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Review Paper

Differences of generalized integration operators from α-Bers space to Bloch-type space

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ABSTRACT: *This paper characterizes the boundedness and compactness of the differences of the generalized integration operators from the α-Bers space to the Bloch-type space. by using the concept of the essential norm, we derive equivalent compactness conditions for these operator differences. Furthermore, we prove that the boundedness and compactness of the differences of the generalized integration operators from the little α-Bers space to the little Bloch-type space are equivalent.*

KEYWORDS: *Generalized integration operators, Bloch-type space; Boundedness, α-Bers space, Compactness*

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I. INTRODUCTION

Let D denote the unit open disk in the complex plane , $H(D)$ be the set of all holomorphic functions on D, and $u(z)$ be a normal function on D, that is, is a positive continuous function on D, and there exist $0 \le \delta < 1$ and $0 < a < b$ such that

$$\frac{u(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{u(r)}{(1-r)^a}=0;$$

$$\frac{u(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{u(r)}{(1-r)^b}=0.$$

If $f \in H(D)$ and

$$\|f\|_{B_u} = |f(0)| + \sup_{z\in D} u(|z|)|f'(z)| < \infty,$$

then f is said to belong to the Bloch-type space B_u. The little Bloch -type space $B_{u,0}$ consists of all $f \in B_u$ such that

$$\lim_{|z|\to 1} u(|z|)|f'(z)| = 0,$$

obviously, B_u and $B_{u,0}$ are Banach spaces under the norm $\|f\|_{B_u} = |f(0)| + \sup_{z\in D} u(|z|)|f'(z)|$.

For $u(|z|) = \left(1-|z|^2\right)^\alpha$, $\alpha > 0$, the Bloch-type space B_u is the α-Bloch space B_α, and the little Bloch-type space $B_{u,0}$ is the little α-Bloch space $B_{\alpha,0}$.

If $f \in H(D)$ and

$$\|f\|_{H_\alpha} = \sup_{z\in D}\left(1-|z|^2\right)^\alpha |f(z)| < \infty, \alpha > 0$$

then f belongs to the α-Bers space H_α, the little α-Bers space $H_{\alpha,0}$ consists of all $f \in H_\alpha$ such that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)^{\alpha}\left|f(z)\right|=0$$

and clearly H_α and $H_{\alpha,0}$ are Banach spaces under the norm $\|f\|_{H_\alpha}=\sup_{z\in D}\left(1-|z|^2\right)^{\alpha}\left|f(z)\right|$.

For arbitrary $z,w\in D$, let $\delta_w(z)=\dfrac{w-z}{1-\bar{w}z}$ be the automorphism of D which interchanges 0 and w. The pseudo-hyperbolic distance $p(z,w)=\left|\delta_w(z)\right|=\left|\dfrac{w-z}{1-\bar{w}z}\right|$ between z and w, and clearly $p(z,w)\le 1$.

Give a linear operator $T: X\to Y$, its essential norm is the distance from the operator T to the set of compact K mapping X to Y, that is,

$$\|T\|_{e,X\to Y}=\inf\left\{\|T-K\|_{X\to Y}: K \text{ is compact}\right\},$$

where X,Y be Banach spaces, Clearly, T is compact if and only if $\|T\|_{e,X\to Y}=0$.

Let $S(D)$ be the set of all analytic self-maps on D, $n\in\mathbb{N}$ and let $f^{(n)}$ denote the n-th derivative of f and $f^{(0)}=f$ .A linear operators $I_{\varphi,g}^{n}$ is defined by

$$I_{\varphi,g}^{n}f(z)=\int_0^z f^{(n)}\left(\varphi(\xi)\right)g(\xi)\,d\xi,\ \varphi\in S(D), g\in H(D), z\in D.$$

The operator $I_{\varphi,g}^{n}$ is called the generalized integration operator. It was first introduced in [1], and further studied in [2,3] . In fact, the generalized integration operator is a generalization of many well-known operators. When $n=1$, the generalized integration operator is the operator studied by Li and Stević in [4,5]. When $n=0$, the generalized integration operator is the integral-type operator studied by Yu-Xia and Cui in [6] .

In recent years, many scholars have been very interested in the differences of two operators acting on various analytic function spaces. Liu and Li [7] studied the boundedness and compactness of differences of the weighted composition operator from the Bloch space to the Bers space. Yu-Xia and Cui [6] estimated the essential norm of differences of the composition operator, and obtained several equivalent conditions for the compactness of differences of the composition operator from the Bloch space to the Bloch space. For other studies on differences of operators, see references [8,9,10,11,12].

Inspired by Liu, Li, Yu-Xia and Cui, this paper studies the boundedness and compactness of differences of the generalized integration operator from α-Bers spaces to Bloch-type spaces.

Throughout this paper, the positive constants are denoted by C, they may be different in different places. The notation $A\lesssim B$ means that there is a positive constant C such that $A\le CB$ and The notation $A\gtrsim B$ means that there is a positive constant C such that $A\ge CB$. Moreover, if both $A\lesssim B$ and $A\gtrsim B$ hold, then one says that $A\approx B$.

Let $n\in\mathbb{N}^{+}, \varphi,\psi\in S(D), g,h\in H(D)$, For the convenience of this paper's research, we define

$$L_{\varphi,g}^{n}(z)=\frac{u(|z|)g(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha+n}};$$

$$L_{\psi,h}^{n}(z)=\frac{u(|z|)h(z)}{\left(1-|\psi(z)|^2\right)^{\alpha+n}}.$$

The following four lemmas run through the proofs of the main results of this paper, which can be derived from Lemmas 2.1 and 2.2 and their proof processes in Reference [13].

