback measure on $\mathbf{D}$ defined by $(\mu \circ \varphi_j^{-1})(E) = \mu[\varphi_j^{-1}(E)]$ for Borel sets $E \subset \mathbf{D}$. With these notation we now consider several pullback measures on $\mathbf{D}$ associated with $\varphi_j, \psi_j, u, v, \epsilon-1$ and $1+\epsilon$. First, we define a pullback measure $\omega = \omega^{\epsilon-1,1+\epsilon}_{\varphi_j,u;\,\psi_j,v}$ by

$$\omega \coloneqq (|\rho u|^{1+\epsilon} dA_{\epsilon-1}) \circ \varphi_j^{-1} + (|\rho v|^{1+\epsilon} dA_{\epsilon-1}) \circ \psi_j^{-1}$$

Also, for $\epsilon \ge 0$, we define a pullback measure $\sigma^{1+\epsilon} = \sigma^{\epsilon-1,1+\epsilon,1+\epsilon}_{\varphi_j,u;\,\psi_j,v}$ by

$$\sigma^{1+\epsilon} \coloneqq [(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon} dA_{\epsilon-1}] \circ \varphi_j^{-1} + [(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon} dA_{\epsilon-1}] \circ \psi_j^{-1}.$$

Finally, for $0 < \epsilon < 1$, we put

$$G_{1-\epsilon} \coloneqq \{z \in \mathbf{D} : \rho(z) < 1-\epsilon\} \tag{1.1}$$

and define a pullback measure $\sigma_{1-\epsilon} = \sigma^{\epsilon-1,1+\epsilon}_{\varphi_j,u,;\psi_j,v;1-\epsilon}$ by

$$\sigma_{1-\epsilon} \coloneqq \left(\chi_{G_{1-\epsilon}}|u-v|^{1+\epsilon} dA_{\epsilon-1}\right) \circ \varphi_j^{-1} + \left(\chi_{G_{1-\epsilon}}|u-v|^{1+\epsilon} dA_{\epsilon-1}\right) \circ \psi_j^{-1}.$$

Let χ denotes the characteristic function of the set specified in its subscript.

Note that $\omega, \sigma^{1+\epsilon}$ and $\sigma_{1-\epsilon}$ are finite measures if $u, v \in L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

The first result is the following Carleson measure criterion for a difference of composition operators with series of symbols to be bounded/compact. For the notion of the (compact) $(\epsilon-1)$-Carleson measures we refer to Section 2.4. When the weights u and v are bounded, there are additional characterizations; see Proposition 4.6.

Theorem 1.1 (see [32]). *Let $\epsilon > 0$, $0 \le \epsilon < \infty$, and $0 < \epsilon < 1$. Let $\varphi_j, \psi_j \in \mathcal{S}(\mathbf{D})$ and $u, v \in L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. Then the following statements are equivalent:*

(a) $C_{\sum_j \varphi_j,u} - C_{\sum_j \psi_j,v} : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ *is bounded (compact, resp.).*

(b) $\omega + \sigma^{1+\epsilon}$ *is a (compact, resp.) $(\epsilon-1)$-Carleson measure.*

(c) $\omega + \sigma_{1-\epsilon}$ *is a (compact, resp.) $(\epsilon-1)$-Carleson measure.*

The authors in [28] studied the isolation phenomena in the space of composition operators acting on the Hardy space $H^2(\mathbf{D})$. In their study they were naturally led to the conjecture that two composition operators are in the same path component if their difference is compact. Their conjecture later turned out to be false (see [2, 13, 15]), but the problems of characterizing the path components and the compact differences attracted broad interest in this field. Here, as a consequence of Theorem 1.1, we obtain an (implicit and abstract) answer to such a long-standing problem in terms of Carleson measures.

Theorem 1.2 (see [32]). *Let $\epsilon > 0$ and $0 < \epsilon < 1$. Let $\varphi_j, \psi_j \in \mathcal{S}(\mathbf{D})$. Then the following statements are equivalent:*

(a) $C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}$ *is compact on* $H^2(\mathbf{D})$.

(b) $\omega + \sigma^{1+\epsilon}$ *is a compact* 1*-Carleson measure.*

(c) $\omega + \sigma_{1-\epsilon}$ *is a compact* 1*-Carleson measure.*

Here, $\omega := \omega^{1,2}_{\varphi_j,\varphi_j';\psi_j,\psi_j'}$, $\sigma^{1+\epsilon} := \sigma^{1,2,1+\epsilon}_{\varphi_j,\varphi_j';\psi_j,\psi_j'}$ *and* $\sigma_{1-\epsilon} := \sigma^{1,2}_{\varphi_j,\varphi_j';\psi_j,\psi_j',1-\epsilon}$.

As a nontrivial application of Theorem 1.2, we will obtain a characterization of the Julia-Caratheodory type for compact differences of composition operators with univalent symbols on the Hardy spaces. As for single composition operators which are always bounded on the Hardy spaces and on the weighted Bergman spaces by the Littlewood Subordination Principle, it is known by MacCluer and Shapiro [20] that compactness of $C_{\sum_j \varphi_j}$ on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is characterized by the Julia-Caratheodory type condition

$$R_{\varphi_j}(z) := \frac{1-|z|^2}{1-\left|\varphi_j(z)\right|^2} \to 0 \tag{1.2}$$

as $|z| \to 1$. However, this characterization does not extend to the Hardy spaces. In fact, while this condition is necessary for compactness of $C_{\sum_j \varphi_j}$ on the Hardy space $H^2(\mathbf{D})$ (see [29]), it is not sufficient by Shapiro's characterization [26]. Nevertheless, when restricted to univalent symbols φ_j, MacCluer and Shapiro [20] noticed that (1.2) is also sufficient for compactness of $C_{\sum_j \varphi_j}$ on $H^2(\mathbf{D})$.

As for differences of composition operators, Moorhouse [22] found quite a natural extension of (1.2). Namely, she characterized compactness of $C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}$ on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ by the Julia-Caratheodory type condition

$$\lim_{|z|\to 1} \left[R_{\varphi_j}(z) + R_{\psi_j}(z)\right]\rho(z) = 0\,; \tag{1.3}$$

actually only the case $\epsilon = 2$ was considered by Moorhouse and then it is relaxed to general $1+\epsilon$ in [6]. Moorhouse [22] also observed that this condition is necessary for compactness of $C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}$ on a class of function spaces including $H^2(\mathbf{D})$.

Motivated by the aforementioned remark about single composition operators with univalent symbols, one may suspect (1.3) to be also sufficient for compactness of $C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}$ on $H^2(\mathbf{D})$ when symbols φ_j and ψ_j are univalent. Using Theorem 1.2, we show that it is actually the case.

Theorem 1.3 (see [32]). *Let φ_j and ψ_j be univalent maps in $\mathcal{S}(\mathbf{D})$. Then $C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}$ is compact on $H^2(\mathbf{D})$ if and only if (1.3) holds.*

In fact it is known by Nieminen and Saksman [23] that compactness of $C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}$ on the Hardy spaces $H^{1+\epsilon}(\mathbf{D})$, $0 \le \epsilon < \infty$, is independent of the parameter $1+\epsilon$. So, Theorems 1.2 and 1.3 also hold with $H^{1+\epsilon}(\mathbf{D})$, $0 \le \epsilon < \infty$, in place of $H^2(\mathbf{D})$.

We collect some basic facts and several technical lemmas to be used in the sequel. We also observe some additional characterizations when the weights are bounded. At the end of the section we include three examples. The first one is to show that boundedness/compactness of operators under consideration depends on parameters $\epsilon - 1$ and $1+\epsilon$. The second one is to show that the exponent $1+\epsilon$ in Theorems 1.1 cannot be smaller than it. The last one is to show that unbounded weighted composition operators can possibly form a compact difference in a nontrivial way. Finally, we prove Theorems as applications of Theorem 1.1. As another application, we show by an explicit example that an earlier work of Acharyya and Wu [1] concerning compactness for differences of weighted composition operators with bounded holomorphic weights does not extend to the case of non-holomorphic weights.

Throughout the sequel we use the same letter $1+\epsilon$ to denote positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants $1+\epsilon$. For nonnegative quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \leq (1+\epsilon)Y$ for some inessential constant $1+\epsilon$. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

1. Preliminaries:

We collect well-known basic standard facts and details in to be used in later sections. See [8] and [27].

2.1. Compact operator.

We clarify the notion of compact operators, since the spaces under consideration are not Banach spaces when $0 < \epsilon < 1$. Let X and Y be topological vector spaces whose topologies are induced by complete metrics. A continuous linear operator $L : X \to Y$ is said to be compact if the image of every bounded sequence in X has a convergent subsequence in Y.

We have the following convenient compactness criterion for a linear combination of weighted composition operators with $L^{1+\epsilon}_{\epsilon-1}$-weights acting on the weighted Bergman spaces.

Lemma 2.1 (see [32]). *Let $\epsilon > 0$ and $0 \leq \epsilon < \infty$. Let T be a linear combination of weighted composition operators with weights in $L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ and assume that $T : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is bounded. Then $T : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is compact if and only if $T(f_j)_n \to 0$ in $L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ for any bounded sequence $\{(f_j)_n\}$ in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ such that $(f_j)_n \to 0$ uniformly on compact subsets of $\mathbf{D}$.*

Proof. Assume that $T : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is compact and let $\{(f_j)_n\}$ be a bounded sequence in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ with $(f_j)_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. Then $\{T(f_j)_n\}$ has a subsequence which converges in $L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ by compactness of T. Since $(f_j)_n \to 0$ on any compact subset of $\mathbf{D}$, we see that $T(f_j)_n(z) \to 0$ at almost every $z \in \mathbf{D}$. Since this is true for any subsequence of $\{(f_j)_n\}$, we conclude that $T(f_j)_n \to 0$ in $L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

Conversely, let $\{(g_j)_n\}$ be any bounded sequence in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. By normality we may find a subsequence $\{(g_j)_{n_k}\}$ converging uniformly on compact subsets of $\mathbf{D}$ to some holomorphic function g_j. Note $g_j \in A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ by Fatou's Lemma. Note also that the sequence $\{(g_j)_{n_k} - g_j\}$ is bounded in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ and converges to 0 uniformly on compact subsets of $\mathbf{D}$. Thus the hypothesis guarantees that $T(g_j)_{n_k} \to Tg_j$ in $L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. The proof is complete.

The proof above is basically the same as the proof of [8, Proposition 3.11] for single composition operators. It is included above for completeness.

2.2 Pseudohyperbolic Distance. The well-known pseudohyperbolic distance between $z, w \in \mathbf{D}$ is given by

$$d(z,w) := \left|\frac{z-w}{1-\bar{z}w}\right|.$$

By a straightforward calculation we have

$$d^2(z,w) = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}. \tag{2.1}$$

for $z, w \in \mathbf{D}$. The pseudohyperbolic disk with center $z \in \mathbf{D}$ and radius $(1-\epsilon) \in (0,1)$ is defined by

$$E_{1-\epsilon}(z) = D\left(\frac{1-(1-\epsilon)^2}{1-(1-\epsilon)^2|z|^2}z, \frac{1-|z|^2}{1-(1-\epsilon)^2|z|^2}(1-\epsilon)\right). \tag{2.2}$$

Here, and elsewhere,

$$D(a,t) := \{w \in \mathbf{C} : |w-a| < t\} \tag{2.3}$$

denotes the Euclidean disk with center $a \in \mathbf{C}$ and radius $t > 0$. Since each holomorphic self-map of $\mathbf{D}$ is a d-contraction by the Schwarz-Pick Lemma, we have

$$\varphi_j(E_{1-\epsilon}(z)) \subset E_{1-\epsilon}\left(\varphi_j(z)\right) \tag{2.4}$$

for $\varphi_j \in \mathcal{S}(\mathbf{D})$.

Given $0 < \epsilon < 1$, we will frequently use the estimate

$$1-|z| \approx |1-\bar{z}w| \approx 1-|w| \tag{2.5}$$

for all $z, w \in \mathbf{D}$ with $d(z,w) < 1-\epsilon$; constants suppressed in these estimates depend only on $1-\epsilon$. Given $\epsilon > 0$ and $0 < \epsilon < 1$, one may use the above estimate to verify

$$A_{\epsilon-1}[E_{1-\epsilon}(z)] \approx (1-|z|)^{\epsilon+1} \tag{2.6}$$

for $z \in \mathbf{D}$; constants suppressed in this estimate depend only on $1-\epsilon$ and $\epsilon > 0$.

All the details for the statements above can be found in [31, Chapter 4].

2.3 Test Functions. Given $\epsilon > 0$ and $0 < \epsilon < 1$, we recall the submean value type inequality

$$\left|f_j(a)\right|^{1+\epsilon} \leq \frac{1+\epsilon}{(1-|a|^2)^{\epsilon+1}} \int_{E_{1-\epsilon}(a)} \sum_j \left|f_j\right|^2 dA_{\epsilon-1}, \quad a \in \mathbf{D} \tag{2.7}$$

valid for functions f_j holomorphic on $\mathbf{D}$ and $0 \leq \epsilon < \infty$ where $\epsilon \geq 0$ is a constant depending only on $\epsilon - 1$ and $1-\epsilon$. This is easily verified via (2.2), (2.5) and the subharmonicity of $\left|f_j\right|^{1+\epsilon}$.

