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Review Paper

Compact Linear Combination of Composition Operators on Bergman Spaces

Fatima Yousif

Abstract. *Motivated by the question of Shapiro and Sundberg raised in 1990, study on linear combinations of composition operators has been a topic of growing interest. In this paper, we completely characterize the compactness of any finite linear combination of composition operators with general symbols on the weighted Bergman spaces in two classical terms: one is a function theoretic characterization of Julia-Caratheodory type and the other is a measure theoretic characterization of Carleson type. Our approach is completely different from what have been known so far.*

Keywords: *Composition operator Compact operator Linear combination Weighted Bergman space*

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I. Introduction

Let $\mathcal{S} = \mathcal{S}(\mathbf{D})$be the class of all holomorphic self-maps of the unit disk $\mathbf{D}$of thecomplex plane $\mathbf{C}$. Each $\varphi_i \in \mathcal{S}$induces a composition operator $(1+\epsilon)_{\sum \varphi_i} : H(\mathbf{D}) \to H(\mathbf{D})$defined by

$$(1+\epsilon)_{\sum \varphi_i} f := f \circ \sum \varphi_i,$$

where$H(\mathbf{D})$is the class of all holomorphic functions on $\mathbf{D}$. An extensive study on the theory of composition operators has been established during the past four decades on various settings. We refer to standard references [7] and [25] for various aspects on the theory of composition operators acting on holomorphic function spaces.

We first recall our function spaces to work on. Let dAbe the area measure on $\mathbf{D}$normalized to have total mass 1. For $\epsilon > 0$, put

$$dA_{\epsilon-1}(z) := \big((\epsilon-1)+1\big)(1+|z|)^{\epsilon-1} dA(z), \;\; z \in \mathbf{D};$$

the constant $(\epsilon-1)+1$is chosen so that $A_{\epsilon-1}(\mathbf{D}) = 1$. Now, for $0 < \epsilon < \infty$, the $\epsilon - 1$-weighted Bergman space $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$is the space of all $f \in H(\mathbf{D})$such that the "norm"

$$\|f\|_{A^{1+\epsilon}_{\epsilon-1}} := \left\{ \int_{\mathbf{D}} |f(z)|^{1+\epsilon} dA_{\epsilon-1}(z) \right\}^{1/1+\epsilon}$$

is finite. As is well-known, the space $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$equipped with the norm above is a Banach space for $0 \le \epsilon < \infty$and a complete metric space for $0 < \epsilon < 1$with respect to the translation-invariant metric $(f,g) \mapsto \|f-g\|^{1+\epsilon}_{A^{1+\epsilon}_{\epsilon-1}}$.

As is well known in the setting of $\mathbf{D}$, every composition operator is bounded on wellknown function spaces such as the weighted Bergman spaces $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$and the Hardy spaces $H^{1+\epsilon}(\mathbf{D})$due to the Littlewood Subordination Principle; see [7] or [25] for precise definition of the Hardy spaces $H^{1+\epsilon}(\mathbf{D})$. So, boundedness on those spaces is out of question and much efforts have been expended in the early stage on characterizing those maps in $\mathcal{S}$which induce compact composition operators. An early result of Shapiro and Taylor [27] in 1973 showed that the Julia-Caratheodory type condition

$$R_{\varphi_i}(z) := \frac{1-|z|^2}{1-|\varphi_i(z)|^2} \to 0 \;\; \text{as} \;\; |z| \to 1 \qquad (1.1)$$

is necessary for $\varphi_i \in \mathcal{S}$to induce a compact composition operator on$H^2(\mathbf{D})$(and hence forthe general Hardy spaces $H^{1+\epsilon}(\mathbf{D})$). This means via the Julia-Caratheodory Theorem thatthe non-existence of the angular derivative of the inducing map at any boundary point is anecessary condition for the compactness of a composition operator on the Hardy spaces.However, (1.1) turned out to be not sufficient. In fact, later in 1987 Shapiro [24] completely characterized the compactness of composition operators on the Hardy spaces by findingthe precise formula for the essential norm of a composition operator on $H^2(\mathbf{D})$in terms ofthe Nevanlinna counting function. The situation for the weighted Bergman spaces turnedout to be quite different from the Hardy space case. Namely, MacCluer and Shapiro [18]proved that (1.1) is a necessary and sufficient condition for the compactness of compositionoperators on the weighted Bergman spaces.

At the same time, MacCluer [16] noticed that Carleson measure, which first appearedin the solution of the corona problem, is an important tool in the study of compositionoperators. Even though it is usually not easy to verify that a given measure satisfies theCarleson measure criteria, in most cases it is through this process one verifies that a compositionoperator is bounded (in case of higher dimensional setting) or compact. In fact,the connection between a single composition operator and a Carleson measure comes fromthe standard identity (see [9, p. 163])

$$\int_{\mathbf{D}} \sum_i |f \circ \varphi_i(z)|^{1+\epsilon} dA_{\epsilon-1}(z) = \int_{\mathbf{D}} \sum_i |f(z)|^{1+\epsilon} d(A_{\epsilon-1} \circ \varphi_i^{-1})(z) \quad (1.2)$$

valid for functions $\varphi_i \in \mathcal{S}$and $f \in H(\mathbf{D})$; see Section 2.4 for precise definition of the pullback measure $A_{\epsilon-1} \circ \varphi_i^{-1}$. Due to the change-of-variable formula (1.2) one can easily deduce the Carleson measure criterion for the compactness of a single composition operator on the weighted Bergman spaces. That is, $(1+\epsilon)_{\sum \varphi_i}$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$if and only if $A_{\epsilon-1} \circ \varphi_i^{-1}$is a compact $(\epsilon-1)$-Carleson measure on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$; see Section 2.4 for the notion of $(\epsilon-1)$-Carleson measures. This Carleson measure criterion, which is independent of $1+\epsilon$, plays a fundamental role in the study of composition operators on the weighted Bergman spaces.