Lemma 1.1. For $\forall n\in\mathbb{N}$, $f\in H_\alpha$ if and only if $f^{(n)}\in H_{\alpha+n}$ and the following asymptotic relationship holds:

$$\|f\|_{H_\alpha}\approx\sum_{k=0}^{n-1}\left|f^{(k)}(0)\right|+\sup_{z\in D}\left(1-|z|^2\right)^{\alpha+n}\left|f^{(n)}(z)\right|.$$

Lemma 1.2. For $\forall n\in\mathbb{N}$, $f\in H_{\alpha,0}$ if and only if

$$\lim_{|z|\to 1}\left(1-|z|^2\right)^{\alpha+n}\left|f^{(n)}(z)\right|=0.$$

Lemma 1.3. Let $z,w\in D$, $n\in\mathbb{N}$, then for any $f\in H_\alpha$, we have

$$\left|\left(1-|z|^2\right)^{\alpha+n} f^{(n)}(z)-\left(1-|w|^2\right)^{\alpha+n} f^{(n)}(w)\right|\le c\|f\|_{H_\alpha}\, p(z,w)$$

Lemma 1.4. Let $z,w\in D_r$, $n\in\mathbb{N}$, where $D_r=\{z\in D:|z|\le r<1\}$, then for any $f\in H_\alpha$, we have

$$\left|\left(1-|z|^2\right)^{\alpha+n} f^{(n)}(z)-\left(1-|w|^2\right)^{\alpha+n} f^{(n)}(w)\right|\le cp(z,w)\max\left\{\sup_{z\in D_r} f^{(n)}(z),\sup_{z\in D_r} f^{(n+1)}(z)\right\}$$

II. THE BOUNDEDNESS OF $I_{\varphi,g}^n-I_{\psi,h}^n$: $H_\alpha\to B_u$

This section will give several equivalent characterizations of $I_{\varphi,g}^n-I_{\psi,h}^n$: $H_\alpha\to B_u$ being bounded. First, we construct two key classes of test functions and prove that they belong to $H_{\alpha,0}$ and are bounded on $H_{\alpha,0}$

Lemma 2.1. For $n\in\mathbb{N}^+, w\in D$, define two families test functions:

$$f_w^{(n)}(z)=\frac{1-|w|^2}{(1-\bar{w}z)^{\alpha+n+1}},$$

$$g_w^{(n)}(z)=\delta_w(z)\, f_w^{(n)}(z),$$

Then $f_w(z), g_w(z)\in H_{\alpha,0}$ and are bounded on $H_{\alpha,0}$.

Proof: Since

$$\begin{aligned}\lim_{|z|\to1}\left(1-|z|^2\right)^{\alpha+n}\left|f_w^{(n)}(z)\right| &= \lim_{|z|\to1}\left(1-|z|^2\right)^{\alpha+n}\left|\frac{1-|w|^2}{(1-\bar{w}z)^{\alpha+n+1}}\right|\\ &\le \lim_{|z|\to1}\left(1-|z|^2\right)^{\alpha+n}\left|\frac{1-|w|^2}{(1-|w|)^{\alpha+n+1}}\right|\\ &=0\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\sup_{z\in D}\left(1-|z|^2\right)^{\alpha+n}\left|f_w^{(n)}(z)\right| &= \sup_{z\in D}\left(1-|z|^2\right)^{\alpha+n}\left|\frac{1-|w|^2}{(1-\bar{w}z)^{\alpha+n+1}}\right|\\ &\le \sup_{z\in D}\left(1-|z|^2\right)^{\alpha+n}\frac{\left(1-|w|^2\right)}{(1-|z|)^{\alpha+n}\left(1-|w|^2\right)}\\ &\le 2^{\alpha+n+1}<\infty,\end{aligned}\tag{2.2}$$

so we know that $f_w(z)\in H_{\alpha,0}$ and is bounded on $H_{\alpha,0}$ from Lemma 1.1 and Lemma 1.2.

Since $|\delta_w(z)|\le1$, so from (2.1), (2.2), Lemma 1.1 and Lemma 1.2, we know that $g_w(z)\in H_{\alpha,0}$ and is bounded on $H_{\alpha,0}$.

Remark 2.1. $f_w(z)$ and $g_w(z)$ are of paramount importance, playing a central role in the study of boundedness and compactness.

Lemma 2.2. Let $n\in\mathbb{N}^+, \varphi,\psi\in S(D), g,h\in H(D)$, then the following inequalities hold:

(i) $\sup_{z\in D}\left|L_{\varphi,g}^n(z)\right| p(\varphi(z),\psi(z)) \circ \sup_{w\in D}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)f_w\right\|_{B_u}+\sup_{w\in D}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)g_w\right\|_{B_u}$;

(ii) $\sup_{z\in D}\left|L_{\psi,h}^n(z)\right| p(\varphi(z),\psi(z)) \circ \sup_{w\in D}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)f_w\right\|_{B_u}+\sup_{w\in D}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)g_w\right\|_{B_u}$;

(iii) $\sup_{z\in D}\left|L_{\varphi,g}^n(z)-L_{\psi,h}^n(z)\right| \circ \sup_{w\in D}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)f_w\right\|_{B_u}+\sup_{w\in D}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)g_w\right\|_{B_u}$.

Proof: For any $z\in D$,

$$\begin{aligned}\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)f_{\varphi(z)}\right\|_{B_u} &= \sup_{z\in D} u(|z|)\left|\left(\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)f_{\varphi(z)}\right)'(z)\right| \\ &\geq u(|z|)\left|f^{(n)}_{\varphi(z)}(\varphi(z))g(z)-f^{(n)}_{\varphi(z)}(\psi(z))h(z)\right| \\ &= \left|L^n_{\varphi,g}(z)-L^n_{\psi,h}(z)\frac{\left(1-|\varphi(z)|^2\right)\left(1-|\psi(z)|^2\right)^{\alpha+n}}{\left(1-\overline{\varphi(z)}\psi(z)\right)^{\alpha+n+1}}\right| \\ &\geq \left|L^n_{\varphi,g}(z)\right|-\left|L^n_{\psi,h}(z)\frac{\left(1-|\varphi(z)|^2\right)\left(1-|\psi(z)|^2\right)^{\alpha+n}}{\left(1-\overline{\varphi(z)}\psi(z)\right)^{\alpha+n+1}}\right|,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)g_{\varphi(z)}\right\|_{B_u} &= \sup_{z\in D} u(|z|)\left|\left(\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)g_{\varphi(z)}\right)'(z)\right| \\ &\geq u(|z|)\left|g^{(n)}_{\varphi(z)}(\varphi(z))g(z)-g^{(n)}_{\varphi(z)}(\psi(z))h(z)\right| \\ &= \left|L^n_{\psi,h}(z)\frac{\left(1-|\varphi(z)|^2\right)\left(1-|\psi(z)|^2\right)^{\alpha+n}}{\left(1-\overline{\varphi(z)}\psi(z)\right)^{\alpha+n+1}}\right| p(\varphi(z),\psi(z)),\end{aligned} \tag{2.4}$$

It can be obtained from the above two formulas that (i) holds.