Note from (2.7) with $\epsilon = 2$ that each point evaluation is a continuous linear functional on the Hilbert space $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. Thus, to each $a \in \mathbf{D}$ corresponds a unique reproducing kernel whose explicit formula is known as $z \mapsto \tau_a^{\epsilon+1}(z)$ where

$$\tau_a(z) := \frac{1}{1-\bar{a}z}. \tag{2.8}$$

Powers of these functions will be the source of test functions in conjunction with Lemma 2.1. The norms of such kernel-type functions are well known. Namely, when $t > 1$, we have

$$\|\tau_a^t\|_{A^{1+\epsilon}_{\epsilon-1}} \approx (1-|a|)^{-t+1}, \quad a \in \mathbf{D}; \tag{2.9}$$

constants suppressed in this estimate are independent of a; see, for example, [31, Lemma 3.10]. Thus

$$\frac{\tau_a^t}{\|\tau_a^t\|_{A^{1+\epsilon}_{\epsilon-1}}} \to 0 \text{ uniformly on compact subsets of } \mathbf{D} \tag{2.10}$$

as $|a| \to 1$.

2.4 Carleson measure. For μ be a positive finite Borel measure on $\mathbf{D}$. For $\epsilon > 0$ and $0 < \epsilon < 1$, setting

$$\hat{\mu}_{\epsilon-1,1-\epsilon}(z) := \frac{\mu[E_{1-\epsilon}(z)]}{A_{\epsilon-1}[E_{1-\epsilon}(z)]}, \qquad z \in \mathbf{D}, \tag{2.11}$$

we recall the following well known characterizations for each $0 \leq \epsilon < \infty$:

$$\text{the embedding } A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \subset L^{1+\epsilon}(d\mu) \text{ is bounded} \Leftrightarrow \sup_{\mathbf{D}} \hat{\mu}_{\epsilon-1,1-\epsilon} < \infty \tag{2.12}$$

and

$$\text{the embedding } A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \subset L^{1+\epsilon}(d\mu) \text{ is compact} \Leftrightarrow \lim_{|z|\to 1} \hat{\mu}_{\epsilon-1,1-\epsilon}(z) = 0. \tag{2.13}$$

We say that μ is an $(\epsilon-1)$-*Carleson measure* if either side of (2.12) holds. Also, we say that μ is a *compact* $(\epsilon-1)$-Carleson measure if either side of (2.13) holds. Note that the notion of (compact) $(\epsilon-1)$-Carleson measures is independent of the parameters $(1+\epsilon)$ and $(1-\epsilon)$. Given $\epsilon > 0$, $0 < \epsilon < 1$ and $0 \leq \epsilon < \infty$, it is also well known that

$$\left\|i_{\mu,1+\epsilon}\right\|^{1+\epsilon} \approx \|\mu\|_{\epsilon-1,1-\epsilon} := \sup_{\mathbf{D}} \hat{\mu}_{\epsilon-1,1-\epsilon} \tag{2.14}$$

for $(\epsilon-1)$-Carleson measures μ; constants suppressed above are independent of μ and $(1+\epsilon)$. Here, $\left\|i_{\mu,1+\epsilon}\right\|$ denotes the operator norm of the embedding
$i_{\mu,1+\epsilon}: A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \subset L^{1+\epsilon}(d\mu)$; see [31, Section 7.2].

The connection between composition operators and Carleson measures comes from the measure theoretic change-of-variable formula (see [14, p. 163])

$$\int_{\mathbf{D}}\sum_j (h\circ\varphi_j)d\mu = \int_{\mathbf{D}}\sum_j hd\left(\mu\circ\varphi_j^{-1}\right) \tag{2.15}$$

valid for $\varphi_j \in \mathcal{S}(\mathbf{D})$ and positive Borel functions h on $\mathbf{D}$. For example, using this formula, one may easily see that $C_{\sum_j \varphi_j,u} : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is bounded (compact, resp.) if and only if $(|u|^{1+\epsilon}dA_{\epsilon-1})\circ\varphi_j^{-1}$ is a (compact, resp.) $(\epsilon-1)$-Carleson measure. Also, in the special case $\mu = A_{\epsilon-1}$, since each composition operator is bounded on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$, it is immediate from (2.15) that $A_{\epsilon-1}\circ\varphi_j^{-1}$ is an $(\epsilon-1)$-Carleson measure for each $\varphi_j \in \mathcal{S}(\mathbf{D})$. Also is well known via (2.15) that $C_{\sum_j\varphi_j}$ is compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ if and only if $A_{\epsilon-1}\circ\varphi_j^{-1}$ is a compact $(\epsilon-1)$-Carleson measure. Moreover, we see from (2.14) and (2.15) that

$$\left\|C_{\sum_j\varphi_j}\right\|^{1+\epsilon}_{A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})} \approx \|\mu\|_{\epsilon-1,1-\epsilon} \tag{2.16}$$

where $\left\|C_{\sum_j\varphi_j}\right\|_{A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})}$ denotes the operator norm of $C_{\sum_j\varphi_j} : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$; see [31, Theorem 7.4].

In connection with the equality (2.15) we also note for easier reference later that there is a constant $C = C(\epsilon-1, 1+\epsilon, 1-\epsilon) > 0$ such that

$$\int_{\mathbf{D}}\sum_j |f_j|^{1+\epsilon}d\mu \le C\int_{\mathbf{D}}\sum_j |f_j|^{1+\epsilon}\hat{\mu}_{\epsilon-1,1-\epsilon}dA_{\epsilon-1} \tag{2.17}$$

for functions $f_j \in A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. In fact one may apply (2.7) in the left hand side of the above, interchange the order of integrations, and then conclude the asserted inequality by (2.5) and (2.6).

3. Technical Lemmas:

Before proceeding, we introduce some notation associated with nonzero $a \in \mathbf{D}$. Put

$$a^* \coloneqq \frac{1}{\bar{a}};$$

this is the inversion of a with respect to the unit circle. For $\epsilon > 0$ and $0 < \epsilon < 1$, put

$$\Omega_{\epsilon,1-\epsilon}(a) \coloneqq \left\{w \in \mathbf{D} : \frac{|1-\bar{a}w|}{(1+\epsilon)^2} < \sup_{z\in E_{1-\epsilon}(a)}|1-\bar{a}z|\right\}.$$

Clearly, we have

$$\overline{E_{1-\epsilon}(a)} \subset \Omega_{\epsilon,1-\epsilon}(a) \subset \Omega_{1,1-\epsilon}(a) \tag{3.1}$$

for any $\epsilon \in (0,1]$. Also, note from (2.2)

$$\sup_{z\in E_{1-\epsilon}(a)}|a^*-z| = \frac{1-|a|^2}{1-(1-\epsilon)^2|a|^2}(1-\epsilon) + \left|a^* - \frac{1-(1-\epsilon)^2}{1-(1-\epsilon)^2|a|^2}a\right|$$
$$= \frac{1-|a|^2}{1-(1-\epsilon)^2|a|^2}(1-\epsilon) + |a|\left(\frac{1}{|a|^2} - \frac{1-(1-\epsilon)^2}{1-(1-\epsilon)^2|a|^2}\right) = \frac{1-|a|^2}{|a|(1-|a|(1-\epsilon))}.$$

We thus have

$$\Omega_{\epsilon,1-\epsilon}(a) = \mathbf{D}\bigcap D\left(a^*, \frac{(1+\epsilon)^2(1-|a|^2)}{|a|(1-|a|(1-\epsilon))}\right). \tag{3.2}$$

Finally, for $N > 0$, we put

$$a_N \coloneqq ae^{-iN(1-|a|)} \text{ where } i = \sqrt{-1}.$$

Note

$$\frac{a^*-a_N^*}{a^*}\cdot\frac{1}{1-|a|} = \frac{1-e^{-iN(1-|a|)}}{1-|a|} \to N_i \tag{3.3}$$

as $|a| \to 1$. We also put

$$\Gamma_N(a) \coloneqq \{a\zeta : |\zeta| = 1 \text{ and } |\operatorname{Arg}\zeta| \le N(1-|a|)\}.$$

Note

$$\left|1-\frac{b}{a}\right| \le \left|1-e^{-iN(1-|a|)}\right| \le N(1-|a|) \tag{3.4}$$

for $b \in \Gamma_N(a)$.

We now prove some technical facts concerning the sets $\Omega_{\epsilon,1-\epsilon}(a)$ and $\Gamma_N(a)$ which will be used in the proof of Theorem 4.1 in the next section. In what follows we denote by

$$C_a^+ := \{w \in 1+\epsilon : \operatorname{Im}(\overline{a}w) > 0\}, \quad a \neq 0$$

the "upper" half-plane with respect to the straight line passing through the origin and a.

Lemma 3.1 (see [32]). *Let* $0 < \epsilon < 1$ *and* $N > 0$. *Put*

$$\theta_{1-\epsilon,N}(a) := \sup_{w \in \Omega_{1,1-\epsilon}(a) \cap C_a^+} \left| \operatorname{Arg}\left(\frac{w - a_N^*}{a^* - a_N^*}\right)\right|$$

for $a \in \mathbf{D}$, $a \neq 0$. *Then*

$$\lim_{|a| \to 1} \theta_{1-\epsilon,N}(a) = \operatorname{Arg}\left[1 + \frac{8i}{N(\epsilon)}\right].$$

Proof. Given nonzero $a \in \mathbf{D}$, setting

$$a_{1-\epsilon} := a^* - \frac{4(1-|a|^2)}{|a|(1-|a|(1-\epsilon))} \frac{a^*}{|a^*|} = a^* - \frac{4(1-|a|^2)}{1-|a|(1-\epsilon)} a^*,$$

we note from (3.2)

$$\theta_{1-\epsilon,N}(a) = \operatorname{Arg}\left(\frac{a_{1-\epsilon} - a_N^*}{a^* - a_N^*}\right). \tag{3.5}$$

In conjunction with this, we note from (3.3)

$$\frac{a_{1-\epsilon} - a_N^*}{a^* - a_N^*} = 1 - \frac{4(1-|a|^2)}{1-|a|(1-\epsilon)} \frac{a^*}{a^* - a_N^*} \to 1 + \frac{8i}{N(\epsilon)}$$

as $|a| \to 1$. So, the lemma holds by this and (3.5). The proof is complete.

Lemma 3.2 (see [32]). *Let* $0 < \epsilon < 1$ *and* $N > 0$. *Then*

$$\limsup_{|a| \to 1} \left[\sup_{\substack{z \in E_{1-\epsilon}(a) \\ w \in \Omega_{1,1-\epsilon}(a) \cap C_a^+}} \left| \operatorname{Arg}\left(\frac{z - a_N^*}{w - a_N^*}\right)\right| \right] \leq 3 \operatorname{Arg}\left[1 + \frac{8i}{N(\epsilon)}\right].$$

Proof. Given $a \in \mathbf{D}$, $a \neq 0$, denote by $\zeta_a = \zeta_{a,1-\epsilon}$ the Euclidean center of the disk $E_{1-\epsilon}(a)$.

Let $z \in E_{1-\epsilon}(a)$ and $\in \Omega_{1,1-\epsilon}(a) \cap C_a^+$. Note

$$\left| \operatorname{Arg}\left(\frac{w - a_N^*}{\zeta_a - a_N^*}\right)\right| \leq \theta_{1-\epsilon,N}(a) \tag{3.6}$$

where $\theta_{1-\epsilon,N}(a)$ is the quantity introduced in Lemma 3.1. Also, note from the first inclusion in (3.1) that to each $z \in E_{1-\epsilon}(a)$ corresponds some $w_z \in \Omega_{1,1-\epsilon}(a) \cap C_a^+$ satisfying

$$\left| \operatorname{Arg}\left(\frac{z - a_N^*}{\zeta_a - a_N^*}\right)\right| = \left| \operatorname{Arg}\left(\frac{w_z - a_N^*}{\zeta_a - a_N^*}\right)\right| \leq \theta_{1-\epsilon,N}(a);$$

the second inequality holds by (3.6). It follows that

$$\begin{aligned} \left| \operatorname{Arg}\left(\frac{z - a_N^*}{w - a_N^*}\right)\right| &\leq \left| \operatorname{Arg}\left(\frac{z - a_N^*}{\zeta_a - a_N^*}\right)\right| + \left| \operatorname{Arg}\left(\frac{\zeta_a - a_N^*}{a^* - a_N^*}\right)\right| + \left| \operatorname{Arg}\left(\frac{a^* - a_N^*}{w - a_N^*}\right)\right| \\ &\leq 2\theta_{1-\epsilon,N}(a) + \left| \operatorname{Arg}\left(\frac{\zeta_a - a_N^*}{a^* - a_N^*}\right)\right| \end{aligned} \tag{3.7}$$

for a with $|a|$ sufficiently close to 1. Note that the second inequality above is independent of $z \in E_{1-\epsilon}(a)$ and $\in \Omega_{1,1-\epsilon}(a) \cap C_a^+$.