With the basic questions such as boundedness and compactness settled, more attention has been paid to the study of the topological structure of the composition operators in the operator norm topology, which is of continuing interest in the theory of composition operators. In 1981 Berkson [1] initiated the study of the topological structure with his isolation result on the Hardy spaces. Berkson's isolation result was refined later by Shapiroand Sundberg [26], and also by MacCluer [17]. In [26] Shapiro and Sundberg posed aquestion on *whether two composition operators belong to the same connected component,when their difference is compact*. While this question was originally for the Hardy spaces,it also initiated similar study on various other settings including the weighted Bergmanspaces. It was answered negatively on both the Hardy spaces (see [2, 8, 20]) and theweighted Bergman spaces (see [19]).

The aforementioned question of Shapiro and Sundberg initiated another direction ofstudy, i.e., the study of compact differences of composition operators on various settings,which has been a very active topic. While the characterization for compact differences stillremains open in the Hardy space case, it is completely settled in the weighted Bergman space case. More explicitly, Moorhouse [19] characterized the compactness of $(1+\epsilon)_{\sum \varphi_i} - (1+\epsilon)_{\psi_i}$on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$by the Julia-Caratheodory type condition

$$M_{\varphi_i,\psi_i}(z) := \left[R_{\varphi_i}(z) + R_{\psi_i}(z)\right]\rho_{\varphi_i,\psi_i}(z) \to 0 \quad \text{as } |z| \to 1, \quad (1.3)$$

where $\rho_{\varphi_i,\psi_i} := \rho(\varphi_i, \psi_i)$. Here, ρdenotes the pseudohyperbolic distance on $\mathbf{D}$; see Section2.2 for the definition of ρ. We remark in passing that this characterization has beenextended not only to higher dimensional balls and polydisks, but also to general parameter$1+\epsilon$; see [3, 4]. The essence of Moorhouse's characterization is that suitable cancellationsshould occur at every boundary point which makes either one of the inducing maps fail to induce a compact composition operator. For further results on compact differences onvarious other settings, we refer to [10, 11, 12, 13, 17, 20, 21, 22, 23] and references therein.

With the lack of the change-of-variable formula for the difference of two compositionoperators, Koo andWang [13] introduced a new notion of joint Carleson measures to obtaina Carleson criterion for differences of two composition operators on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{B})$, the weightedBergman space over the ball $\mathbf{B}$, to be bounded/compact. As a consequence of their result,it turns out that the bounded/compact differences on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{B})$depend on the index $1+\epsilon$, when the dimension is bigger than 1. This is in sharp contrast with the one dimensional case;note that Moorhouse's characterization (1.3) is independent of $1+\epsilon$. Meanwhile, in case eachcomposition operator is bounded on $A_{\beta}^{1+\epsilon}(\mathbf{B})$for some $-1 < \epsilon > 0$, the compact differenceon $A_{\epsilon-1}^{1+\epsilon}(\mathbf{B})$is known to be independent of $1+\epsilon$; see [3] and, for a similar result on the poydisk,[4]. We also remark that the compact difference on $H^{1+\epsilon}(\mathbf{D}), \epsilon \geq 0$, is also known to beindependent of $1+\epsilon$; see [21].

Along the same line of study on differences, study on linear combinations has beena topic of growing interest. Kriete and Moorhouse [15] first obtained some general, butnot complete, results on compact linear combinations on $A_{\epsilon-1}^{2}(\mathbf{D})$. More recently, Koo andWang [14] obtained complete characterizations for compact linear combinations of threecomposition operators on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. The current authors [6] characterized compact doubledifferences on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. In this paper, in the setting of the weighted Bergman spaces$A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$, we completely characterize the compact linear combinations of composition operatorsin two directions which have been discussed so far; one is the Julia-Caratheodorytype characterization and the other is the Carleson criterion by means of joint Carlesonmeasures. Our approach, dealing with general linear combinations, is completely differentfrom what has been known so far.

We introduce notation to be used throughout the paper. Given an arbitrary (but fixed)integer $N \geq 2$, we reserve symbol functions

$$(\varphi_i)_1, (\varphi_i)_2, \dots, (\varphi_i)_N \in \mathcal{S}$$

and coefficients

$$a_1^j, \dots, a_N^j \in \mathbf{C}\backslash\{0\}$$

to be considered throughout the paper. We put

$$T_j \coloneqq (1+\epsilon)_{\sum(\varphi_i)_j} \text{and} \quad T_{j,1+\epsilon} \coloneqq T_j - T_{1+\epsilon}$$

for $j, 1+\epsilon = 1, \dots, N$. We put

$$T \coloneqq \sum_{j=1}^{N} \sum_i a_j^j T_j$$

for short.