Similarly, it can be obtained that (ii) holds.

On the other hand, from (2.3) and Lemma 1.3, we have

$$\begin{aligned}\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)f_{\varphi(z)}\right\|_{B_u} &\geq \left|L^n_{\varphi,g}(z)-L^n_{\psi,h}(z)\right|-\left|L^n_{\psi,h}(z)\right|\left|1-\frac{\left(1-|\varphi(z)|^2\right)\left(1-|\psi(z)|^2\right)^{\alpha+n}}{\left(1-\overline{\varphi(z)}\psi(z)\right)^{\alpha+n+1}}\right| \\ &= \left|L^n_{\varphi,g}(z)-L^n_{\psi,h}(z)\right| \\ &\quad -\left|L^n_{\psi,h}(z)\right|\left|f^{(n)}_{\varphi(a)}(\varphi(z))\left(1-|\varphi(z)|^2\right)^{\alpha+n}-f^{(n)}_{\varphi(a)}(\psi(z))\left(1-|\psi(z)|^2\right)^{\alpha+n}\right| \\ &\pm \left|L^n_{\varphi,g}(z)-L^n_{\psi,h}(z)\right|-\left|L^n_{\psi,h}(z)\right|p(\varphi(z),\psi(z)).\end{aligned} \tag{2.5}$$

It can be obtained from (2.5) and (ii) that (iii) holds.

Lemma 2.3. Let $n\in\mathbb{N}^+,\varphi,\psi\in S(D),g,h\in H(D)$, then the following inequalities hold:

(i) $\sup_{w\in D}\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)f_w\right\|_{B_u} \circ \sup_{j\in\mathbb{N}^+} j^\alpha\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)z^j\right\|_{B_u}$;

(ii) $\sup_{w\in D}\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)g_w\right\|_{B_u} \circ \sup_{j\in\mathbb{N}^+} j^\alpha\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)z^j\right\|_{B_u}$.

Proof: For $f^{(n)}_w(z)$: when $w=0$, $f^{(n)}_w(z)=1$, then

$$\begin{aligned}\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)f_w\right\|_{B_u} &= \sup_{z\in D} u(|z|)|g(z)-h(z)| \\ &\leq n^\alpha \sup_{z\in D} n!u(|z|)|g(z)-h(z)| = n^\alpha\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)z^n\right\|_{B_u} \\ &\leq \sup_{j\in\mathbb{N}^+} j^\alpha\left\|\left(I^n_{\varphi,g}-I^n_{\psi,h}\right)z^j\right\|_{B_u}.\end{aligned}$$

When $w\neq 0$, $f^{(n)}_w(z)=\dfrac{1-|w|^2}{(1-\bar{w}z)^{\alpha+n+1}}=\left(1-|w|^2\right)\sum_{k=0}^{\infty}\dfrac{\Gamma(k+\alpha+n+1)}{k!\Gamma(\alpha+n+1)}\bar{w}^k z^k$.

when $k\to\infty$, by Stirling's formula, we have $\frac{\Gamma(k+\alpha+n+1)}{k!\Gamma(\alpha+n+1)}\approx k^{\alpha+n}$.Therefore

$$\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_{w}\right\|_{B_{u}}\le\left(1-|w|^{2}\right)\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha+n+1)}{k!\Gamma(\alpha+n+1)}|\bar{w}|^{k}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|$$

$$\circ\ \left(1-|w|^{2}\right)\sum_{k=0}^{\infty}|\bar{w}|^{k}k^{\alpha}k^{n}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|$$

$$\circ\ \sup_{j\ge n}(j-n)^{\alpha}(j-n)^{n}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{j-n}g(z)-(\psi(z))^{j-n}h(z)\right|$$

$$\le\sup_{j\in\mathbb{N}^{+}}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_{u}}.$$

It is known from the above two cases that (i) holds.

For $g_{w}^{(n)}(z)$: when $w=0$, $g_{w}^{(n)}(z)=-z$, then

$$\begin{aligned}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)g_{w}\right\|_{B_{u}}&=\sup_{z\in D}u(|z|)\left|\varphi(z)g(z)-\psi(z)h(z)\right|\\&\le(n+1)^{\alpha}\sup_{z\in D}(n+1)!u(|z|)\left|\varphi(z)g(z)-\psi(z)h(z)\right|\\&=(n+1)^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{n+1}\right\|_{B_{u}}\le\sup_{j\in\mathbb{N}^{+}}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_{u}}.\end{aligned}$$

When $w\ne0$,

$$\begin{aligned}g_{w}^{(n)}(z)&=\frac{1-|w|^{2}}{(1-\bar{w}z)^{\alpha+n+1}}\cdot\frac{w-z}{1-\bar{w}z}=\frac{1-|w|^{2}}{(1-\bar{w}z)^{\alpha+n+1}}\cdot\frac{w(1-\bar{w}z)+z\left(|w|^{2}-1\right)}{1-\bar{w}z}\\&=wf_{w}^{(n)}(z)+\frac{1-|w|^{2}}{(1-\bar{w}z)^{\alpha+n+1}}\cdot\frac{z\left(|w|^{2}-1\right)}{1-\bar{w}z},\end{aligned}$$

then

$$\begin{aligned}g_{w}^{(n)}(z)&=wf_{w}^{(n)}(z)-\left(1-|w|^{2}\right)^{2}\left(\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha+n+1)}{k!\Gamma(\alpha+n+1)}\bar{w}^{k}z^{k}\right)\left(\sum_{k=0}^{\infty}\bar{w}^{k}z^{k+1}\right)\\&=wf_{w}^{(n)}(z)-\left(1-|w|^{2}\right)^{2}\sum_{k=1}^{\infty}\left(\sum_{L=0}^{k-1}\frac{\Gamma(k+\alpha+n+1)}{L!\Gamma(\alpha+n+1)}\right)\bar{w}^{k-1}z^{k}.\end{aligned}$$