Meanwhile, we have by (2.2)

$$\zeta_a - a^* = \left(\frac{1-(1-\epsilon)^2}{1-(1-\epsilon)^2|a|^2} - \frac{1}{|a|^2}\right) a = \frac{|a|^2 - 1}{(1-(1-\epsilon)^2|a|^2)} a^*$$

and hence by (3.3)

$$\frac{\zeta_a - a_N^*}{a^* - a_N^*} = 1 + \frac{\zeta_a - a_N^*}{a^* - a_N^*} \to 1 + \frac{2i}{N(1-(1-\epsilon)^2)}$$

as $|a| \to 1$. This, together with (3.7) and Lemma 3.1, implies the lemma. The proof is complete.

Lemma 3.3 (see [32]). *Let* $0 < \epsilon < 1$ *and* $N > 0$. *Then there is a constant* $C = C(1-\epsilon, N) > 0$ *such that*

$$1 \le \frac{|1-\bar{b}w|}{1-|a|} \le 1+\epsilon \tag{3.8}$$

for $a \in \mathbf{D}\backslash\{0\}$ *with* $N(1-|a|) < \pi$, $b \in \Gamma_N(a)$ *and* $w \in \Omega_{1,1-\epsilon}(a)$.

Proof. The first inequality is clear. On the other hand, for $a, b \in \mathbf{D}$ with $|a| = |b|$, we note

$$|1-\bar{b}w| = |a||b^* - w| \le |a||b^* - a^*| + |a||a^* - w| = \left|1 - \frac{b}{a}\right| + |a||a^* - w|$$

for all $w \in \mathbf{D}$. Thus the second inequality holds by (3.4) and (3.2). The proof is complete.

Lemma 3.4 (see [32]). *Let* $0 < \epsilon < 1$ *and* $N > 0$. *Then there is a constant* $C = C(1-\epsilon, N) > 0$ *such that*

$$1 - d(z,w) \le (1+\epsilon)\left|\frac{1-\bar{b}z}{1-\bar{b}w}\right|$$

for $a \in \mathbf{D}\backslash\{0\}$ *with* $N(1-|a|) < \pi$, $b \in \Gamma_N(a)$, $z \in E_{1-\epsilon}(a)$ *and* $w \in \mathbf{D}$.

Proof. Note from (2.1)

$$1 - d(z,w) \le 4 \cdot \frac{1-|z|}{|1-\bar{z}w|}$$

and thus

$$[1 - d(z,w)]\left|\frac{1-\bar{b}w}{1-\bar{b}z}\right| \le 4\left|\frac{1-\bar{b}w}{1-\bar{z}w}\right|$$

for all $b, z, w \in \mathbf{D}$. Moreover, we have

$$\left|\frac{1-\bar{b}w}{1-\bar{z}w}\right| = \left|1 + \frac{w(\bar{z}-\bar{b})}{1-\bar{z}w}\right| \le 1 + \left|\frac{z-b}{1-\bar{z}w}\right| = 1 + d(b,z)\left|\frac{1-\bar{b}z}{1-\bar{z}w}\right| \le 1 + \frac{|1-\bar{b}z|}{1-|z|}$$

for all $b, z, w \in \mathbf{D}$. Now, one may conclude the lemma by (2.5) and Lemma 3.3. The proof is complete.

Lemma 3.5 (see [32]). *Let* $0 < \epsilon < 1$ *and* $N > 0$. *Then there is a constant* $C = C(s, N) > 0$ *such that*

$$d(z,w) \le \frac{1+\epsilon}{|a|}\left|1 - \frac{1-\bar{b}z}{1-\bar{b}w}\right|$$

for $a \in \mathbf{D}\backslash\{0\}$ *with* $N(1-|a|) < \pi$, $b \in \Gamma_N(a)$, $z \in E_{1-\epsilon}(a)$, *and* $w \in \Omega_{1,1-\epsilon}(a)$.

Proof. Note

$$\begin{aligned}\left|1 - \frac{1-\bar{b}z}{1-\bar{b}w}\right| &= \left|\frac{\bar{b}(w-z)}{1-\bar{b}w}\right| \\ &= |b|d(z,w)\left|\frac{1-\bar{z}w}{1-\bar{b}w}\right| \\ &\ge |b|d(z,w)\frac{1-|z|}{|1-\bar{b}w|}.\end{aligned}$$

for all $b, z, w \in \mathbf{D}$. Note $|b| = |a|$ for $b \in \Gamma_N(a)$. We thus conclude the lemma by (2.5) and Lemma 3.3. The proof is complete.

4. Characterizations:

We first prove Theorem 1.1. In fact we will prove a more detailed version of Theorem 1.1. We then notice that there are several other versions when the weights are bounded. We also exhibit a couple of examples related to Theorem 1.1 at the end of the section.

Before proceeding, we decompose the measures $\omega, \sigma^{1+\epsilon}$ and $\sigma_{1-\epsilon}$ (associated with $\varphi_j, \psi_j, u, v, \epsilon - 1, 1+\epsilon$) defined in the Introduction into two parts as follows:

$$\omega = \omega_{\varphi_j,u} + \omega_{\psi_j,v},$$

$$\sigma^{1+\epsilon} = \sigma_{\varphi_j}^{1+\epsilon} + \sigma_{\psi_j}^{1+\epsilon},$$

$$\sigma_{1-\epsilon} = \sigma_{\varphi_j,1-\epsilon} + \sigma_{\psi_j,1-\epsilon}$$

where measures $\omega_{\varphi_j,u}, \sigma_{\varphi_j}^{1+\epsilon}, \sigma_{\varphi_j,1-\epsilon}$ are defined by

$$\omega_{\varphi_j,u} := (|\rho u|^{1+\epsilon} dA_{\epsilon-1}) \circ \varphi_j^{-1},$$
$$\sigma_{\varphi_j}^{1+\epsilon} := [(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon} dA_{\epsilon-1}] \circ \varphi_j^{-1},$$
$$\sigma_{\varphi_j,1-\epsilon} := \left(\chi_{G_{1-\epsilon}}|u-v|^{1+\epsilon} dA_{\epsilon-1}\right) \circ \varphi_j^{-1};$$

measures $\omega_{\psi_j,u}, \sigma_{\psi_j}^{1+\epsilon}, \sigma_{\psi_j,1-\epsilon}$ are defined similarly. Parameters omitted in these notation should be clear from the context.

Note that the equivalences of Assertions (a), (b) and (d) below are the content of Theorem 1.1.

Theorem 4.1 (see [32]). *Let $\epsilon > 0$, $0 \le \epsilon < \infty$, and $0 < \epsilon < 1$. Let $\varphi_j, \psi_j \in \mathcal{S}(\mathbf{D})$ and $u, v \in L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. Then the following statements are equivalent:*

(a) $C_{\sum_j \varphi_j,u} - C_{\sum_j \psi_j,v} : A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \to L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ *is bounded(compact, resp.).*

(b) $\omega + \sigma_{\varphi_j}^{1+\epsilon}$ *and* $\omega + \sigma_{\psi_j}^{1+\epsilon}$ *are (compact, resp.)* $(\epsilon-1)$*-Carleson measures.*

(c) $\omega + \sigma_{\varphi_j}^{1+\epsilon}$ *or* $\omega + \sigma_{\psi_j}^{1+\epsilon}$ *is a (compact, resp.)* $(\epsilon-1)$*-Carleson measure.*

(d) $\omega + \sigma_{\varphi_j,1-\epsilon}$ *and* $\omega + \sigma_{\psi_j,1-\epsilon}$ *are (compact, resp.)* $(\epsilon-1)$*-Carleson measures.*

(e) $\omega + \sigma_{\varphi_j,1-\epsilon}$ *or* $\omega + \sigma_{\psi_j,1-\epsilon}$ *is a (compact, resp.)* $(\epsilon-1)$*-Carleson measure.*

We will complete the proof of Theorem 4.1 by proving the sequences of implications

$$\text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(e)} \Longrightarrow \text{(a)} \Longrightarrow \text{(b)}$$

and

$$\text{(b)} \Longrightarrow \text{(d)} \Longrightarrow \text{(e)}.$$

Note that the implications (b) $\Longrightarrow$ (c) and (d) $\Longrightarrow$ (e) are trivial. Also, since

$$\rho \le 1-\epsilon \quad \text{on} \quad G_{1-\epsilon}$$

for each $0 < \epsilon < 1$, the implications (b) $\Longrightarrow$ (d) and (c) $\Longrightarrow$ (e) are clear for any $\epsilon > 0$. Thus it remains to prove the implications

$$\text{(e)} \Longrightarrow \text{(a)} \Longrightarrow \text{(b)}.$$

The following lemma, which is to be used in the proof of the implication (e) $\Longrightarrow$ (a), can be found in [18, Lemma 2.2].

Lemma 4.2 (see [32]). *Let $\epsilon > 0$, $0 \le \epsilon < \infty$ and $0 < \epsilon < 1$. Then there is a constant $C = C(\epsilon-1, 1+\epsilon, 1-\epsilon) > 0$ such that*

$$\left|f_j(z) - f_j(w)\right|^{1+\epsilon} \le C \frac{d^{1+\epsilon}(z,w)}{(1-|z|)^{(\epsilon+1)}} \int_{B_{1-\epsilon}(z)} \sum_j \left|f_j\right|^{1+\epsilon} dA_{\epsilon-1}$$

for functions $f_j \in A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ and $z, w \in \mathbf{D}$ with $d(z,w) < 1-\epsilon$.

We now prove the implication (e) $\Longrightarrow$ (a).

Proof of (e) $\Longrightarrow$ (a). We first consider boundedness. Assume (e). By symmetry we may assume that $\omega + \sigma_{\varphi_j,1-\epsilon}$ is an $(\epsilon-1)$-Carleson measure. Put

$$\mu := \omega + \sigma_{\varphi_j,1-\epsilon} \ \text{ and } \ T := C_{\sum_j \varphi_j,u} - C_{\sum_j \psi_j,v}$$

for simplicity. We claim that there is a constant $C = C(\epsilon-1, 1+\epsilon, 1-\epsilon) > 0$ such that

$$\left\|Tf_j\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} \le C \int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} \hat{\mu}_{\epsilon-1,1-\epsilon} dA_{\epsilon-1} \tag{4.1}$$

for functions $f_j \in A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$; recall that $\hat{\mu}_{\epsilon-1,1-\epsilon}$ is the function introduced in (2.11). With this claim granted, we see from (2.12) that $T : A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \to L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ is bounded and

$$\|T\|_{A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \to L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})}^{1+\epsilon} \le (1+\epsilon)\|\mu\|_{\epsilon-1,1-\epsilon}$$

where $\|T\|_{A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \to L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})}^{1+\epsilon}$ denotes the operator norm of $T : A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \to L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$.