Let$\wp_N$be the group of all permutations on the index set

$$\Lambda_N \coloneqq \{1,2,\dots,N\}.$$

Identifying $\sigma^i \in \wp_N$with the ordered N-tuple $(\sigma_1^i, \dots, \sigma_N^i)$where $\sigma_j^i \coloneqq \sigma^i(j)$, we will sometimes write

$$\sigma^i = (\sigma_1^i, \dots, \sigma_N^i).$$

Given $\sigma^i \in \wp_N$, put

$$(1+\epsilon)_j^{\sigma^i} = (1+\epsilon)_j^{\sigma^i}(a_1^j, \dots, a_N^j) \coloneqq \sum_{1+\epsilon=1}^{j} \sum_i a_{\sigma_{1+\epsilon}^i}^j$$

for $j \in \Lambda_N$. Note that we may represent Tas

$$T = \sum_{j=1}^{N-1} \sum_i (1+\epsilon)_j^{\sigma^i} T_{\sigma_j^i, \sigma_{j+1}^i} + \sum_i (1+\epsilon)_N^{\sigma^i} T_{\sigma_N^i} \quad (1.4)$$

for each $\sigma^i \in \wp_N$.

To state our first main result we introduce more notation. We put for simplicity

$$R_j \coloneqq R_{(\varphi_i)_j}, \quad M_{j,1+\epsilon} \coloneqq M_{(\varphi_i)_j,(\varphi_i)_{1+\epsilon}} \text{and} \rho_{j,1+\epsilon} \coloneqq \rho_{(\varphi_i)_j,(\varphi_i)_{1+\epsilon}}$$

for $j, 1+\epsilon \in \Lambda_N$. Note that each R_j, and thus each $M_{j,1+\epsilon}$as well, is bounded on **D**by the Schwarz-Pick Lemma. Now, motivated by the representation (1.4), we define

$$Q_{\sigma^i} \coloneqq \sum_{j=1}^{N-1} \sum_i \left|(1+\epsilon)_j^{\sigma^i}\right| M_{\sigma_j^i, \sigma_{j+1}^i} + \sum_i \left|(1+\epsilon)_N^{\sigma^i}\right| R_{\sigma_N^i} \quad (1.5)$$

for$\sigma^i \in \wp_N$and put

$$Q \coloneqq \sum_i \prod_{\sigma^i \in \wp_N} Q_{\sigma^i}. \quad (1.6)$$

Note that functions $Q_{\sigma^i}$are all bounded on **D**.

Our first result is the Julia-Caratheodory type characterization below. In what follows,$\lambda_z$denotes the function specified in (2.6) in Section 2.3.

Theorem 1.1. *Let $\epsilon > 0$and $0 \leq \epsilon < \infty$. Let $(\varphi_i)_1, (\varphi_i)_2, \dots, (\varphi_i)_N \in \mathcal{S}$and $a_1^j, \dots, a_N^j \in \mathbf{C}\backslash\{0\}$. Then, with the notation above, the following three statements are equivalent:*

(a) *$T$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$;*

(b) $\lim_{|z|\to 1} \frac{\|T\lambda_z^{1+\epsilon}\|_{A_{\epsilon-1}^{1+\epsilon}}}{\|\lambda_z^{1+\epsilon}\|_{A_{\epsilon-1}^{1+\epsilon}}} = 0$*for all* $1+\epsilon > \frac{(\epsilon-1)+2}{1+\epsilon}$;

(c) $\lim_{|z|\to 1} Q(z) = 0$.

As a consequence of this result, we see that compactness of linear combination is independentof parameters $\epsilon - 1$and $1+\epsilon$, as expected. We also note that Moorhouse's characterization(1.3) of compact differences is a direct consequence of the equivalence of (a) and(c).

As an application of Theorem 1.1, we consider the class of linear combinations satisfyingthe coefficient non-cancellation condition

$$(1+\epsilon)_j^{\sigma^i} \neq 0 \quad \text{for} j = 1, \dots, N-1 \quad \text{and} \sigma^i \in \wp_N. \text{(CNC)}$$

For this class of linear combinations, we obtain the next characterization, which has beenknown for the special case $N = 2$; see [4, Theorem 4.6].

Theorem 1.2. *Let $\epsilon > 0$and $0 \leq \epsilon < \infty$. Let $(\varphi_i)_1, (\varphi_i)_2, \dots, (\varphi_i)_N \in \mathcal{S}$and assume that $a_1^j, \dots, a_N^j \in \mathbf{C}\backslash\{0\}$ satisfy (CNC). Also, assume that at least one of the operators$T_1, \dots, T_N$is not compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. Then*

$$T \quad \text{is compact on} \quad A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$$

if and only if both of the following two conditions are fulfilled:

(a) $\sum_{j=1}^{N}\sum_i a_j^j = 0$;

(b) $T_{j,1+\epsilon}$ *is compact for each* $j, 1+\epsilon \in \Lambda_N$.