When $k\to\infty$, by Stirling's formula and the Stoltz theorem, we have

$$\sum_{L=0}^{k-1}\frac{\Gamma(L+\alpha+n+1)}{L!\Gamma(\alpha+n+1)}\approx\sum_{L=0}^{k-1}L^{\alpha+n}\approx k^{\alpha+n+1},$$

therefore

$$\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)g_{w}\right\|_{B_{u}}$$

$$\circ\ \left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_{w}\right\|_{B_{u}}+\left(1-|w|^{2}\right)^{2}\sum_{k=1}^{\infty}\left(\sum_{L=0}^{k-1}\frac{\Gamma(L+\alpha+n+1)}{L!\Gamma(\alpha+n+1)}\right)|\bar{w}|^{k-1}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|$$

$$\circ\ \left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_{w}\right\|_{B_{u}}+\left(1-|w|^{2}\right)^{2}\sum_{k=1}^{\infty}k|\bar{w}|^{k-1}\sup_{z\in D}k^{\alpha}k^{n}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|$$

$$\circ\ \left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_{w}\right\|_{B_{u}}+\sup_{j\ge n}(j-n)^{\alpha}(j-n)^{n}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{j-n}g(z)-(\psi(z))^{j-n}h(z)\right|$$

$$\circ\ \left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_{w}\right\|_{B_{u}}+\sup_{j\in N^{+}}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_{u}}.$$

It is known from the above two cases that (ii) holds.

The following theorem is our main result, which combines Lemma 2.2 and Lemma 2.3.

Theorem 2.1. Let $n \in \mathbb{N}^+, \varphi, \psi \in S(D), g, h \in H(D)$, then the following conditions are equivalent:

(i) $I_{\varphi,g}^n - I_{\psi,h}^n : H_\alpha \to B_u$ is bounded;

(ii) $I_{\varphi,g}^n - I_{\psi,h}^n : H_{\alpha,0} \to B_u$ is bounded;

(iii) $\sup_{j\in\mathbb{N}^+} j^\alpha \left\| \left(I_{\varphi,g}^n - I_{\psi,h}^n \right) z^j \right\|_{B_u} < \infty$;

(iv) $\sup_{w\in D} \left\| \left(I_{\varphi,g}^n - I_{\psi,h}^n \right) f_w \right\|_{B_u} + \sup_{w\in D} \left\| \left(I_{\varphi,g}^n - I_{\psi,h}^n \right) g_w \right\|_{B_u} < \infty$;

(v) $\sup_{z\in D} \left| L_{\psi,h}^n(z) \right| p(\varphi(z), \psi(z)) + \sup_{z\in D} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| < \infty$;

(vi) $\sup_{z\in D} \left| L_{\varphi,g}^n(z) \right| p(\varphi(z), \psi(z)) + \sup_{z\in D} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| < \infty$.

Proof: It is obvious that(i) ⇒ (ii) .

(ii) ⇒ (iii). Suppose (ii) holds. Obviously $z^j \in H_{\alpha,0}$, where $j \in \mathbb{N}^+$, and when $j \to \infty$,

$$\left\| z^j \right\|_{H_\alpha} = \sup_{z\in D} \left(1-|z|^2\right)^\alpha \left| z^j \right| \approx (j+1)^{-\alpha}, \tag{2.6}$$

so

$$\infty > \left\| \left(I_{\varphi,g}^n - I_{\psi,\mathrm{h}}^n \right) \frac{z^j}{\left\| z^j \right\|_{H_\alpha}} \right\|_{B_u} \approx (j+1)^\alpha \left\| \left(I_{\varphi,g}^n - I_{\psi,\mathrm{h}}^n \right) z^j \right\|_{B_u} > j^\alpha \left\| \left(I_{\varphi,g}^n - I_{\psi,\mathrm{h}}^n \right) z^j \right\|_{B_u}.$$

Therefore $\sup_{j\in\mathbb{N}^+} j^\alpha \left\| \left(I_{\varphi,g}^n - I_{\psi,h}^n \right) z^j \right\|_{B_u} < \infty$, then (c) holds.

It can be known from Lemma 2.1 and Lemma 2.2 that (iii) ⇒(iv)⇒(v).

(v) ⇒ (vi). Suppose (v) holds, then

$$\begin{aligned}
\left| L_{\varphi,g}^n(z) \right| p(\varphi(z), \psi(z)) &= \sup_{z\in D} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) + L_{\psi,h}^n(z) \right| p(\varphi(z), \psi(z)) \\
&\le \sup_{z\in D} \left| L_{\psi,h}^n(z) \right| p(\varphi(z), \psi(z)) + \sup_{z\in D} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| p(\varphi(z), \psi(z)) \\
&\le \sup_{z\in D} \left| L_{\psi,h}^n(z) \right| p(\varphi(z), \psi(z)) + \sup_{z\in D} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| < \infty,
\end{aligned}$$

so (vi) holds.