Let $f_j \in A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. To establish (4.1) we write

$$\left\|Tf_j\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} = \int_{\mathbf{D}\setminus G_{1-\epsilon}} + \int_{G_{1-\epsilon}} \sum_j \left|u(f_j \circ \varphi_j) - v(f_j \circ \psi_j)\right|^{1+\epsilon} dA_{\epsilon-1} =: I_{1-\epsilon} + II_{1-\epsilon};$$

recall that $G_{1-\epsilon}$ is the set specified in (1.1). We will estimate each integral separately. For the integral $I_{1-\epsilon}$, we note $\rho \geq 1-\epsilon$ on $\mathbf{D}\backslash G_{1-\epsilon}$ and therefore

$$\begin{aligned} I_{1-\epsilon} &\lesssim \int_{\mathbf{D}\backslash G_{1-\epsilon}} \sum_j [\left|u(f_j\circ\varphi_j)\right|^{1+\epsilon} + \left|v(f_j\circ\psi_j)\right|^{1+\epsilon}] dA_{\epsilon-1} \\ &\leq \frac{1}{(1-\epsilon)^{1+\epsilon}} \int_{\mathbf{D}} \sum_j [\left|\rho u(f_j\circ\varphi_j)\right|^{1+\epsilon} + \left|\rho v(f_j\circ\psi_j)\right|^{1+\epsilon}] dA_{\epsilon-1} \\ &= \frac{1}{(1-\epsilon)^{1+\epsilon}} \int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} d\omega \\ &\leq \frac{1}{(1-\epsilon)^{1+\epsilon}} \int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} d\mu\,; \end{aligned}$$

the equality above holds by (2.15). By this and (2.17) we conclude

$$I_{1-\epsilon} \leq C \int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} \hat{\mu}_{\epsilon-1,1-\epsilon} dA_{\epsilon-1} \tag{4.2}$$

for some constant $\epsilon \geq 0$ independent of f_j. Meanwhile, for the integral $II_{1-\epsilon}$, we note

$$\begin{aligned} II_{1-\epsilon} &\lesssim \int_{G_{1-\epsilon}} \sum_j [\left|(u-v)(f_j\circ\varphi_j)\right|^{1+\epsilon} + \left|v(f_j\circ\varphi_j - f_j\circ\psi_j)\right|^{1+\epsilon}] dA_{\epsilon-1} \\ &= \int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} d\sigma_{\varphi_j,1-\epsilon} + \int_{G_{1-\epsilon}} \sum_j \left|v(f_j\circ\varphi_j - f_j\circ\psi_j)\right|^{1+\epsilon} dA_{\epsilon-1}; \end{aligned}$$

the equality holds again by (2.15). For the first term of the above, we have by (2.17)

$$\int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} d\sigma_{\varphi_j,1-\epsilon} \leq \int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} d\mu \leq C \int_{\mathbf{D}} \sum_j \left|f_j\right|^{1+\epsilon} \hat{\mu}_{\epsilon-1,1-\epsilon} dA_{\epsilon-1}$$

for some constant $\epsilon \geq 0$ independent of f_j. To estimate the second integral, we recall $\rho < 1-\epsilon$ on $G_{1-\epsilon}$

$$\left|(f_j\circ\varphi_j)(z) - (f_j\circ\psi_j)(z)\right|^{1+\epsilon} \lesssim \sum_j \frac{\rho^{1+\epsilon}(z)}{\left(1-\left|\psi_j(z)\right|\right)^{\epsilon+1}} \int_{E_{1-\epsilon}(\psi_j(z))} \sum_j \left|f_j\right|^{1+\epsilon} dA_{\epsilon-1}$$

for all $z \in G_{1-\epsilon}$. Now, integrating over $G_{1-\epsilon}$ both sides of the above against measure $|v|^{1+\epsilon} dA_{\epsilon-1}$ and then applying Fubini's Theorem, we obtain

$$\begin{aligned} &\int_{G_{1-\epsilon}} \sum_j \left|v(f_j\circ\varphi_j - f_j\circ\psi_j)\right|^{1+\epsilon} dA_{\epsilon-1} \\ &\qquad \lesssim \int_{G_{1-\epsilon}} \sum_j \frac{\rho^{1+\epsilon}(z)|v(z)|^{1+\epsilon}}{\left(1-\left|\psi_j(z)\right|\right)^{\epsilon+1}} \left\{ \int_{E_{1-\epsilon}(\psi_j(z))} \left|f_j(w)\right|^{1+\epsilon} dA_{\epsilon-1}(w) \right\} dA_{\epsilon-1}(z) \\ &\leq \int_{\mathbf{D}} \sum_j \left|f_j(w)\right|^{1+\epsilon} \left\{ \int_{\psi_j^{-1}(E_{1-\epsilon}(w))} \frac{\rho^{1+\epsilon}(z)|v(z)|^{1+\epsilon}}{\left(1-\left|\psi_j(z)\right|\right)^{\epsilon+1}} dA_{\epsilon-1}(z) \right\} dA_{\epsilon-1}(w) \\ &\approx \int_{\mathbf{D}} \sum_j \left|f_j(w)\right|^{1+\epsilon} \left(\widehat{\omega_{\psi_j,v}}\right)_{\epsilon-1,1-\epsilon}(w) dA_{\epsilon-1}(w); \end{aligned}$$

we used (2.5) and (2.6) for the last estimate. Combining these observations, we obtain

$$II_{1-\epsilon} \leq C \int_{\mathbf{D}} \sum \left|f_j\right|^{1+\epsilon} \hat{\mu}_{\epsilon-1,1-\epsilon} dA_{\epsilon-1} \tag{4.3}$$

for some constant $\epsilon \geq 0$ independent of f_j. Finally, we conclude (4.1) by (4.2) and (4.3), as asserted. This completes the proof for boundedness.

We now turn to the proof of compactness. By symmetry again, we may assume that μ is a compact $(\epsilon-1)$-Carleson measure. By what we have proved above, $T : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is bounded. To prove compactness of $T : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$, we consider an arbitrary sequence $\left\{(f_j)_n\right\}$ in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ such that $\left\|(f_j)_n\right\|_{A^{1+\epsilon}_{\epsilon-1}} \le 1$ and $(f_j)_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. We claim

$$T(f_j)_n \to 0 \quad \text{in } L^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \tag{4.4}$$

as $n \to \infty$. With this claim granted, we conclude by Lemma 2.1 that $T : A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \to L^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ is compact.

It remains to prove (4.4). Let $t \in (0,1)$. We have by (4.1)

$$\int_{\mathbf{D}} \sum_j \left|T(f_j)_n\right|^{1+\epsilon} dA_{\epsilon-1} \lesssim \int_{\mathbf{D}} \sum_j \left|(f_j)_n\right|^{1+\epsilon} \hat{\mu}_{\epsilon-1,1-\epsilon} dA_{\epsilon-1} = \int_{t\mathbf{D}} + \int_{\mathbf{D}\setminus t\mathbf{D}}$$

for all n. Note that $\mu\alpha,s$ is bounded on $\mathbf{D}$ by (2.12). So, since $(f_j)_n \to 0$ uniformly on $t\mathbf{D}$, we have

$$\lim_{n\to\infty} \int_{t\mathbf{D}} \sum_j \left|(f_j)_n\right|^{1+\epsilon} \hat{\mu}_{\epsilon-1,1-\epsilon} dA_{\epsilon-1} = 0.$$

On the other hand, since $\left\|(f_j)_n\right\|_{A^{1+\epsilon}_{\epsilon-1}} \le 1$ for all n, we have

$$\int_{\mathbf{D}\setminus t\mathbf{D}} \sum_j \left|(f_j)_n\right|^{1+\epsilon} \hat{\mu}_{\epsilon-1,1-\epsilon} dA_{\epsilon-1} \le \sup_{\mathbf{D}\setminus t\mathbf{D}} \hat{\mu}_{\epsilon-1,1-\epsilon}$$

for all n. Accordingly, we obtain

$$\limsup_{n\to\infty} \int_{\mathbf{D}} \sum_j \left|T(f_j)_n\right|^{1+\epsilon} dA_{\epsilon-1} \le \sup_{\mathbf{D}\setminus t\mathbf{D}} \hat{\mu}_{\epsilon-1,1-\epsilon}.$$

Note that this holds for arbitrary $t \in (0,1)$. Also, since μ is a compact $(\epsilon-1)$-Carleson measure by assumption, we note from (2.13) that the right hand side of the above tends to 0 as $t \to 1$.

Thus, taking the limit $t \to 1$, we conclude (4.4), as claimed. The proof is complete.

We now proceed to the proof of the implication (a) $\Longrightarrow$ (b), which is the hardest step. The next lemma, which is immediate from the triangle inequality, will be repeatedly used in the proof of the implication (a) $\Longrightarrow$ (b). It is included here for easier references.

Lemma 4.3 (see [32]). *Let $\epsilon > 0$. If λ and ζ are nonzero complex numbers such that $(1+\epsilon)|\zeta| \le |\lambda|$, then*

$$\frac{\epsilon}{1+\epsilon} \le \frac{|\lambda-\zeta|}{|\lambda|} \le \frac{2+\epsilon}{1+\epsilon}.$$

In what follows S_ϵ denotes the truncated angular sector consisting of all $\lambda \in \mathbf{C}$ such that $|\mathrm{Arg}\,\lambda| \le \frac{\pi}{4}$ and $\le |\lambda| \le \frac{1}{\epsilon}$.

Lemma 4.4 (see [32]). *Let $0 < \epsilon < 1$ and $\delta > 0$. Then there is a constant $C = C(\epsilon,\delta) > 0$ such that*

$$\frac{1}{C} \le \frac{|1-\lambda^\delta|}{|1-\lambda|} \le C$$

whenever $\lambda, \lambda^\delta \in S_\epsilon\setminus\{1\}$.

Proof. Let $\lambda \in S_\epsilon\setminus\{1\}$. Note that the line segment connecting 1 and λ is contained in S_ϵ. Thus, noting

$$1-\lambda^\delta = \delta\int_\lambda^1 \zeta^{\delta-1} d\zeta,$$

we see

$$\frac{|1-\lambda^\delta|}{|1-\lambda|} \le \delta\left(\sup_{\zeta\in S_\epsilon}|\zeta|^{\delta-1}\right) \le \delta\left(\epsilon^{\delta-1} + \frac{1}{\epsilon^{\delta-1}}\right).$$

Now, further assume $\lambda^\delta \in S_\epsilon\setminus\{1\}$ and put $\zeta := \lambda^\delta$. Then $\zeta^{1/\delta} = \lambda \in S_\epsilon\setminus\{1\}$. It thus follows from what we have observed above that

$$\frac{|1-\lambda|}{|1-\lambda^\delta|} = \frac{|1-\zeta^{1/\delta}|}{|1-\zeta|} \le \frac{1}{\delta}\left(\epsilon^{1/\delta-1} + \frac{1}{\epsilon^{1/\delta-1}}\right).$$

This completes the proof.

Now we prove the implication (a) $\Longrightarrow$ (b).

Proof of (a) $\Longrightarrow$ (b). Fix $0 < \epsilon < 1$. Also, fix $\gamma > 0$ such that

$$\gamma = 1$$

and choose $N > 0$ so large that

$$\operatorname{Arg}\left[1 + \frac{8i}{N(2-\epsilon)}\right] < \min\left\{\frac{\pi}{12}, \frac{\pi}{12\gamma}\right\}.$$

Note from Lemma 3.2

$$\sup_{\substack{z \in E_{1-\epsilon}(a) \\ w \in \Omega_{1,1-\epsilon}(a) \cap C_a^+}} \left|\operatorname{Arg}\left(\frac{1-\overline{a_N}z}{1-\overline{a_N}w}\right)\right| < \min\left\{\frac{\pi}{4}, \frac{\pi}{4\gamma}\right\} \tag{4.5}$$

for all $a \in \mathbf{D}$ with $|a|$ sufficiently close to 1.

We now introduce our test functions. Let $a \in \mathbf{D}$ and assume $N(1-|a|) < \pi$. For $t > 1$ to be fixed later, we put

$$(f_j)_{b,t} := \frac{\tau_b^t}{\|\tau_b^t\|_{A_{\epsilon-1}^{1+\epsilon}}}, \quad b \in \Gamma_N(a)$$

where τ_b is the function specified in (2.8). Put

$$\mu := \omega_{\varphi_j,u} + \sigma_{\varphi_j}^{1+\epsilon}$$

for simplicity. Since $u, v \in L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ by assumption, we see that μ is a finite measure.

We claim that there is a constant $C = C(\epsilon-1, 1+\epsilon,\ 1-\epsilon) > 0$ satisfying

$$\hat{\mu}_{\epsilon-1,1-\epsilon}(a) \le C \sum_j \left[\left\|T(f_j)_{a_N,\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} + \left\|T(f_j)_{a_N,2\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} + \left\|T(f_j)_{\overline{a_N},\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} + \left\|T(f_j)_{\overline{a_N},2\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} + \left\|T(f_j)_{a,\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} + \left\|T(f_j)_{a,\gamma+1}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon}\right] \tag{4.6}$$

for all a with $|a|$ sufficiently close to 1. Here $\|\cdot\|_{L_{\epsilon-1}^{1+\epsilon}}$ denotes the "norm" on $L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. With this claim granted, we deduce by (2.12) and (2.13) that μ is a (compact, resp.) $(\epsilon-1)$-Carleson measure from the boundedness(compactness, resp.) of $T : A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \to L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$; for compactness we also use (2.10) and Lemma 2.1. By symmetry the same assertions also hold for the measure $\omega_{\psi_j,v} + \sigma_{\psi_j}^{1+\epsilon}$, which completes the proof of the implication (a) $\Longrightarrow$ (b).

It remains to establish (4.6). For the rest of the proof we assume that $a \in \mathbf{D}$ is an arbitrary point with $|a|$ sufficiently close to 1. Setting

$$Q_b := \frac{1-\overline{b}\varphi_j}{1-\overline{b}\psi_j}, \quad b \in \Gamma_N(a),$$

we have by (2.9)

$$\begin{aligned}\left\|T(f_j)_{b,t}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} &\approx (1-|a|)^{(1+\epsilon)t-1-\epsilon} \int_{\mathbf{D}} \sum_j \left|\frac{u}{(1-\overline{b}\varphi_j)^t} - \frac{v}{(1-\overline{b}\psi_j)^t}\right|^{1+\epsilon} dA_{\epsilon-1} \\ &\ge (1-|a|)^{(1+\epsilon)t-1-\epsilon} \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \frac{|u - vQ_b^t|^{1+\epsilon}}{|1-\overline{b}\varphi_j|^{(1+\epsilon)t}} dA_{\epsilon-1} \\ &\approx \frac{1}{(1-|a|)^{\epsilon+1}} \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j |u - vQ_b^t|^{1+\epsilon} dA_{\epsilon-1};\end{aligned} \tag{4.7}$$

the last estimate holds by Lemma 3.3. In conjunction with this, we note from Lemma 3.4

$$|Q_b| \gtrsim 1-\rho > [1-\rho]^{1/\gamma} \quad \text{on} \quad \varphi_j^{-1}(E_{1-\epsilon}(a)); \tag{4.8}$$

the constant suppressed in this estimate depends only on $1-\epsilon$ and N.