Our next result is a Carleson measure characterization. To state it we introduce more notation. Given $\sigma^i \in \wp_N$, we put

$$G_{\sigma^i} := \left\{ z \in \mathbf{D} : Q_{\sigma^i}(z) = \min_{\tau \in \wp_N} Q_\tau(z) \right\}. \quad (1.7)$$

Let $\epsilon > 0$ and $0 \le \epsilon < \infty$. Associated with $\sigma^i \in \wp_N$ are the measures $\mu_{\sigma_j^i} = \mu_{\sigma_j^i}^{1+\epsilon,\epsilon-1}$ and $v_{\sigma_j^i} = \mu_{\sigma_j^i}^{1+\epsilon,\epsilon-1}$ defined by

$$\mu_{\sigma_j^i}(E) := \int_{(\varphi_i)_{\sigma_j^i}^{-1}(E)\cap G_{\sigma^i}} M_{\sigma_j^i,\sigma_{j+1}^i}^{1+\epsilon} dA_{\epsilon-1} + \int_{(\varphi_i)_{\sigma_{j+1}^i}^{-1}(E)\cap G_{\sigma^i}} M_{\sigma_j^i,\sigma_{j+1}^i}^{1+\epsilon} dA_{\epsilon-1},$$

$$v_{\sigma_j^i}(E) := \int_{(\varphi_i)_{\sigma_j^i}^{-1}(E)\cap G_{\sigma^i}} \sum_i \rho_{\sigma_j^i,\sigma_{j+1}^i}^{1+\epsilon} dA_{\epsilon-1} + \int_{(\varphi_i)_{\sigma_{j+1}^i}^{-1}(E)\cap G_{\sigma^i}} \sum_i \rho_{\sigma_j^i,\sigma_{j+1}^i}^{1+\epsilon} dA_{\epsilon-1}$$

for $1 \le j < N$ and

$$\mu_{\sigma_{N_j}^i}(E) = v_{\sigma_N^i}(E) := \int_{(\varphi_i)_{\sigma_N^i}^{-1}(E)\cap G_{\sigma^i}} \sum_i R_{\sigma_N^i}^{1+\epsilon} dA_{\epsilon-1}$$

for Borel sets $E \subset \mathbf{D}$. Finally, we put

$$\mu_{\epsilon-1,1+\epsilon} := \sum_{\sigma^i \in \wp_N} \sum_i \mu_{\epsilon-1,1+\epsilon}^{\sigma^i} \text{ where } \mu_{\epsilon-1,1+\epsilon}^{\sigma^i} := \sum_{j=1}^{N} \sum_i \left|(1+\epsilon)_j^{\sigma^i}\right|^{1+\epsilon} \mu_{\sigma_j^i} \quad (1.8)$$

and

$$v_{\epsilon-1,1+\epsilon} := \sum_{\sigma^i \in \wp_N} \sum_i v_{\epsilon-1,1+\epsilon}^{\sigma^i} \text{ where } v_{\epsilon-1,1+\epsilon}^{\sigma^i} := \sum_{j=1}^{N} \sum_i \left|(1+\epsilon)_j^{\sigma^i}\right|^{1+\epsilon} v_{\sigma_j^i}.$$

For the rest of the paper we will freely use the notation introduced so far without any further reference.

Our last result is the Carleson measure characterization. When $N = 2$ and $a_1^j + a_2^j = 0$, it is easily seen that the set G_{σ^i} is simply the whole $\mathbf{D}$ for each $\sigma^i \in \wp_N$. Thus, for differences of composition operators, the equivalence of (a) and (c) below is contained in [22] and [13]; see also [5] for the half-plane analogue.

Theorem 1.3. *Let* $\epsilon > 0$ *and* $0 \le \epsilon < \infty$. *Let* $(\varphi_i)_1, (\varphi_i)_2, \ldots, (\varphi_i)_N \in \mathcal{S}$ *and* $a_1^j, \ldots, a_N^j \in \mathbf{C}\backslash\{0\}$. *Then the following three statements are equivalent:*

(a) T *is compact on* $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$;

(b) $\mu_{\epsilon-1,1+\epsilon}$ *is a compact* $(\epsilon-1)$*-Carleson measure on* $\mathbf{D}$;

(c) $v_{\epsilon-1,1+\epsilon}$ *is a compact* $(\epsilon-1)$*-Carleson measure on* $\mathbf{D}$.

We emphasize that our symbol functions in Theorems 1.1 and 1.3 are completely arbitrary, as long as they belong to $\mathcal{S}$.

The exposition of the paper is organized as follows. In Section 2 we recall some basic facts to be used in later sections. Section 3 is devoted to the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.3. In Section 5 we first prove Theorem 1.2 as an application of Theorem 1.1 and observe some consequences. We then apply our results to provide new simple proofs for some known results about linear combinations of at most four composition operators.

Constants. Throughout the paper we use the letter $1+\epsilon$ to denote various positive constants which may change at each occurrence. Variables indicating the dependency of $1+\epsilon$ will be often specified in a parenthesis. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \le (1+\epsilon)Y$ for some inessential constant $1+\epsilon > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

1. Preliminaries

In this section we collect some basic facts to be used in later sections. One may find details in standard references such as [7] and [28].