(vi) ⇒ (i). Suppose (vi) holds. Then by Lemma 1.1 and Lemma 1.3, for any $f \in H_\alpha$,

$$\begin{aligned}
\left\| \left(I_{\varphi,g}^n - I_{\psi,\mathrm{h}}^n \right) f \right\|_{B_u} &= \sup_{z\in D} u(|z|) \left| \left(\left(I_{\varphi,g}^n - I_{\psi,\mathrm{h}}^n \right) f \right)'(z) \right| = \sup_{z\in D} u(|z|) \left| f^{(n)}(\varphi(z)) g(z) - f^{(n)}(\psi(z)) h(z) \right| \\
&= \sup_{z\in D} \left| L_{\varphi,g}^n(z) f^{(n)}(\varphi(z)) \left(1-|\varphi(z)|^2\right)^{\alpha+n} - L_{\psi,h}^n(z) f^{(n)}(\psi(z)) \left(1-|\psi(z)|^2\right)^{\alpha+n} \right| \\
&\le \sup_{z\in D} \left| L_{\varphi,g}^n(z) \right| \left\| f^{(n)}(\varphi(z)) \left(1-|\varphi(z)|^2\right)^{\alpha+n} - f^{(n)}(\psi(z)) \left(1-|\psi(z)|^2\right)^{\alpha+n} \right\| \\
&\quad + \sup_{z\in D} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| \left| f^{(n)}(\psi(z)) \left(1-|\psi(z)|^2\right) \right| \\
&\le c\|f\|_{H_\alpha} \sup_{z\in D} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| + c\|f\|_{H_\alpha} \sup_{z\in D} \left| L_{\varphi,g}^n(z) \right| p(\varphi(z), \psi(z)),
\end{aligned}$$

therefore $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_\alpha \to B_u$ is bounded, that is, (i) holds.

III. THE COMPACTNESS OF $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_\alpha \to B_u$

In this part, using the essential norm as a bridge, we obtain several equivalent conditions for $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_\alpha \to B_u$ to be compact.

Using a method similar to Proposition 3.11 in [14], the following lemmas can be obtained.

Lemma 3.1. Let $n\in\mathbb{N}^{+},\varphi,\psi\in S(D),g,h\in H(D)$, then $I_{\varphi,g}^{n}-I_{\psi,h}^{n}$: $H_{\alpha}\to B_{u}$ is compact if and only if $I_{\varphi,g}^{n}-I_{\psi,h}^{n}$: $H_{\alpha}\to B_{u}$ is bounded and for any sequence $\{f_k\}_{k\in\mathbb{N}^{+}}$ which is bounded on H_{α} and converges uniformly to zero on compact subsets of D, we have $\lim\limits_{k\to\infty}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_k\right\|_{B_u}=0$.

The following two lemmas are two lemmas parallel to Lemma 2.2 and Lemma 2.3.

Lemma 3.2. Let $n\in\mathbb{N}^{+},\varphi,\psi\in S(D),g,h\in H(D)$, then the following inequalities hold:

(i) $\lim\limits_{r\to1}\sup\limits_{|\varphi(z)|>r}\left|L_{\varphi,g}^{n}(z)\right|p(\varphi(z),\psi(z))\lesssim\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}+\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)g_w\right\|_{B_u}$;

(ii) $\lim\limits_{r\to1}\sup\limits_{|\psi(z)|>r}\left|L_{\psi,h}^{n}(z)\right|p(\varphi(z),\psi(z))\lesssim\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}+\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)g_w\right\|_{B_u}$;

(iii) $\lim\limits_{r\to1}\sup\limits_{|\varphi(z)|>r,|\psi(z)|>r}\left|L_{\varphi,g}^{n}(z)-L_{\psi,h}^{n}(z)\right|\lesssim\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}+\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)g_w\right\|_{B_u}$.

Proof: By the proof process of Lemma 2.2, we know that (i), (ii), (iii) hold.

Lemma 3.3. let $n\in\mathbb{N}^{+},\varphi,\psi\in S(D),g,h\in H(D)$, suppose that $I_{\varphi,g}^{n}-I_{\psi,h}^{n}$: $H_{\alpha}\to B_{u}$ is bounded, then the following inequalities hold:

(i) $\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}\lesssim\limsup\limits_{j\to\infty}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_u}$;

(ii) $\limsup\limits_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)g_w\right\|_{B_u}\lesssim\limsup\limits_{j\to\infty}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_u}$.

Proof: It is known from the proof of Lemma 2.3 that when $w\neq0$ and $m\in\mathbb{N}^{+}$,

$$
\begin{aligned}
&\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}\\
&\lesssim\left(1-|w|^2\right)\sum_{k=0}^{m}\frac{\Gamma(k+\alpha+n+1)}{k!\Gamma(\alpha+n+1)}|\bar{w}|^{k}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|\\
&+\left(1-|w|^2\right)\sum_{k=m+1}^{\infty}|\bar{w}|^{k}\sup_{j\geq m+n+1}(j-n)^{\alpha}(j-n)^{n}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{j-n}g(z)-(\psi(z))^{j-n}h(z)\right|\\
&\lesssim\left(1-|w|^2\right)\sum_{k=0}^{m}\frac{\Gamma(k+\alpha+n+1)}{k!\Gamma(\alpha+n+1)}|\bar{w}|^{k}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|+\sup_{j\geq m+n+1}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_u}
\end{aligned}
$$

and because $\limsup\limits_{|w|\to1}\left(1-|w|^2\right)\sum\limits_{k=0}^{m}\frac{\Gamma(k+\alpha+n+1)}{k!\Gamma(\alpha+n+1)}|\bar{w}|^{k}\sup\limits_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|=0$, so

$$
\limsup_{|w|\to1}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}\lesssim\sup_{j\geq m+n+1}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_u}. \tag{3.1}
$$

Let $j\to\infty$ to get (i) holds.

It is known from the proof of Lemma 2.3 that when $w\neq0$ and $m\in\mathbb{N}^{+}$,

$$
\begin{aligned}
&\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)g_w\right\|_{B_u}\\
&\lesssim\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}+\left(1-|w|^2\right)^2\sum_{k=1}^{\infty}\left(\sum_{L=0}^{k-1}\frac{\Gamma(L+\alpha+n+1)}{L!\Gamma(\alpha+n+1)}\right)|\bar{w}|^{k-1}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|\\
&+\left(1-|w|^2\right)^2\sum_{k=m+1}^{\infty}k|\bar{w}|^{k-1}\sup_{j\geq m+n+1}(j-n)^{\alpha+n}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{j-n}g(z)-(\psi(z))^{j-n}h(z)\right|\\
&\lesssim\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)f_w\right\|_{B_u}+\left(1-|w|^2\right)^2\sum_{k=1}^{\infty}\left(\sum_{L=0}^{k-1}\frac{\Gamma(L+\alpha+n+1)}{L!\Gamma(\alpha+n+1)}\right)|\bar{w}|^{k-1}\sup_{z\in D}u(|z|)\left|(\varphi(z))^{k}g(z)-(\psi(z))^{k}h(z)\right|\\
&+\sup_{j\in\mathbb{N}^{+}}j^{\alpha}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)z^{j}\right\|_{B_u}.
\end{aligned}
$$