In order to estimate the integrals in (4.7), we introduce several auxiliary notation. We first choose $\epsilon = \epsilon(1-\epsilon, N, \gamma) \in (0, 1]$ such that

$$\sup_{\substack{z\in E_{1-\epsilon}(a)\\ w\in\Omega_{1,1-\epsilon}(a)\cap C_a^+}} \left|\frac{1-\overline{a_N}z}{1-\overline{a_N}w}\right|^{\gamma} \le \frac{1}{\sqrt{2}[(1+\epsilon)^2-1]} \tag{4.9}$$

as $|a|\to 1$. Existence of such ϵ is guaranteed by Lemma 3.3 and (2.5). Now, using such ϵ, we put

$$\Omega^+_{\epsilon,1-\epsilon}(a) := \Omega_{\epsilon,1-\epsilon}(a)\cap \boldsymbol{C}^+_a,$$
$$\Omega^-_{\epsilon,1-\epsilon}(a) := \Omega_{\epsilon,1-\epsilon}(a)\backslash \Omega^+_{\epsilon,1-\epsilon}(a),$$
$$\Omega'_{\epsilon,1-\epsilon}(a) := \mathbf{D}\backslash \Omega_{\epsilon,1-\epsilon}(a)$$

so that

$$\mathbf{D} = \Omega^+_{\epsilon,1-\epsilon}(a)\cup \Omega^-_{\epsilon,1-\epsilon}(a)\cup \Omega'_{\epsilon,1-\epsilon}(a)$$

for each a. Accordingly, setting

$$F_1(a) := \varphi_j^{-1}(E_{1-\epsilon}(a))\cap \psi_j^{-1}\left(\Omega^+_{\epsilon,1-\epsilon}(a)\right),$$
$$F_2(a) := \varphi_j^{-1}(E_{1-\epsilon}(a))\cap \psi_j^{-1}\left(\Omega^-_{\epsilon,1-\epsilon}(a)\right),$$
$$F_3(a) := \varphi_j^{-1}(E_{1-\epsilon}(a))\cap \psi_j^{-1}\left(\Omega'_{\epsilon,1-\epsilon}(a)\right),$$

we have by (4.7)

$$\left\|T(f_j)_{b,t}\right\|^{1+\epsilon}_{L^{1+\epsilon}_{\epsilon-1}} \gtrsim \frac{1}{(1-|a|)^{(\epsilon+1)}}\int_{F_j(a)} |u-vQ_b^t|^{1+\epsilon}dA_{\epsilon-1} \tag{4.10}$$

for $j = 1, 2, 3$. We will obtain lower estimates for the above integrals with suitably chosen b and t depending on the regions $F_j(a)$.

`First, for the integral over $F_1(a)$, we choose $b = a_N$ and $t=\gamma$. Noting

$$u - vQ^{\gamma}_{a_N} = u\left(1-Q^{\gamma}_{a_N}\right) + (u-v)Q^{\gamma}_{a_N}, \tag{4.11}$$

we further decompose the region $F_1(a)$ into two parts. Namely, we set

$$F_{1,1}(a) := \left\{z\in F_1(a) : |u(z)|\left|1-Q^{\gamma}_{a_N}(z)\right| \le \frac{1}{1+\epsilon}|u(z)-v(z)|\left|Q^{\gamma}_{a_N}(z)\right|\right\}$$

and

$$F_{1,2}(a) := F_1(a)\backslash F_{1,1}(a).$$

In conjunction with the integral over $F_{1,1}(a)$, we note from (4.11) and Lemma 4.3 that

$$\left|u-vQ^{\gamma}_{a_N}\right|^{1+\epsilon} \approx |u|^{1+\epsilon}\left|1-Q^{\gamma}_{a_N}\right|^{1+\epsilon} + |u-v|^{1+\epsilon}\left|Q_{a_N}\right|^{(1+\epsilon)\gamma} \text{ on } F_{1,1}(a). \tag{4.12}$$

In conjunction with the integral over $F_{1,2}(a)$, note that

$$\left|\mathrm{Arg}\left[Q^{\gamma}_{a_N}\right]\right| \le \frac{\pi}{4} \text{ and } \left|Q^{\gamma}_{a_N}\right| \le \frac{1}{\sqrt{2}[(1+\epsilon)^2-1]} \text{ on } F_1(a)$$

by (4.5) and (4.9). So, we have

$$\mathrm{Re}\left[\frac{1}{Q^{\gamma}_{a_N}}\right] \ge (1+\epsilon)^2 - 1 \quad\text{on}\quad F_1(a)$$

and therefore

$$\frac{|u|\left|1-Q^{2\gamma}_{a_N}\right|}{|u-v|\left|Q^{2\gamma}_{a_N}\right|} \ge \frac{1}{1+\epsilon}\left|1+\frac{1}{Q^{\gamma}_{a_N}}\right| \ge 1+\epsilon \quad \text{on } F_{1,2}(a).$$

We also note from Lemma 3.3

$$\left|Q_{a_N}\right| \ge \frac{1-|a|}{\left|1-\overline{a_N}\psi_j\right|} \gtrsim 1 \text{ on } \psi_j^{-1}\left(\Omega_{1,1-\epsilon}(a)\right)$$

and from (4.5)

$$\left|1-Q^{\gamma}_{a_N}\right| \ge 1 \quad \text{on } F_1(a).$$

Combining these observations with (4.11), we obtain by Lemma 4.3

$$\begin{aligned}\left|u-vQ^{2\gamma}_{a_N}\right|^{1+\epsilon} &\approx |u|^{1+\epsilon}\left|1-Q^{2\gamma}_{a_N}\right|^{1+\epsilon} + |u-v|^{1+\epsilon}\left|Q_{a_N}\right|^{(1+\epsilon)\gamma}\\ &\gtrsim |u|^{1+\epsilon}\left|1-Q^{\gamma}_{a_N}\right|^{1+\epsilon} + |u-v|^{1+\epsilon}\left|Q_{a_N}\right|^{(1+\epsilon)\gamma} \text{ on } F_{1,2}(a).\end{aligned} \tag{4.13}$$

We also note from (4.5), Lemma 4.4 and Lemma 3.5 that

$$\left|1-Q_{a_N}^{\gamma}\right| \approx \left|1-Q_{a_N}\right| \gtrsim \rho \quad \text{on } F_1(a). \tag{4.14}$$

Combining the observations in (4.12), (4.13), (4.8) and (4.14), we obtain

$$\begin{aligned}\int_{F_{1,1}(a)}\left|u-vQ_{a_N}^{\gamma}\right|^{1+\epsilon} dA_{\epsilon-1} &+ \int_{F_{1,2}(a)}\left|u-vQ_{a_N}^{2\gamma}\right|^{1+\epsilon} dA_{\epsilon-1}\\ &\gtrsim \int_{F_1(a)}\left(|u|^{1+\epsilon}\left|1-Q_{a_N}^{\gamma}\right|^{1+\epsilon}+|u-v|^{1+\epsilon}\left|Q_{a_N}\right|^{(1+\epsilon)\gamma}\right) dA_{\epsilon-1}\\ &\gtrsim \int_{F_1(a)}\left[|\rho u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon}\right] dA_{\epsilon-1}\end{aligned}$$

so that

$$\begin{aligned}\left\|T(f_j)_{a_N,\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} &+ \left\|T(f_j)_{a_N,2\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon}\\ &\gtrsim \frac{1}{(1-|a|)^{\epsilon+1}}\int_{F_1(a)}\left[|\rho u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon}\right] dA_{\epsilon-1}\end{aligned} \tag{4.15}$$

by (4.10).

Next, for the integral over $F_2(a)$, we also have by an symmetric argument

$$\begin{aligned}\left\|T(f_j)_{\overline{a_N},\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} &+ \left\|T(f_j)_{\overline{a_N},2\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon}\\ &\gtrsim \frac{1}{(1-|a|)^{\epsilon+1}}\int_{F_2(a)}\left[|\rho u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon}\right] dA_{\epsilon-1}.\end{aligned} \tag{4.16}$$

One may keep track of the constants suppressed in the estimates (4.15) and (4.16) to see that they are independent of a.

Finally, we consider the integral over $F_3(a)$. We decompose the region $F_3(a)$ into two parts as in the case of the integral over $F_1(a)$. We set

$$F_{3,1}(a) := \left\{z \in F_3(a) : |u(z)| \le \frac{1}{1+\epsilon}|v(z)|\left|Q_a^{\gamma}(z)\right|\right\}$$

and

$$F_{3,2}(a) := F_3(a)\backslash F_{3,1}(a).$$

Note from Lemma 4.3 and (4.8)

$$\left|u-vQ_a^{\gamma}\right|^{1+\epsilon} \approx |u|^{1+\epsilon}+\left|vQ_a^{\gamma}\right|^{1+\epsilon} \gtrsim |u|^{1+\epsilon}+|v|^{1+\epsilon}(1-\rho)^{1+\epsilon} \quad \text{on } F_{3,1}(a).$$

Also, note

$$\begin{aligned}|\rho u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon} &\lesssim |u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}\left(|u|^{1+\epsilon}+|v|^{1+\epsilon}\right)\\ &\le 2|u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}|v|^{1+\epsilon}.\end{aligned} \tag{4.17}$$

Accordingly, we obtain

$$\left\|T(f_j)_{a,\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} \gtrsim \frac{1}{(1-|a|)^{\epsilon+1}}\int_{F_{3,1}(a)}\left[|\rho u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon}\right] dA_{\epsilon-1} \tag{4.18}$$

by (4.10). In conjunction with the integral over $F_{3,2}(a)$, we note

$$|Q_a| \le (1+\epsilon)^{-2} \quad \text{on } F_3(a)$$

by definition of the set $\Omega_{\epsilon,1-\epsilon}(a)$ and therefore

$$\frac{\left|vQ_a^{\gamma+1}\right|}{|u|} < (1+\epsilon)|Q_a| \le \frac{1}{1+\epsilon} \quad \text{on } F_{3,2}(a). \tag{4.19}$$

In addition, we have by (4.8)

$$(1-\rho)^{1+\epsilon}|v|^{1+\epsilon} \le \left|vQ_a^{\gamma}\right|^{1+\epsilon} < (1+\epsilon)^{1+\epsilon}|u|^{1+\epsilon} \quad \text{on } F_{3,2}(a). \tag{4.20}$$

It follows from (4.19), (4.20) and Lemma 4.3 that

$$\left|u-vQ_a^{\gamma+1}\right|^{1+\epsilon} \approx |u|^{1+\epsilon} \gtrsim |u|^{1+\epsilon}+(1-\rho)^{1+\epsilon}|v|^{1+\epsilon} \quad \text{on } F_{3,2}(a).$$

Accordingly, we obtain

$$\left\|T(f_j)_{a,\gamma+1}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} \gtrsim \frac{1}{(1-|a|)^{\epsilon+1}} \int_{F_{3,2}(a)} [|\rho u|^{1+\epsilon} + (1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon}]\, dA_{\epsilon-1}$$

by (4.10) and (4.17). This, together with (4.18), yields

$$\begin{aligned} &\left\|T(f_j)_{a,\gamma}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} + \left\|T(f_j)_{a,\gamma+1}\right\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon} \\ &\qquad \gtrsim \frac{1}{(1-|a|)^{\epsilon+1}} \int_{F_3(a)} [|\rho u|^{1+\epsilon} + (1-\rho)^{1+\epsilon}|u-v|^{1+\epsilon}]\, dA_{\epsilon-1}. \end{aligned} \quad (4.21)$$

Consequently, combining (4.15), (4.16), (4.21) and (2.6), we conclude (4.6), as required. The proof is complete.

Having completed the proof of Theorem 4.1, we now proceed to observe additional Carleson measure characterizations with bounded weights. We first recall the following estimate which is implicit in the proof of [22, Lemma 1] or [6, Lemma 4.3].

Lemma 4.5 (see [32]). *Let $\epsilon > 0$ and $0 \le \epsilon < \infty$. Put $\gamma := \min\left\{\frac{\epsilon}{2}, 1\right\}$. Let $\varphi_j \in \mathcal{S}(\mathbf{D})$, $\epsilon > 0$ and $W : \mathbf{D} \to [0,1]$ be a Borel function. If*

$$\sup_{\mathbf{D}} \left(W R_{\varphi_j}\right) \le \epsilon, \quad (4.22)$$

then there is a constant $C = C(\epsilon - 1) > 0$ such that

$$\int_{\mathbf{D}} \sum_j \left|f_j \circ \varphi_j\right|^{1+\epsilon} W dA_{\epsilon-1} \le C\epsilon^{\gamma} \sum_j \|f_j\|_{L_{\epsilon-1}^{1+\epsilon}}^{1+\epsilon}$$

for $f_j \in A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$.