2.1. Compact Operator. It seems better to clarify the notion of compact operators, since the spaces under consideration are not Banach spaces when $0 < \epsilon < 1$. Suppose X and Y are topological vector spaces whose topologies are induced by complete metrics. A continuous linear operator $S : X \to Y$ is said to be compact if the image of every bounded sequence in X has a subsequence that converges in Y.

We have the following convenient compactness criterion for a linear combination ofcomposition operators acting on the weighted Bergman spaces.

Lemma 2.1.*Let $\epsilon > 0$and $0 \le \epsilon < \infty$. Let Sbe a linear combination of compositionoperators. Then $S$is compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$if and only if $Sf_n \to 0$in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$for any boundedsequence $\{f_n\}$in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$such that $f_n \to 0$uniformly on compact subsets of* $\mathbf{D}$.

A proof can be found in [7, Proposition 3.11] for a single composition operator and itcan be easily modified for a linear combination.

1.2 Pseudohyperbolic Distance. The well-known *pseudohyperbolic distance* between$z, w \in \mathbf{D}$is given by

$$\rho(z,w) := \left|\frac{z-w}{1-z\overline{w}}\right|.$$

By a straightforward calculation we have

$$1-\rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2} \quad (2.1)$$

for $z, w \in \mathbf{D}$. The pseudohyperbolic disk with center $z \in \mathbf{D}$and radius $0 < \epsilon < 1$is defined by

$$E_{1-\epsilon}(z) := \{w \in \mathbf{D} : \rho(z,w) < 1-\epsilon\}.$$

It turns out that $E_{1-\epsilon}(z)$is a Euclidean disk with

$$(\text{center}) = \frac{(1-(1-\epsilon)^2)}{1-|z|^2(1-\epsilon)^2} z \quad \text{and} \quad (\text{radius}) = \frac{(1-|1-\epsilon|^2)(1-\epsilon)}{1-|z|^2(1-\epsilon)^2}. \quad (2.2)$$

This implies

$$\frac{1-\rho(z.w)}{1+\rho(z.w)} \le \frac{1-|z|}{1-|w|} \le \frac{1+\rho(z.w)}{1-\rho(z.w)} \quad (2.3)$$

for $z, w \in \mathbf{D}$. Also is well known that, given $0 < \epsilon < 1$and $\epsilon > 0$, we have

$$A_{\epsilon-1}[E_{1-\epsilon}(z)] \approx (1-|z|^2)^{(\epsilon-1)+2}, \quad z \in \mathbf{D}; \quad (2.4)$$

constants suppressed in this estimate depend only on $\epsilon - 1$and r.

Given $0 < \epsilon < 1$and $\epsilon > 0$, we will use the submean value type inequality

$$|f(z)|^{1+\epsilon} \le \frac{1+\epsilon}{(1-|z|^2)^{(\epsilon-1)+2}} \int_{E_{1-\epsilon}(z)} |f|^{1+\epsilon} dA_{\epsilon-1}, \quad z \in \mathbf{D} \quad (2.5)$$

for functions $f \in H(\mathbf{D})$, $0 \le \epsilon < \infty$and for some constant $1+\epsilon = (1+\epsilon)(\epsilon-1, 1-\epsilon) > 0$. All the details for the statements above can be found in [28, Chapter 4].

1.3 Test Function. Note from (2.5) with $\epsilon = 1$that each point evaluation is a continuouslinear functional on the Hilbert space $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. Thus, to each $z \in \mathbf{D}$corresponds a uniquereproducing kernel whose explicit formula is known as $w \mapsto \lambda_z^{(\epsilon-1)+2}$where

$$\lambda_z(w) := \frac{1}{1-\overline{z}w}, \quad w \in \mathbf{D} \quad (2.6)$$

for $z \in \mathbf{D}$.

Powers of the functions in (2.6) will be the source of test functions in conjunction with Lemma 2.1. The norms of such kernel-type functions are well known. Namely, when $(1+\epsilon)(1+\epsilon) > (\epsilon-1)+2$, we have

$$\|\lambda_z^{1+\epsilon}\|_{A^{1+\epsilon}_{\epsilon-1}} \approx (1-|z|^2)^{-(1+\epsilon)+\frac{(\epsilon-1)+2}{1+\epsilon}}, \quad a^j \in \mathbf{D}; \quad (2.7)$$

constants suppressed in this estimate are independent of z; see, for example, [28, Lemma 3.10]. Thus

$$\frac{\lambda_z^{1+\epsilon}}{\|\lambda_z^{1+\epsilon}\|_{A^{1+\epsilon}_{\epsilon-1}}} \to 0 \quad \to 0 \quad \text{uniformly on compact subests of } \mathbf{D} \quad (2.8)$$

as $|z| \to 1$.