Moreover, since

$$\limsup_{|w|\to 1}\left(1-|w|^2\right)^2\sum_{k=1}^{\infty}\left(\sum_{L=0}^{k-1}\frac{\Gamma(L+\alpha+n+1)}{L!\,\Gamma(\alpha+n+1)}\right)|\overline{w}|^{k-1}\sup_{z\in D}u(|z|)\left|(\varphi(z))^k g(z)-(\psi(z))^k h(z)\right|=0,$$

therefore, from (3.1), we can get

$$\limsup_{|w|\to 1}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)g_w\right\|_{B_u}\circ\sup_{j\geq m+n+1}j^{\alpha}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)z^j\right\|_{B_u},$$

let $j\to\infty$ to get (ii) holds.

The following is our main theorem in this section, the proof utilizes the three lemmas mentioned above.

Theorem 3.1. Let $n\in\mathbb{N}^{+},\varphi,\psi\in S(D),g,h\in H(D)$, suppose that $I_{\varphi,g}^n-I_{\psi,h}^n$: $H_\alpha\to B_u$ is bounded, then the following conditions are equivalent:

(i) $I_{\varphi,g}^n-I_{\psi,h}^n$: $H_\alpha\to B_u$ is compact;

(ii) $I_{\varphi,g}^n-I_{\psi,h}^n$: $H_{\alpha,0}\to B_u$ is compact;

(iii) $\left\|I_{\varphi,g}^n-I_{\psi,h}^n\right\|_{e,H_{\alpha,0}\to B_u}=0$;

(iv) $\limsup_{j\to\infty}j^{\alpha}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)z^j\right\|_{B_u}=0$;

(v) $\lim_{|w|\to 1}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)f_w\right\|_{B_u}=\lim_{|w|\to 1}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)g_w\right\|_{B_u}=0$;

(vi)

$$\begin{aligned}\lim_{r\to 1}\sup_{|\varphi(z)|>r}\left|L_{\varphi,g}^n(z)\right|p(\varphi(z),\psi(z))&=\lim_{r\to 1}\sup_{|\psi(z)|>r}\left|L_{\psi,h}^n(z)\right|p(\varphi(z),\psi(z))\\&=\lim_{r\to 1}\sup_{|\varphi(z)|>r,|\psi(z)|>r}\left|L_{\varphi,g}^n(z)-L_{\psi,h}^n(z)\right|=0;\end{aligned}$$

Proof: (i) ⇒ (ii) ⇒ (iii) is obviously established.

(iii) ⇒ (iv). Suppose (iii) holds, then $I_{\varphi,g}^n-I_{\psi,h}^n$: $H_{\alpha,0}\to B_u$ be compact. Let $\mathrm{T}:H_{\alpha,0}\to B_u$ is a compact operators and $h_j(z)=\dfrac{z^j}{\left\|z^j\right\|_{H_\alpha}}$, where $j\in\mathbb{N}^{+}$, then $h_j(z)$ is bounded on $H_{\alpha,0}$ and converges uniformly to zero on compact subsets of D, so $\lim_{j\to\infty}\left\|\mathrm{T}h_j\right\|_{B_u}=0$. Then from (2.6) we have

$$\begin{aligned}\left\|I_{\varphi,g}^n-I_{\psi,h}^n\right\|_{e,H_{\alpha,0}\to B_u}&=\inf_{\mathrm{T}}\left\|I_{\varphi,g}^n-I_{\psi,h}^n-\mathrm{T}\right\|\geq\inf_{\mathrm{T}}\limsup_{j\to\infty}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n-\mathrm{T}\right)h_j\right\|_{B_u}\\&\geq\inf_{\mathrm{T}}\limsup_{j\to\infty}\left(\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)h_j\right\|_{B_u}-\left\|\mathrm{T}h_j\right\|_{B_u}\right)\\&=\limsup_{j\to\infty}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)h_j\right\|_{B_u}\pm\limsup_{j\to\infty}(j+1)^{\alpha}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)z^j\right\|_{B_u}\\&\geq\limsup_{j\to\infty}j^{\alpha}\left\|\left(I_{\varphi,g}^n-I_{\psi,h}^n\right)z^j\right\|_{B_u},\end{aligned}$$

therefore (iv) is established.

From Lemma 3.3 and Lemma 3.3 we know that (iv) ⇒ (v) ⇒ (vi).

(vi) ⇒ (i). Suppose (vi) is established. By the definitions of limit and supremum for any $\varepsilon>0$, there exists $\delta\in[0,1)$ such that

$$\left|L_{\varphi,g}^n(z)\right|p(\varphi(z),\psi(z))<\varepsilon,\text{ where }|\varphi(z)|>\delta;$$

$$\left|L_{\psi,h}^n(z)\right|p(\varphi(z),\psi(z))<\varepsilon,\text{ where }|\psi(z)|>\delta;$$

$$\left|L_{\varphi,g}^n(z)-L_{\psi,h}^n(z)\right|<\varepsilon,\text{ where }|\varphi(z)|>\delta,|\psi(z)|>\delta.$$