When the weights are bounded, one may now use the next proposition to obtain several other versions of Theorems 1.1 and 4.1.

Proposition 4.6 (see [32]). *Let $\epsilon > 0$ and $\gamma > 0$. Let $\varphi_j \in \mathcal{S}(\mathbf{D})$ and $W : \mathbf{D} \to [0,1]$ be a Borel function. Put*

$$\mu := (W dA_{\epsilon-1}) \circ \varphi_j^{-1} \ \ and \ \ v := \left(R_{\varphi_j}^{\gamma} W dA_{\epsilon-1}\right) \circ \varphi_j^{-1}.$$

Then μ is a compact $(\epsilon-1)$-Carleson measure if and only if v is a compact α-Carleson measure.

Proof. The "only if" part is clear, because R_{φ_j} is bounded on $\mathbf{D}$ by the Schwarz-Pick Lemma. We now prove the "if" part. Assume that v is a compact $(\epsilon-1)$-Carleson measure. Let $\varepsilon \in (0,1)$ and put

$$K_{\varepsilon} := \left\{z \in \mathbf{D} : R_{\varphi_j}(z) \le \epsilon\right\}.$$

Note

$$\begin{aligned} \mu(E) &= \int_{\varphi_j^{-1}(E)\cap K_\varepsilon} \sum_j + \int_{\varphi_j^{-1}(E)\setminus K_\varepsilon} \sum_j W dA_{\epsilon-1} \\ &\le \int_{\varphi_j^{-1}(E)} \sum_j \chi_{K_\varepsilon} W dA_{\epsilon-1} \frac{1}{\varepsilon^{\gamma}} v(E) \end{aligned}$$

for any Boreal set $E \subset \mathbf{D}$. It follows that

$$\mu \le \mu_{\epsilon} + \frac{1}{\varepsilon^{\gamma}} v$$

where $\mu_\epsilon := \left(\chi_{K_\varepsilon} W dA_{\epsilon-1}\right) \circ \varphi_j^{-1}$. Accordingly, fixing $0 < \epsilon < 1$ and using the notation introduced in (2.14), we obtain

$$\hat{\mu}_{\epsilon-1,1-\epsilon}(z) \le \|\mu_\epsilon\|_{\epsilon-1,1-\epsilon} + \frac{1}{\varepsilon^{\gamma}} \hat{v}_{\epsilon-1,1-\epsilon}(z)$$

for all $z \in \mathbf{D}$. Now, since v is a compact $(\epsilon-1)$-Carleson measure, we obtain by (2.13)

$$\limsup_{|z|\to 1} \hat{\mu}_{\epsilon-1,1-\epsilon}(z) \le \|\mu_\epsilon\|_{\epsilon-1,1-\epsilon}$$

for each $\epsilon > 0$. Note from Lemma 4.5 and (2.14) that $\|\mu_\epsilon\|_{\epsilon-1,1-\epsilon} \to 0$ as $\epsilon \to 0$. Thus, taking the limit $\epsilon \to 0$, we obtain

$$\lim_{|z|\to 1} \hat{\mu}_{\epsilon-1,1-\epsilon}(z) = 0$$

and thus conclude the lemma by (2.13). The proof is complete.

We now exhibit some examples related to our characterizations. First, we provide an example showing that boundedness/compactness for the operators under consideration may depend on the parameters $(\epsilon - 1)$ and $(1 + \epsilon)$. In fact our example below is a single weighted composition operator with holomorphic weight. Such an example, which might have been known to experts, is included here for completeness.

Example (see [32]). Let $\epsilon > 0$ and $0 \leq \epsilon < \infty$. Let $0 < \epsilon < 1$. Consider functions φ_j and u_ϵ given by

$$\varphi_j(z) := 1 - (1 - z)^{1/2} \quad \textbf{and} \quad u_\epsilon(z) = \frac{1}{(1 - z)^\epsilon}$$

for $z \in \mathbf{D}$. It is elementary to check that $\varphi_j \in \mathcal{S}(\mathbf{D})$. Also note $u_\epsilon \in A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ by [31, Lemma 3.10], because $0 < \epsilon < 1$. We claim the following:

(i) $C_{\sum_j \varphi_j, u_\epsilon}$ is bounded on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ if and only if $\epsilon \leq \frac{1}{2}$.

(ii) $C_{\sum_j \varphi_j, u_\epsilon}$ is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ if and only if $\epsilon < \frac{1}{2}$.

In order to see this, we use the Carleson measure criteria for the pullback measure

$$\mu := (|u_\epsilon|^{1+\epsilon} dA_{\epsilon-1}) \circ \varphi_j^{-1}.$$

Fix $0 < \epsilon < 1$. Note that $\varphi_j(\mathbf{D})$ is contained in a non-tangential region with vertex at 1. Thus it is enough to investigate the behavior of $\hat{\mu}_{\epsilon-1,1-\epsilon}(a)$ as $a \to 1$. In addition, there is $t = t(1 - \epsilon) \in (0, 1)$ such that

$$\varphi_j^{-1}\big(E_{1-\epsilon}(a)\big) \subset \varphi_j^{-1}\big(E_t(|a|)\big)$$

for a near 1. So, we assume $0 < a < 1$ for the rest of the proof.

Note from (2.2)

$$\frac{(1 - a)(\epsilon)}{1 + a(1 - \epsilon)} < \big|1 - \varphi_j(z)\big| < \frac{(1 - a)(2 - \epsilon)}{1 - a(1 - \epsilon)} \tag{4.23}$$

and thus $|1 - z| \approx (1 - a)^2$ for $z \in \varphi_j^{-1}\big(E_{1-\epsilon}(a)\big)$. We thus obtain

$$\mu[E_{1-\epsilon}(a)] = \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \frac{dA_{\epsilon-1}(z)}{|1 - z|^{\epsilon(1+\epsilon)}} \approx \sum_j \frac{A_{\epsilon-1}\big[\varphi_j^{-1}\big(E_{1-\epsilon}(a)\big)\big]}{(1 - a)^{2\epsilon(1+\epsilon)}}.$$

In conjunction with this, we note from (4.23)

$$A_{\epsilon-1}\big[\varphi_j^{-1}\big(E_{1-\epsilon}(a)\big)\big] \leq \int_{|1-z|\leq c^2(1-a)^2} dA_{\epsilon-1}(z) \approx (1 - a)^{2(\epsilon+1)} \tag{4.24}$$

where $c = \frac{(2-\epsilon)}{\epsilon}$; see [8, Exercise 2.2.8] for this estimate. On the other hand, since

$$E_{1-\epsilon}(b) \subset \varphi_j^{-1}\big(E_{1-\epsilon}(a)\big) \quad \text{where} \quad b := \varphi_j^{-1}(a) = 1 - (1 - a)^2$$

by (2.4), we obtain by (2.6)

$$A_{\epsilon-1}\big[\varphi_j^{-1}\big(E_{1-\epsilon}(a)\big)\big] \geq A_{\epsilon-1}[E_{1-\epsilon}(b)] \approx (1 - b)^{(\epsilon+1)} = (1 - a)^{2(\epsilon+1)} \tag{4.25}$$

for $0 < a < 1$.

Now, combining (4.24) and (4.25), we obtain by (2.6)

$$\hat{\mu}_{\epsilon-1,1-\epsilon}(a) \approx \frac{(1 - a)^{(\epsilon+1)}}{(1 - a)^{2\epsilon(1+\epsilon)}}$$

for $0 < a < 1$; constants suppressed in this estimate are independent of a. We thus conclude (i) and (ii) by (2.12) and (2.13), respectively.

Next, we provide an example showing that the exponent $(1 + \epsilon)$ in Theorem 4.1 cannot be strictly less than $(\epsilon + 1)$. However, we do not know whether it can be reduced to $(\epsilon + 1)$.

Example (see [32]). Let $\epsilon > 0$ and $0 \leq \epsilon < \infty$. Let $\epsilon > -1$ and pick t_0 such that

$$t_0 = 1.$$

Define

$$u(z) := \frac{1}{(1-z)^{t_0}}, \quad z \in \mathbf{D}.$$

Since $t_0 < 1$, we have $u \in A^p_{\epsilon-1}(\mathbf{D})$; see [31, Lemma 3.10].

Now, consider the weighted composition operator $C_{0,u}$ given by

$$C_{0,u} f_j = u f_j(0)$$

for $f_j \in A^p_{\epsilon-1}(\mathbf{D})$. Taking $\psi_j := id$, the identity map on $\mathbf{D}$, and $v \equiv 0$, we may view $C_{0,u}$ as a difference of weighted composition operators. That is, we have $C_{0,u} = C_{0,u} - C_{\psi_j,0}$. Since the point evaluation $f_j \mapsto f_j(0)$ is compact on $A^p_{\epsilon-1}(\mathbf{D})$, we note that $C_{0,u}$ is also compact on $A^p_{\epsilon-1}(\mathbf{D})$.

On the other hand, with $\varphi_j = 0$ and $\psi_j = id$, note $\rho(z) = d(0,z) = |z|$. Thus, fixing $1-\epsilon \in (0,1)$, we have by (2.6), (2.5) and (2.7)

$$\begin{aligned}\left(\widetilde{\sigma^{1+\epsilon}_{\psi_j}}\right)_{\epsilon-1,1-\epsilon}(a) &\approx \frac{1}{(1-|a|)^{\epsilon+1}} \int_{E_{1-\epsilon}(a)} (1-|z|)^{1+\epsilon} |u(z)|^{1+\epsilon} dA_{\epsilon-1}(z) \\ &\gtrsim (1-|a|)^{1+\epsilon} |u(a)|^{1+\epsilon} \\ &= \frac{(1-|a|)^{1+\epsilon}}{|1-a|^{(1+\epsilon)t_0}}\end{aligned}$$

for $a \in \boldsymbol{D}$. Since $t_0 > 1$, the last expression diverges to ∞ as $a \to 1$ along the real axis. It follows from (2.13) that $\sigma^{1+\epsilon}_{\psi_j}$ is not even an $(\epsilon-1)$-Carleson measure.

Finally, we provide an example showing that unbounded weighted composition operators can form a compact difference in a nontrivial way.

Example (see [32]). Put

$$u(z) := \frac{1}{1-z} \text{ and } \varphi_j(z) := \frac{1+z}{2}$$

for $z \in \mathbf{D}$. Also, put

$$v_\epsilon := u + \epsilon(1-\varphi_j)^4 \text{ and } (\psi_j)_\epsilon := \varphi_j + \epsilon(1-\varphi_j)^4$$

where $\epsilon > 0$ is a small number to be chosen in a moment. Note

$$1 - |\varphi_j(z)|^2 = |1-\varphi_j(z)|^2 + \frac{1-|z|^2}{2} \geq |1-\varphi_j(z)|^2 \tag{4.26}$$

for $z \in \mathbf{D}$. Using this, one may check $(\psi_j)_\epsilon \in \mathcal{S}(\mathbf{D})$ for $\epsilon < \frac{1}{4}$. So, we fix $\epsilon \in \left(0, \frac{1}{8}\right]$ for the rest of the proof. Let $\epsilon \geq 0, 0 \leq \epsilon < \infty$ and assume $\epsilon > -1$ so that $u, v_\epsilon \in A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ by [31, Lemma 3.10]. We claim the following:

(i) $C_{\sum_j \varphi_j, u} - C_{\sum_j (\psi_j)_\epsilon, v_\epsilon}$ is compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

(ii) $C_{\sum_j \varphi_j, u}$ and $C_{\sum_j (\psi_j)_\epsilon, v_\epsilon}$ are not even bounded on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

In order to see this we will use again the Carleson measure criteria. For that purpose we note from (4.26)

$$\begin{aligned}\left|1-\varphi_j(z)\overline{(\psi_j)_\epsilon(z)}\right| &= \left|1-|\varphi_j(z)|^2 - \epsilon\varphi_j(z)\left(1-\overline{\varphi_j(z)}\right)^4\right| \\ &\geq |1-\varphi_j(z)|^2 - \frac{|1-\varphi_j(z)|^4}{8} \\ &\geq \frac{|1-\varphi_j(z)|^2}{2}\end{aligned}$$

and thus

$$\rho(z) = \frac{\epsilon|1-\varphi_j(z)|^4}{\left|1-\varphi_j(z)\overline{(\psi_j)_\epsilon(z)}\right|} \leq \frac{|1-\varphi_j(z)|^2}{4} \tag{4.27}$$

for $z \in \mathbf{D}$; we used $\epsilon \leq \frac{1}{8}$ for the inequality above.