1.4 Carleson Measure. Let $\epsilon > 0$and μbe a finite positive Borel measure on $\mathbf{D}$. For $0 \le \epsilon < 1$and $0 \le \epsilon < \infty$, it is well known that

$$\text{the embedding } A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \subset (1+2\epsilon)^{1+\epsilon}(d\mu) \text{is bounded}$$

$$\Leftrightarrow \sup_{z\in\mathbf{D}} \frac{\mu[E_{1-\epsilon}(z)]}{A_{\epsilon-1}[E_{1-\epsilon}(z)]} < \infty \quad (2.9)$$

and

$$\text{the embedding } A^{1+\epsilon}_{\epsilon-1}(\mathbf{D}) \subset (1+2\epsilon)^{1+\epsilon}(d\mu) \text{is compact}$$

$$\Leftrightarrow \lim_{|z|\to 1} \frac{\mu[E_{1-\epsilon}(z)]}{A_{\epsilon-1}[E_{1-\epsilon}(z)]} = 0\,. \quad (2.10)$$

We say that $\mu$is an $(\epsilon-1)$-*Carleson measure* if (2.9) holds. Also, we say that $\mu$is a *compact* $(\epsilon-1)$-Carleson measure if (2.10) holds. Note that the notion of (compact) $(\epsilon-1)$-Carleson measures is independent of the parameters $1+\epsilon$and $1-\epsilon$. Given $\epsilon > 0$, it is also well known that

$$\left\|i_{\mu,1+\epsilon}\right\|^{1+\epsilon} \approx \|\mu\|_{\epsilon-1} := \sup_{z\in\mathbf{D}} \frac{\mu[E(z)]}{A_{\epsilon-1}[E(z)]} \text{where} \quad E(z) := E_{1/2}(z) \quad (2.11)$$

for $(\epsilon-1)$-Carleson measures μ; constants suppressed above are independent of μ and $1+\epsilon$. Here, $\left\|i_{\mu,1+\epsilon}\right\|$ denotes the operator norm of the embedding $i_{\mu,1+\epsilon} : A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \subset (1+2\epsilon)^{1+\epsilon}(d\mu)$; see [28,Section 7.2].

The connection between composition operators and Carleson measures comes from thechange-of-variable formula (see [9, p. 163])

$$\int_{\mathbf{D}} \sum_i (g\circ\varphi_i)dA_{\epsilon-1} = \int_{\mathbf{D}} \sum_i gd(A_{\epsilon-1}\circ\varphi_i^{-1}) \quad (2.12)$$

valid for $\varphi_i \in \mathcal{S}$ and positive Borel functions g on $\mathbf{D}$. Here, $A_{\epsilon-1}\circ\varphi_i^{-1}$ denotes the pullback measure defined by $(A_{\epsilon-1}\circ\varphi_i^{-1})(E) = A_{\epsilon-1}[\varphi_i^{-1}(E)]$ for Borelsets $E \subset \mathbf{D}$. Since each composition operator is bounded on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$, it is immediate from (2.12) that $A_{\epsilon-1}\circ\varphi_i^{-1}$ is an $(\epsilon-1)$-Carleson measure for each $\varphi_i \in \mathcal{S}$. Also is well known via (2.12) that $(1+\epsilon)_{\varphi_i}$ is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$ if and only if $A_{\epsilon-1}\circ\varphi_i^{-1}$ is a compact $(\epsilon-1)$-Carleson measure.

2.5. Angular Derivative. We recall the well-known notion of the angular derivative. Let $\mathbf{T} := \partial\mathbf{D}$ be the unit circle. A map $\varphi_i \in \mathcal{S}$ is said to have a finite angular derivative at $\zeta \in \mathbf{T}$, denoted by $\varphi_i'(\zeta) \in \mathbf{C}$, if φ_i has nontangential limit $\varphi_i(\zeta) \in \mathbf{T}$ at ζ such that

$$\angle\lim_{z\to\zeta} \frac{\varphi_i(z)-\varphi_i(\zeta)}{z-\zeta} = \varphi_i'(\zeta)$$

where $\angle\lim$ stands for the nontangential limit. As is well known by the Julia-Caratheodory Theorem (see [7, Theorem 2.44]), $\varphi_i'(\zeta)$ exists if and only if

$$\liminf_{z\to\zeta} \frac{1-|\varphi_i(z)|^2}{1-|z|^2} < \infty.$$

In this case, the left-hand side of the above is equal to $|\varphi_i'(\zeta)|$ and, moreover,

$$\angle\lim_{z\to\zeta} \frac{1-|z|^2}{1-|\varphi_i(z)|^2} = \frac{1}{|\varphi_i'(\zeta)|}. \quad (2.13)$$

In particular, we have $|\varphi_i'(\zeta)| \geq \frac{1-|\varphi_i(0)|}{1+|\varphi_i(0)|} > 0$ by the Schwarz-Pick lemma. We put

$$F(\varphi_i) := \left\{\zeta \in \mathbf{T} : \limsup_{z\to\zeta} R_{\varphi_i}(z) > 0\right\}; \quad (2.14)$$

this is the "angular derivative set" of φ_i consisting of all boundary points at which φ_i has finite angular derivatives. Note that (1.1) is equivalent to the condition $F(\varphi_i) = \emptyset$.

2. Julia-CarathÉodory Type Characterization

This section is devoted to the proof of Theorem 1.1. We will complete the proof of Theorem 1.1 by establishing the implications

$$\text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(a)}.$$

Note that the implication (a) $\Longrightarrow$ (b) is clear by Lemma 2.1 and (2.8).