Let $\{M_k\}_{k\in\mathbb{N}^{+}}$ be a bounded sequence on H_α and converge uniformly to 0 on compact subsets of D. Without loss of generality, assume $\left\|M_k\right\|_{H_\alpha}\leq 1$, then

$$\begin{aligned}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)M_{k}\right\|_{B_{u}} &=\sup_{z\in D}u(|z|)\left|\left(\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)M_{k}\right)'(z)\right|=\sup_{z\in D}u(|z|)\left|M_{k}^{(n)}(\varphi(z))g(z)-M_{k}^{(n)}(\psi(z))h(z)\right|\\ &=\sup_{z\in D}\left|L_{\varphi,g}^{n}(z)M_{k}^{(n)}(\varphi(z))\left(1-|\varphi(z)|^{2}\right)^{\alpha+n}-L_{\psi,h}^{n}(z)M_{k}^{(n)}(\psi(z))\left(1-|\psi(z)|^{2}\right)^{\alpha+n}\right| \qquad (3.2)\\ &\le\sup_{z\in D}E_{K}(z)+\sup_{z\in D}F_{k}(z),\end{aligned}$$

where

$$E_{K}(z)=\left|L_{\varphi,g}^{n}(z)-L_{\psi,h}^{n}(z)\right|\left|M_{k}^{(n)}(\varphi(z))\left(1-|\varphi(z)|^{2}\right)^{\alpha+n}\right|,$$

$$F_{k}(z)=\left|L_{\psi,h}^{n}(z)\right|\left|M_{k}^{(n)}(\varphi(z))\left(1-|\varphi(z)|^{2}\right)^{\alpha+n}-M_{k}^{(n)}(\psi(z))\left(1-|\psi(z)|^{2}\right)^{\alpha+n}\right|.$$

When $\varphi(z)>\delta,\psi(z)>\delta$, by Lemma 1.1 and Lemma 1.3, we have

$$E_{K}(z)\le c\|M_{k}\|_{H_{\alpha}}\left|L_{\varphi,g}^{n}(z)-L_{\psi,h}^{n}(z)\right|\le c\varepsilon,$$

$$F_{k}(z)\le c\|M_{k}\|_{H_{\alpha}}\left|L_{\psi,h}^{n}(z)\right|p(\varphi(z),\psi(z))<c\varepsilon.$$

When $\varphi(z)\le\delta,\psi(z)>\delta$, by Lemma 1.3, we have

$$E_{K}(z)\le cM_{k}^{(n)}(\varphi(z)),$$

$$F_{k}(z)\le c\|M_{k}\|_{H_{\alpha}}\left|L_{\psi,h}^{n}(z)\right|p(\varphi(z),\psi(z))<c\varepsilon.$$

When $\varphi(z)\le\delta,\psi(z)\le\delta$, by Lemma 1.4, we have

$$E_{K}(z)\le cM_{k}^{(n)}(\varphi(z)),$$

$$\begin{aligned}F_{k}(z)&\le c\left|L_{\psi,h}^{n}(z)\right|p(\varphi(z),\psi(z))\max\left\{\sup_{|z|\le\delta}\left|M_{k}^{(n)}(z)\right|,\sup_{|z|\le\delta}\left|M_{k}^{(n+1)}(z)\right|\right\}\\ &\le c\max\left\{\sup_{|z|\le\delta}\left|M_{k}^{(n)}(z)\right|,\sup_{|z|\le\delta}\left|M_{k}^{(n+1)}(z)\right|\right\}.\end{aligned}$$

When $\varphi(z)>\delta,\psi(z)\le\delta$, by Lemma 1.1 , Lemma 1.3 and (3.2), we have

$$\begin{aligned}&\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)M_{k}\right\|_{B_{u}}\\ &\le\sup_{z\in D}\left|L_{\varphi,g}^{n}(z)\right|\left|M_{k}^{(n)}(\varphi(z))\left(1-|\varphi(z)|^{2}\right)^{\alpha+n}-M_{k}^{(n)}(\psi(z))\left(1-|\psi(z)|^{2}\right)^{\alpha+n}\right|\\ &\quad+\sup_{z\in D}\left|L_{\varphi,g}^{n}(z)-L_{\psi,h}^{n}(z)\right|\left|M_{k}^{(n)}(\psi(z))\left(1-|\psi(z)|^{2}\right)^{\alpha+n}\right|\\ &\le\sup_{z\in D}c\|M_{k}\|_{H_{\alpha}}\left|L_{\varphi,g}^{n}(z)\right|p(\varphi(z),\psi(z))+\sup_{z\in D}\left|L_{\varphi,g}^{n}(z)-L_{\psi,h}^{n}(z)\right|\left|M_{k}^{(n)}(\psi(z))\right|\\ &\le c\varepsilon+c\left|M_{k}^{(n)}(\psi(z))\right|.\end{aligned}$$

So

$$\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)M_{k}\right\|_{B_{u}}\le c\varepsilon+\sup_{|\varphi(z)|\le\delta}cM_{k}^{(n)}(\varphi(z))+\sup_{|\psi(z)|\le\delta}cM_{k}^{(n)}(\psi(z))+\max\left\{\sup_{|z|\le\delta}\left|M_{k}^{(n)}(z)\right|,\sup_{|z|\le\delta}\left|M_{k}^{(n+1)}(z)\right|\right\}.$$

and because M_k converges uniformly to 0 on the compact subsets of D , so

$$\lim_{k\to\infty}\left\|\left(I_{\varphi,g}^{n}-I_{\psi,h}^{n}\right)M_{k}\right\|_{B_{u}}=0,$$

by Lemma 3.1, (i) is established.

IV. THE BOUNDEDNESS AND COMPACTNESS OF $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_{\alpha,0} \to B_{u,0}$

Compactness is a stronger property than boundedness. Using the following two lemmas, we can know that the compactness and boundedness of $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_{\alpha,0} \to B_{u,0}$ are equivalent.

Applying a method similar to Lemma 1 in [15], we can obtain the following lemma.