Fix $0 < \epsilon < 1$. Note from (4.26) and (2.5) that

$$|1-z|^2 \leq 4\left(1-|\varphi_j(z)|^2\right) \approx 1-|a|, \quad z \in \varphi_j^{-1}(E_{1-\epsilon}(a)) \tag{4.28}$$

for $a \in \mathbf{D}$. This, together with (4.27), yields

$$|\rho(z)u(z)|^{1+\epsilon} \le \left|\frac{1-z}{16}\right|^2 \lesssim (1-|a|)^{1+\epsilon/2}, \quad z \in \varphi_j^{-1}\big(E_{1-\epsilon}(a)\big)$$

for $a \in \mathbf{D}$. As a consequence, we obtain

$$\omega_{\varphi_j,u}[E_{1-\epsilon}(a)] = \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j |\rho(z)u(z)|^{1+\epsilon} dA_{\epsilon-1}(z)$$
$$\lesssim (1-|a|)^{1+\epsilon/2} \sum_j \big(A_{\epsilon-1} \circ \varphi_j^{-1}\big)[E_{1-\epsilon}(a)]$$

so that

$$\left(\widehat{\omega_{\varphi_j,u}}\right)_{\epsilon-1,1-\epsilon}(a) \lesssim \sum_j (1-|a|)^{1+\epsilon/2} \left\|A_{\epsilon-1} \circ \varphi_j^{-1}\right\|_{\epsilon-1,1-\epsilon}$$

for $a \in \mathbf{D}$; recall $\left\|A_{\epsilon-1} \circ \varphi_j^{-1}\right\|_{\epsilon-1,1-\epsilon} < \infty$by (2.16). Similarly, we have the same estimates for $\left(\widehat{\omega_{(\psi_j)_\epsilon,v_\epsilon}}\right)_{\epsilon-1,1-\epsilon}$. Using the fact that $|u(z) - v_\epsilon(z)| \lesssim |1-z|^4$ instead of $|\rho(z)u(z)| \lesssim |1-z|$, we also obtain similar estimates for the measures $\sigma_{\varphi_j}^{1+\epsilon}$and $\sigma_{(\psi_j)_\epsilon}^{1+\epsilon}$. Thus we see that Assertion (b) of Theorem 1.1 holds and hence conclude (i).

Meanwhile, note from (2.4)

$$E_{1-\epsilon}(b) \subset \varphi_j^{-1}\big(E_{1-\epsilon}(a)\big) \text{ where } b := \varphi_j^{-1}(a) = 2a - 1$$

for $a \in \mathbf{D}$. It follows from (2.6) that

$$\big(A_{\epsilon-1} \circ \varphi_j^{-1}\big)[E_{1-\epsilon}(a)] \ge A_{\epsilon-1}[E_{1-\epsilon}(b)] \approx A_{\epsilon-1}[E_{1-\epsilon}(a)] \tag{4.29}$$

for $0 < a < 1$. Now, setting

$$\mu := (|u|^{1+\epsilon} dA_{\epsilon-1}) \circ \varphi_j^{-1},$$

we have by (4.28) and (4.29)

$$\hat{\mu}_{\epsilon-1,1-\epsilon}(a) = \frac{1}{A_{\epsilon-1}[E_{1-\epsilon}(a)]} \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \frac{dA_{\epsilon-1}(z)}{|1-z|^{1+\epsilon}} \gtrsim \frac{1}{(1-a)^{1+\epsilon/2}}$$

for $0 < a < 1$. This shows that μ is not an $\epsilon - 1$-Carleson measure, or equivalently, that $C_{\sum_j \varphi_j,u}$ is not bounded on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. Using this and (i), we conclude (ii), as required.

5. Applications:

We observe some consequences of Theorem 1.1. We first verify Theorem 1.2 and then use it to prove Theorem 1.3.

We begin with some preliminary observations. For $0 \le \epsilon < \infty$, recall that $H^{1+\epsilon}(\mathbf{D})$ is the Banach space of all holomorphic functions f_j on $\mathbf{D}$ whose norm is given by

$$\|f_j\|_{H^{1+\epsilon}} := \sup_{0<1-\epsilon<1} \left\{\frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|f_j\left((1-\epsilon)e^{i\theta}\right)\right|^{1+\epsilon} d\theta\right\}^{1/1+\epsilon}.$$

When $\epsilon = 1$, the well-known Littlewood-Paley Identity asserts that the H^2-norm can be converted to an area integral:

$$\|f_j\|_{H^2} = |f_j(0)|^2 + \int_{\mathbf{D}} \sum_j |f_j'(z)|^2 \log\frac{1}{|z|} dA(z).$$

Since $\log|z|^{-1}$ is integrable near 0 and comparable to $1-|z|^2$ near boundary, this yields

$$\|f_j - f_j(0)\|_{H^2} \approx \|f_j'\|_{A_1^2}; \tag{5.1}$$

constants suppressed here are independent of $f_j \in H^2(\mathbf{D})$. Thus, denoting by ∂ the differentiation operator $f_j \mapsto f_j'$, we see from (5.1) that the operator

$$\partial : H^2(\mathbf{D}) \to A_1^2(\mathbf{D})$$

is bounded. Conversely, denoting by $\mathfrak{I}$ the integration operator $f_j \mapsto \int_0^z f_j(\zeta)d\zeta$, we also see that the operator

$$\mathfrak{I} : A_1^2(\mathbf{D}) \to H^2(\mathbf{D})$$

is bounded.

Now, given $\varphi_j, \psi_j \in \mathcal{S}(\mathbf{D})$, note that

$$C_{\sum_j \varphi_j, \sum_j \varphi_j'} - C_{\sum_j \psi_j, \sum_j \psi_j'} = \partial \circ \left(C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}\right) \circ \mathfrak{I} \quad \text{on } A_1^2(\mathbf{D})$$

and

$$C_{\sum_j \varphi_j} - C_{\sum_j \psi_j} = \mathfrak{I} \circ \left(C_{\sum_j \varphi_j, \sum_j \varphi_j'} - C_{\sum_j \psi_j, \sum_j \psi_j'}\right) \circ \partial + \Lambda_0 \circ \left(C_{\sum_j \varphi_j} - C_{\sum_j \psi_j}\right) \quad \text{on } H^2(\mathbf{D})$$

where $\Lambda_0 : H^2(\mathbf{D}) \to H^2(\mathbf{D})$ is the point evaluation at 0. Since Λ_0 is a compact operator on $H^2(\mathbf{D})$, it follows that

$$\begin{aligned} & C_{\sum_j \varphi_j} - C_{\sum_j \psi_j} : \text{compact } H^2(\mathbf{D}) \\ \Leftrightarrow \quad & C_{\sum_j \varphi_j, \sum_j \varphi_j'} - C_{\sum_j \psi_j, \sum_j \psi_j'} : \text{compact on } A_1^2(\mathbf{D}). \end{aligned}$$

Thus Theorem 1.2 is now immediate from Theorem 1.1.

We now turn to the proof of Theorem 1.3. We begin with a lemma relying on the Koebe One-quarter Theorem (see [11, Theorem 2.3]): If η is a univalent holomorphic function on $\mathbf{D}$ normalized so that $\eta(0) = 0$ and $\eta'(0) = 1$, then $\frac{1}{4}\mathbf{D} \subset \eta(\mathbf{D})$. In what follows recall that R_{φ_j} denotes the function introduced in (1.2).

Lemma 5.1 (see [32]). *Let $\varphi_j, \psi_j \in \mathcal{S}(\mathbf{D})$ be univalent maps satisfying* (1.3). *Let $K \subset \mathbf{D}$ and assume*

$$\inf_K R_{\varphi_j} > 0. \tag{5.2}$$

Then

$$\lim_{|z| \to 1, z \in K} \sum_j \left|\varphi_j'(z) - \psi_j'(z)\right| = 0. \tag{5.3}$$

Proof. From (1.3) and (5.2) we note $\rho(z) \to 0$ as $|z| \to 1$ inside K and hence

$$1 - \left|\varphi_j(z)\right|^2 \approx 1 - \left|\psi_j(z)\right|^2$$

for all $z \in K$ by (2.5). Thus we also have

$$\inf_K R_{\psi_j} > 0.$$

Now, suppose that (5.3) fails. We will complete the proof by deriving a contradiction. Since (5.3) fails, there is a sequence $\{z_n\}$ in K and $\epsilon > 0$ such that $|z_n| \to 1$ and

$$\left|\varphi_j'(z_n) - \psi_j'(z_n)\right| \geq 8\epsilon \tag{5.4}$$

for all n. By passing to a subsequence if necessary, we may assume by symmetry

$$\left|\varphi_j'(z_n)\right| \geq 4\epsilon \tag{5.5}$$

for all n. Note that R_{φ_j} is bounded on $\mathbf{D}$ by the Schwarz-Pick Lemma. Thus, we may further assume

$$R_{\varphi_j}(z_n) \leq \frac{1}{4\epsilon} \tag{5.6}$$

for all n.

Meanwhile, applying the Koebe One-quarter Theorem to the normalized univalent functions

$$\lambda \in \mathbf{D} \mapsto \frac{\varphi_j\left(\frac{\lambda + z_n}{1 + \bar{z}_n \lambda}\right) - \varphi_j(z_n)}{\varphi_j'(z_n)(1 - |z_n|^2)},$$

we obtain

$$D\left(\varphi_j(z_n), \frac{|\varphi_j'(z_n)|(1 - |z_n|^2)}{4}\right) \subset \varphi_j(\mathbf{D})$$

for each n; recall that $D(\cdot,\cdot)$ is the Euclidean disk defined in (2.3). This, together with (5.5), shows that the inverse function φ_j^{-1} is defined on the disk $D\left(\varphi_j(z_n), \epsilon(1 - |z_n|^2)\right)$ for each n. Thus, applying once again the Koebe One-quarter Theorem to the normalized univalent functions

$$\lambda \in \mathbf{D} \mapsto \frac{\varphi_j^{-1}(\varphi_j(z_n) + \epsilon(1-|z_n|^2)\lambda) - z_n}{\frac{\epsilon(1-|z_n|^2)}{\varphi_j'(z_n)}},$$

we obtain

$$D\left(z_n, \frac{\epsilon(1-|z_n|^2)}{4|\varphi_j'(z_n)|}\right) \subset \varphi_j^{-1}\left[D\left(\varphi_j(z_n), \epsilon(1-|z_n|^2)\right)\right].$$

In connection with this, choosing $\delta \in (0,1)$ with $4\delta \le \inf_K R_{\varphi_j}$, we note by the Schwarz-Pick Lemma

$$\frac{1}{|\varphi_j'(z_n)|} \ge R_{\varphi_j}(z_n) \ge 4\delta$$

and thus by (5.6)

$$D(z_n, \delta\epsilon(1-|z_n|)) \subset \varphi_j^{-1}\left[D\left(\varphi_j(z_n), \frac{1-|\varphi_j(z_n)|}{2}\right)\right]$$

for each n. This yields

$$1-|z| \approx 1-|z_n| \quad \text{and} \quad 1-|\varphi_j(z)| \approx 1-|\varphi_j(z_n)| \tag{5.7}$$

for $z \in D(z_n, \delta\epsilon(1-|z_n|))$. As a consequence we have

$$R_{\varphi_j}(z) \approx R_{\varphi_j}(z_n) \approx 1, \quad z \in D(z_n, \delta\epsilon(1-|z_n|))$$

for all n; constants suppressed in these estimates are independent of z and n. It follows from this and (1.3) that

$$M_n := \sup_{D(z_n, \delta\epsilon(1-|z_n|))} \rho \to 0 \tag{5.8}$$

as $n \to \infty$. In view of this, we note from (2.5) and (5.7)

$$\left|1-\overline{\varphi_j(z)}\psi_j(z)\right| \approx 1-|\varphi_j(z)| \approx 1-|z_n|, \quad z \in D(z_n, \delta\epsilon(1-|z_n|))$$

for all n. Consequently, we conclude by the Cauchy Estimates

$$|\varphi_j'(z_n) - \psi_j'(z_n)| \le \frac{1}{\delta\epsilon(1-|z_n|)}\left(\sup_{D(z_n, \delta\epsilon(1-|z_n|))} |\varphi_j - \psi_j|\right) \approx \frac{M_n}{\delta\epsilon}$$

for all n; constants suppressed in this estimate are independent of n. This, together with (5.8), yields a contradiction to (5.4), as desired. The proof is complete.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. As mentioned in the Introduction, Moorhouse [22] already noticed that the necessity is true for general φ_j and ψ_j, not necessarily univalent. So, we only need to prove the sufficiency. Assume that (1.3) holds. Let ω and $\sigma_{1-\epsilon}$ be the measures in Theorem 1.2. We will show that $\omega + \sigma_{1-\epsilon}$ is a compact 1-Carleson measure for any $0 < \epsilon < 1$, which
concludes the theorem by Theorem 1.2.