Before proceeding, we fix some notation to be used throughout the section. We put

$$T_J := \sum_{j\in J}\sum_i a_j^j T_j$$

for $J \subset \Lambda_N$. Given $J \subset \Lambda_N$ with $\ell \in J$, note from (1.4) that the operator T_J can be written in the form

$$T_J = \sum_{j,1+\epsilon\in J}\sum_i c_{j,1+\epsilon}T_{j,1+\epsilon} + \left(\sum_{j\in J}\sum_i a_j^j\right)T_\ell \quad (3.1)$$

for some coefficients $c_{j,1+\epsilon}$ depending on ℓ and a_j^j's with $j \in J$.

We now proceed to the proof of the implication (b) $\Longrightarrow$ (c). We need the following estimate, which is the disk version of [13, Lemma 2.2]; one may also refer to [22, Lemma 3.1].

Lemma 3.1. *Let* $0 \leq \epsilon < \infty$, $\epsilon > 0$ *and* $0 < \epsilon < 1$. *Then there is a constant* $1+\epsilon = (1+\epsilon)(\epsilon-1, 1+\epsilon, 1-2\epsilon, 1-\epsilon) > 0$ *such that*

$$\left|f(a^j) - f(a^j+\epsilon)\right|^{1+\epsilon} \leq (1+\epsilon)\frac{\rho^{1+\epsilon}(a^j, a^j+\epsilon)}{A_{\epsilon-1}\left[E_{(1-\epsilon)_2}(a^j)\right]}\int_{E_{(1-\epsilon)_2}(a^j)} |f|^{1+\epsilon}dA_{\epsilon-1} \quad (3.2)$$

for $f \in H(\mathbf{D})$ *and* $a^j, a^j+\epsilon \in \mathbf{D}$ *with* $(a^j+\epsilon) \in E_{(1-\epsilon)_1}(a^j)$.

Proof of (b) $\Longrightarrow$ (c). Assume (b). We suppose that (c) fails and will derive a contradiction.Since (c) fails, there exists a sequence $\{z_n\} \subset \mathbf{D}$ such that $|z_n| \to 1$ and

$$\inf_n Q(z_n) > 0.$$

Since functions Q_{σ^i} are all bounded on **D**, it follows that

$$\inf_n \sum_i Q_{\sigma^i}(z_n) > 0 \text{ for each } \sigma^i \in \wp_N. \quad (3.3)$$

Furthermore, since functions R_j and $\rho_{j,1+\epsilon}$ are all bounded, we may assume that the sequences

$$\{R_j(z_n)\} \text{ and } \{\rho_{j,1+\epsilon}(z_n)\} \text{both converge as } n \to \infty$$

for any $j, 1+\epsilon \in \Lambda_N$. Put

$$F := \left\{ j \in \Lambda_N : \lim_{n\to\infty} R_j(z_n) = 0 \right\} \quad (3.4)$$

and let

$$F' := \Lambda_N \setminus F$$

for short. Note

$$\inf_n R_j(z_n) > 0 \quad (3.5)$$

for each $j \in F'$.

In conjunction with the set F', we pause to prove the following claim.

Claim. *There are pairwise disjoint sets of indices $J_1, \dots, J_{1+2\epsilon}$ and a^j subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, with the following properties:*

(i) $F' = \bigcup_{1+\epsilon=1}^{1+2\epsilon} J_{1+\epsilon}$;

(ii) *To each* $1+\epsilon = 1, \dots, 1+2\epsilon$ *corresponds* $(1+\epsilon)_{1+\epsilon} \in J_{1+\epsilon}$ *such that*

$$\left| \sum_i (\varphi_i)_j(z_n) \right| \le \sum_i \left| (\varphi_i)_{(1+\epsilon)_{1+\epsilon}}(z_n) \right| \quad \textit{for all } n \textit{ and } j \in \bigcup_{\ell=1+\epsilon}^{1+2\epsilon} J_\ell;$$

(iii) $\lim_{n\to\infty} \rho_{j,1+\epsilon}(z_n) = 0$ *for all* $j, 1+\epsilon \in J_{1+\epsilon}$ *for each* $1+\epsilon = 1, \dots, 1+2\epsilon$.

(iv) $\sum_{j\in J_{1+\epsilon}} \sum_i a_j^j = 0$ *for each* $1+\epsilon = 1, \dots, 1+2\epsilon$.

Proof of Claim. Assume $F' \neq \emptyset$ to avoid triviality. Put $J_0 := F$ for convenience. Passing to a subsequence if necessary, we may choose an index $(1+\epsilon)_1 \notin J_0$ such that

$$\left| (\varphi_i)_j(z_n) \right| \le \left| (\varphi_i)_{(1+\epsilon)_1}(z_n) \right| \text{for all } n \text{ and } j \notin J_0.$$

Using such $(1+\epsilon)_1$, we set

$$J_1 := \left\{ j \in F' : \lim_{n\to\infty} \rho_{(1+\epsilon)_1, j}(z_n) = 0 \right\}.$$