Lemma 4.1: L is a tight subset on $B_{u,0}$ if and only if L is a bounded closed subset on $B_{u,0}$ and

$$\lim_{|z|\to 1}\sup_{f\in L} u(|z|)|f'(z)| = 0.$$

Lemma 4.2: Let $n\in \mathbb{N}^+$, φ, $\psi \in S(D)$, g, $h\in H(D)$, $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_{\alpha,0} \to B_{u,0}$ is bounded, then

$$\lim_{|z|\to 1}\left|L_{\varphi,g}^n(z)\right| p(\varphi(z),\psi(z)) + \lim_{|z|\to 1}\left|L_{\psi,h}^n(z)\right| p(\varphi(z),\psi(z)) + \lim_{|z|\to 1}\left|L_{\varphi,g}^n(z) - L_{\psi,h}^n(z)\right| = 0.$$

Proof: From (2.3) and (2.4), we have

$$\begin{aligned} 0 &= \lim_{|z|\to 1} u(|z|)\left|\left(\left(I_{\varphi,g}^n - I_{\psi,h}^n\right) f_{\varphi(z)}\right)'(z)\right| \\ &= \lim_{|z|\to 1} u(|z|)\left| f_{\varphi(z)}^{(n)}(\varphi(z))g(z) - f_{\varphi(z)}^{(n)}(\psi(z))h(z)\right| \\ &\geq \lim_{|z|\to 1}\left|L_{\varphi,g}^n(z)\right| - \lim_{|z|\to 1}\left|L_{\psi,h}^n(z)\frac{\left(1-|\varphi(z)|^2\right)\left(1-|\psi(z)|^2\right)^{\alpha+n}}{\left(1-\overline{\varphi(z)}\psi(z)\right)^{\alpha+n+1}}\right|, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} 0 &= \lim_{|z|\to 1} u(|z|)\left|\left(\left(I_{\varphi,g}^n - I_{\psi,h}^n\right) g_{\varphi(z)}\right)'(z)\right| \\ &= \lim_{|z|\to 1} u(|z|)\left| g_{\varphi(z)}^{(n)}(\varphi(z))g(z) - g_{\varphi(z)}^{(n)}(\psi(z))h(z)\right| \\ &= \lim_{|z|\to 1}\left|L_{\psi,h}^n(z)\frac{\left(1-|\varphi(z)|^2\right)\left(1-|\psi(z)|^2\right)^{\alpha+n}}{\left(1-\overline{\varphi(z)}\psi(z)\right)^{\alpha+n+1}}\right| p(\varphi(z),\psi(z)). \end{aligned} \tag{4.2}$$

therefore, multiplying both sides of (4.1) by $p(\varphi(z),\psi(z))$ and combining with (4.2), we can get

$$\lim_{|z|\to 1}\left|L_{\varphi,g}^n(z)\right| p(\varphi(z),\psi(z)) = 0.$$

Similarly, we can also get

$$\lim_{|z|\to 1}\left|L_{\psi,h}^n(z)\right| p(\varphi(z),\psi(z)) = 0. \tag{4.3}$$

From (2.5), we can get

$$\begin{aligned} 0 &= \lim_{|z|\to 1} u(|z|)\left|\left(\left(I_{\varphi,g}^n - I_{\psi,h}^n\right) f_{\varphi(z)}\right)'(z)\right| \\ &= \lim_{|z|\to 1} u(|z|)\left| f_{\varphi(z)}^{(n)}(\varphi(z))g(z) - f_{\varphi(z)}^{(n)}(\psi(z))h(z)\right| \\ &\pm \lim_{|z|\to 1}\left|L_{\varphi,g}^n(z) - L_{\psi,h}^n(z)\right| - \lim_{|z|\to 1}\left|L_{\psi,h}^n(z)\right| p(\varphi(z),\psi(z)), \end{aligned} \tag{4.4}$$

so, from (4.3) and(4.4), we can get

$$\lim_{|z|\to 1}\left|L_{\varphi,g}^n(z) - L_{\psi,h}^n(z)\right| = 0.$$

In summary, Lemma 4.2 holds.

Theorem 4.1: Let $n \in \mathbb{N}^+$, $\varphi,\ \psi \in S(D)$, g, $h \in H(D)$, then $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_{\alpha,0} \to B_{u,0}$ is bounded if and only if $I_{\varphi,g}^n - I_{\psi,h}^n : H_{\alpha,0} \to B_{u,0}$ is compact.

Proof: The sufficiency is obvious established. Therefore, only the necessity needs to be proved. by the closed - graph theorem and Lemma 4.1, we know that $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_{\alpha,0} \to B_{u,0}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{H_\alpha} \le 1} u(|z|) \left\| \left(\left(I_{\varphi,g}^n - I_{\psi,h}^n \right) f \right)'(z) \right\| = 0,$$

where $f \in H_{\alpha,0}$.By Lemma 4.2, we have

$$\begin{aligned}
&\lim_{|z|\to 1} \sup_{\|f\|_{H_\alpha} \le 1} u(|z|) \left\| \left(\left(I_{\varphi,g}^n - I_{\psi,h}^n \right) f \right)'(z) \right\| = \lim_{|z|\to 1} \sup_{\|f\|_{H_\alpha} \le 1} u(|z|) \left| f^{(n)}(\varphi(z)) g(z) - f^{(n)}(\psi(z)) h(z) \right| \\
&= \lim_{|z|\to 1} \sup_{\|f\|_{H_\alpha} \le 1} \left| L_{\varphi,g}^n(z) f^{(n)}(\varphi(z)) \left(1 - |\varphi(z)|^2\right)^{\alpha+n} - L_{\psi,h}^n(z) f^{(n)}(\psi(z)) \left(1 - |\psi(z)|^2\right)^{\alpha+n} \right| \\
&\le \lim_{|z|\to 1} \sup_{\|f\|_{H_\alpha} \le 1} \left| L_{\psi,h}^n(z) \right| \left| f^{(n)}(\varphi(z)) \left(1 - |\varphi(z)|^2\right)^{\alpha+n} - f^{(n)}(\psi(z)) \left(1 - |\psi(z)|^2\right)^{\alpha+n} \right| \\
&+ \lim_{|z|\to 1} \sup_{\|f\|_{H_\alpha} \le 1} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| \left| f^{(n)}(\varphi(z)) \left(1 - |\varphi(z)|^2\right)^{\alpha+n} \right| \\
&\le c \lim_{|z|\to 1} \left| L_{\varphi,g}^n(z) - L_{\psi,h}^n(z) \right| + c \lim_{|z|\to 1} \left| L_{\psi,h}^n(z) \right| p(\varphi(z), \psi(z)) \\
&= 0,
\end{aligned}$$

therefore, $I_{\varphi,g}^n - I_{\psi,h}^n$: $H_{\alpha,0} \to B_{u,0}$ is compact.

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