First, we consider the measure ω. Let $0 < \epsilon < 1$. Setting

$$M_{\varphi_j}(a) := \sup_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \left[\rho^2 R_{\varphi_j}\right]$$

for short, we have

$$\begin{aligned}\int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j |\rho\varphi_j'|^2 dA_1 &\le 2M_{\varphi_j}(a)\int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \left(1-|\varphi_j(z)|^2\right)|\varphi_j'(z)|^2 dA(z) \\ &= 2\sum_j M_{\varphi_j}(a)\int_{E_{1-\epsilon}(a)\cap\varphi_j(\mathbf{D})} (1-|w|^2)dA(w) \\ &\le \sum_j M_{\varphi_j}(a) A_1[E_{1-\epsilon}(a)]\end{aligned}$$

for $a \in \mathbf{D}$; the first equality holds by the change of variables $w = \varphi_j(z)$. By symmetry the same estimate with ψ_j in place of φ_j also holds. Consequently, we obtain

$$\widehat{\omega}_{1,1-\epsilon}(a) \leq M_{\varphi_j}(a) + M_{\psi_j}(a) \tag{5.9}$$

for $a \in \mathbf{D}$. Meanwhile, note from (2.5) that $1 - |\varphi_j(z)| \to 0$ as $|a| \to 1$ uniformly in $z \in \varphi_j^{-1}(E_{1-\epsilon}(a))$. Thus we see from the Schwarz-Pick Lemma that $1 - |z| \to 0$ as $|a| \to 1$ uniformly in $z \in \varphi_j^{-1}(E_{1-\epsilon}(a))$. This, together with (1.3), yields $M_{\varphi_j}(a) + M_{\psi_j}(a) \to 0$ as $|a| \to 1$. Consequently, we deduce from (5.9) and (2.13) that ω is a compact 1-Carleson measure, as required.

Now, we let $0 < \epsilon < 1$ and consider the measure $\sigma_{1-\epsilon}$; recall

$$\sigma_{1-\epsilon} = \sigma_{\varphi_j,1-\epsilon} + \sigma_{\psi_j,1-\epsilon}$$

by definition. We will show that $\sigma_{\varphi_j,1-\epsilon}$ is a compact 1-Carleson measure; the same is true for $\sigma_{\psi_j,1-\epsilon}$ by symmetry. Pick $1 - \epsilon \in (0, \epsilon)$ and let $\epsilon > 0$. Put

$$\Delta_{1,\epsilon} := \left\{z \in \mathbf{D} : R_{\varphi_j}(z) + R_{\psi_j}(z) < 2\epsilon\right\}$$

$$\Delta_{2,\epsilon} := \left\{z \in \mathbf{D} : R_{\varphi_j}(z) > 2\epsilon\right\}$$

$$\Delta_{3,\epsilon} := \left\{z \in \mathbf{D} : R_{\psi_j}(z) \geq 2\epsilon\right\}$$

so that

$$\varphi_j^{-1}(E_{1-\epsilon}(a)) = \bigcup_{j=1}^{3} [\varphi_j^{-1}(E_{1-\epsilon}(a)) \cap \Delta_{j,\epsilon}].$$

We have

$$\sigma_{\varphi_j,1-\epsilon}[E_{1-\epsilon}(a)] = \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \chi_{G_{1-\epsilon}} |\varphi_j' - \psi_j'|^2 dA_1 \leq I_1 + I_2 + I_3$$

for $a \in \mathbf{D}$ where

$$I_j := \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \chi_{G_{1-\epsilon}} \chi_{\Delta_{j,\epsilon}} |\varphi_j' - \psi_j'|^2 dA_1$$

for $j = 1, 2, 3$; recall that χ denotes the characteristic function of the set specified in its subscript. To estimate I_1, we first note

$$\int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \chi_{\Delta_{1,\epsilon}} |\varphi_j'|^2 dA_1 \leq 2\epsilon \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \left(1 - |\varphi_j|^2\right) |\varphi_j'|^2 dA$$
$$\leq \epsilon A_1[E_{1-\epsilon}(a)];$$

the last inequality holds by a change of variables, as above. Also, since $\varphi_j^{-1}(E_{1-\epsilon}(a)) \cap G_{1-\epsilon}$ is contained in $\psi_j^{-1}\left(E_{2(1-\epsilon)}(a)\right)$, we have similarly

$$\int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \chi_{G_{1-\epsilon}} \chi_{\Delta_{1,\epsilon}} |\psi_j'|^2 dA_1 \leq 2\epsilon \int_{\psi_j^{-1}\left(E_{2(1-\epsilon)}(a)\right)} \sum_j \left(1 - |\psi_j|^2\right) |\psi_j'|^2 dA$$
$$\leq \epsilon A_1\left[E_{2(1-\epsilon)}(a)\right].$$

It follows that

$$\frac{I_1}{A_1[E_{1-\epsilon}(a)]} \leq \frac{2}{A_1[E_{1-\epsilon}(a)]} \int_{\varphi_j^{-1}(E_{1-\epsilon}(a))} \sum_j \chi_{G_{1-\epsilon}} \chi_{\Delta_{1,\epsilon}} \left(|\varphi_j'|^2 + |\psi_j'|^2\right) dA_1 \lesssim \epsilon \tag{5.10}$$

for all $a \in \mathbf{D}$.

For the second integral I_2, we have by (2.16) (with $\epsilon = 1$) and Lemma 5.1

$$\frac{I_2}{A_1[E_{1-\epsilon}(a)]} \approx \sum_j \frac{I_2}{(A_1 \circ \varphi_j^{-1})[E_{1-\epsilon}(a)]} \leq \left(\sup_{\varphi_j^{-1}(E_{1-\epsilon}(a)) \cap \Delta_{2,\epsilon}} \sum_j |\varphi_j' - \psi_j'| \right) \to 0 \tag{5.11}$$

as $|a| \to 1$; we used (2.2) to apply Lemma 5.1. Similarly, we have

$$\frac{I_3}{A_1[E_{1-\epsilon}(a)]} \lesssim \sum_j \left(\sup_{\varphi_j^{-1}(E_{1-\epsilon}(a)) \cap \Delta_{3,\epsilon}} |\varphi_j' - \psi_j'| \right) \to 0 \tag{5.12}$$

as $|a| \to 1$. One may check that constants suppressed in the estimates (5.10), (5.11) and (5.12) are independent of a and ϵ. Thus, combining these estimates and then taking the limit $|a| \to 1$, we obtain

$$\limsup_{|a|\to 1} \sum_j \widehat{\left(\sigma_{\varphi_j,1-\epsilon}\right)}_{1,1-\epsilon}(a) \le C\epsilon$$

for some constant $C > 0$ independent of ϵ. Since ϵ is arbitrary, we conclude

$$\lim_{|a|\to 1} \sum_j \widehat{\left(\sigma_{\varphi_j,1-\epsilon}\right)}_{1,1-\epsilon}(a) = 0$$

and hence by (2.13) that $\sigma_{\varphi_j,1-\epsilon}$ is a compact 1-Carleson measure, as required. The proof is complete.

We now turn to the special case when the weight functions u and v belong to $H^\infty(\mathbf{D})$, the class of all bounded holomorphic functions on $\mathbf{D}$. In such a case, boundedness is out of question, because weighted composition operators with bounded weights are always bounded on the spaces $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. Acharyya and Wu have recently obtained a compactness characterization for differences of composition operators with bounded holomorphic symbols acting from a weighted Bergman space into another. The theorem below is their characterization for those operators acting from a weighted Bergman space into itself. We remark in passing that, as a consequence of this characterization, compactness for differences of weighted composition operators with bounded holomorphic weights is independent of parameters $\epsilon - 1$ and $1 + \epsilon$.

Theorem 5.2 (see [32]) (Acharyya and Wu [1]). *Let $\epsilon > 0$, $0 \le \epsilon < \infty$ and $\epsilon > -1$. Let $\varphi_j, \psi_j \in \mathcal{S}(\mathbf{D})$ and $u, v \in H^\infty(\mathbf{D})$. Then the following statements are equivalent:*

(a) *$C_{\sum_j \varphi_j,u} - C_{\sum_j \psi_j,v}$ is compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.*

(b) *The following two conditions are fulfilled;*

$$\lim_{|z|\to 1} \rho^{1+\epsilon}(z)\left[|u(z)|^{1+\epsilon} R_{\varphi_j}(z) + |v(z)|^{1+\epsilon} R_{\psi_j}(z)\right] = 0$$

and

$$\lim_{|z|\to 1} \left(1 - \rho(z)\right)^{1+\epsilon} |u(z) - v(z)|^{1+\epsilon}\left[R_{\varphi_j}(z) + R_{\psi_j}(z)\right] = 0.$$

In fact Acharyya and Wu prove that Assertion (a) is equivalent to the following two conditions:

$$\lim_{|z|\to 1} \rho(z)\left[|u(z)| R^1_{\sum_j \varphi_j}(z) + |v(z)| R^1_{\sum_j \psi_j}(z)\right] = 0$$

and

$$\lim_{|z|\to 1} \left(1 - \rho(z)\right)^1 |u(z) - v(z)|\left[R^1_{\sum_j \varphi_j}(z) + R^1_{\sum_j \psi_j}(z)\right] = 0.$$

Note that functions $\rho, u, v, R_{\varphi_j}, R_{\psi_j}$ are all bounded on $\mathbf{D}$. Thus it is elementary to verify that the above two conditions are equivalent to the ones in Assertion (b).

In conjunction with Theorem 5.2, we recall the following lemma taken from [22, Lemma1].

Lemma 5.3 (see [32]). *Let $\epsilon > 0$. Let $\varphi_j \in \mathcal{S}(\mathbf{D})$ and $W : \mathbf{D} \to [0, 1]$ be a Borel function. If*

$$\lim_{|z|\to 1} \sum_j W(z) R_{\varphi_j}(z) = 0,$$

then $(W dA_{\epsilon-1}) \circ \varphi_j^{-1}$ is a compact $\epsilon - 1$-Carleson measure.

Now we note by Lemma 5.3 and Theorem 1.1 that the implication (b) $\Longrightarrow$ (a) in Theorem 5.2 remains valid for $u, v \in L^\infty(\mathbf{D})$. In view of this, one may ask whether Theorem 5.2 extends to the case of arbitrary bounded weights. The answer is *no*, as the next example shows.

Example (see [32]). Put

$$u = \sum_{n=2}^{\infty} \chi_{D_n}$$

where $D_n := D\left(1 - \frac{1}{n}, \frac{1}{3n^2}\right)$. Note that the disks D_n's are pairwise disjoint. Taking $\varphi_j = id$, the identity map on $\mathbf{D}$, and $v = \psi_j \equiv 0$, we see that

$$C_{\sum_j \varphi_j,u} - C_{\sum_j \psi_j,v} = C_{id,u}$$

is the multiplication operator with symbol u.

We first check the compactness of $C_{id,u} : A_0^1(\mathbf{D}) \to L_0^1(\mathbf{D})$. Fix $0 < \epsilon < 1$. Given $a \in \boldsymbol{D}$, let $J(a)$ be the set of all indices n such that $D_n \cap E_{1-\epsilon}(a) \neq \emptyset$. For $n \in J(a)$ and $z \in D_n \cap E_{1-\epsilon}(a) \neq \emptyset$, we note $1 - \frac{1}{n} - \frac{1}{3n^2} < |z| < 1 - \frac{1}{n} + \frac{1}{3n^2}$ and thus $\frac{2}{3} < n(1-|z|) < \frac{4}{3}$. We thus have $n(1-|a|) \approx 1$ by (2.2). It follows that the number of elements of $J(a)$, denoted by $|J(a)|$, is bounded by some constant times $(1-|a|)^{-1}$. We also have $A(D_n) = \frac{1}{9n^4} \approx (1-|a|)^4$. Accordingly, setting $\mu := (|u|dA) \circ \varphi_j^{-1} = |u|dA$, we obtain

$$\begin{aligned}\hat{\mu}_{0,1-\epsilon}(a) &\approx \frac{1}{(1-|a|)^2} \sum_{n \in J(a)} A[D_n \cap E_{1-\epsilon}(a)] \\ &\leq \frac{|J(a)|}{(1-|a|)^2} \left[\sup_{n \in J(a)} A(D_n) \right] \\ &\lesssim 1 - |a|\end{aligned}$$

for all $a \in \mathbf{D}$. One may check that the constants suppressed above are independent of a. Accordingly, μ is a compact 0-Carleson measure. This, together with Theorem 1.1, implies that $C_{id,u} : A_0^1(\mathbf{D}) \to L_0^1(\mathbf{D})$ is compact, as asserted.

On the other hand, since $\rho(z) = |z|$ and $R_{\varphi_j} \equiv 1$, we see that conditions in (b) of Theorem 5.2 reduce to the condition

$$\lim_{|z| \to 1} |u(z)| = 0,$$

which is certainly not possible. This shows that the implication (a) $\Longrightarrow$ (b) in Theorem 5.2 is no longer true for arbitrary bounded weights.

Before closing the paper, we remark that the arguments of the current paper extend to differences of weighted composition operators acting from $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ to $L_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ for $0 \leq \epsilon < \infty$.

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