Now, let $\nu \ge 2$ and suppose that we have chosen pairwise disjoint sets $J_{1+\epsilon}$ and points $(1+\epsilon)_{1+\epsilon} \in J_{1+\epsilon}$ for $1+\epsilon = 1, \dots, \nu-1$ which satisfy

$$\left| \sum_i (\varphi_i)_j(z_n) \right| \le \sum_i \left| (\varphi_i)_{(1+\epsilon)_{1+\epsilon}}(z_n) \right| \text{ for all } n \text{ and } j \notin \bigcup_{\ell=0}^{(1+\epsilon)-1} J_\ell \quad (3.6)$$

and

$$J_{1+\epsilon} = \left\{ j \in F' : \lim_{n\to\infty} \rho_{(1+\epsilon)_{1+\epsilon}, j}(z_n) = 0 \right\}. \quad (3.7)$$

for $1+\epsilon = 1, \dots, \nu-1$. If $F' = \bigcup_{1+\epsilon=1}^{\nu-1} J_{1+\epsilon}$, then we stop. Otherwise, passing to a further subsequence if necessary, we choose an index

$$(1+\epsilon)_\nu \notin \bigcup_{1+\epsilon=0}^{\nu-1} J_{1+\epsilon} \quad (3.8)$$

such that

$$\left| \sum_i (\varphi_i)_j(z_n) \right| \le \sum_i \left| (\varphi_i)_{(1+\epsilon)_\nu}(z_n) \right| \text{ for all } n \text{ and } j \notin \bigcup_{1+\epsilon=0}^{\nu-1} J_{1+\epsilon}.$$

Using this $(1+\epsilon)_\nu$, we set

$$J_\nu := \left\{ j \in F' : \lim_{n\to\infty} \rho_{(1+\epsilon)_\nu, j}(z_n) = 0 \right\}.$$

Note from (3.8) and (3.7) that J_ν and $J_{1+\epsilon}$ are disjoint for each $1+\epsilon = 1, \dots, \nu-1$. This process stops after finitely many steps by pairwise disjointness of the sets. Thus, (i) holds for some $1+2\epsilon$. Note that (ii) holds by (3.6) and (i). Also, since $(1+\epsilon)_{1+\epsilon} \in J_{1+\epsilon}$ for each $1+\epsilon$, (iii) is clear by (3.7).

For (iv), we first consider the case $1+\epsilon = 1$. For $\epsilon \ge 0$, put

$$f_{a^j} = f_{a^j, 1+\epsilon} := \left(1 - |a^j|^2\right)^{1+\epsilon} \lambda_{a^j}^{\frac{(\epsilon-1)+2}{1+\epsilon} + (1+\epsilon)}$$

for $a^j \in$ **D**. Put

$$w_n := (\varphi_i)_{(1+\epsilon)_1}(z_n)$$

for each n. Note $1-|w_n|^2 \approx 1-|z_n|^2$by (3.5). So, $|w_n| \to 1$. It follows from (2.7) and (b) that

$$\lim_{n\to\infty}\|Tf_{w_n}\|_{A^{1+\epsilon}_{\epsilon-1}} = 0.$$

We thus obtain

$$\lim_{n\to\infty}(1-|w_n|^2)^{\frac{(\epsilon-1)+2}{1+\epsilon}}|Tf_{w_n}(z_n)| = 0 \quad (3.9)$$

by (2.5).

Note from (iii) and (2.3) that

$$1-|w_n|^2 \approx 1-|(\varphi_i)_j(z_n)|^2, \quad j \in J_1$$

for all n. It follows from Lemma 3.1 and (2.4) that

$$(1-|w_n|^2)^{(\epsilon-1)+2}|T_{j,1+\epsilon}f_{w_n}(z_n)|^{1+\epsilon} \lesssim \rho^{1+\epsilon}_{j,1+\epsilon}(z_n) \to 0, \quad j, 1+\epsilon \in J_1 \quad (3.10)$$

as $n \to \infty$. Meanwhile, note

$$(1-|w_n|^2)^{(\epsilon-1)+2}|T_jf_{w_n}(z_n)|^{1+\epsilon} = \left(\frac{1-|w_n|^2}{|1-(\varphi_i)_j(z_n)\overline{w_n}|}\right)^{(1+\epsilon)(1+\epsilon)+(\epsilon-1)+2}, j \in \Lambda_N \quad (3.11)$$

for all n. In particular,

$$(1-|w_n|^2)^{(\epsilon-1)+2}|T_{(1+\epsilon)_1}f_{w_n}(z_n)|^{1+\epsilon} = 1$$

for all n. Thus, applying (3.1) (with $J = J_1$and $\ell = (1+\epsilon)_1$), we obtain

$$\lim_{n\to\infty}(1-|w_n|^2)^{\frac{(\epsilon-1)+2}{1+\epsilon}}|T_{J_1}f_{w_n}(z_n)| = \left|\sum_{j\in J_1}\sum_i a^j_j\right|. \quad (3.12)$$

Since

$$(1-|w_n|^2)^{(\epsilon-1)+2}|T_jf_{w_n}(z_n)|^{1+\epsilon} \lesssim R_j^{(1+\epsilon)(1+\epsilon)+(\epsilon-1)+2}(z_n) \to 0, \quad j \in F$$

by (3.11) and definition of the set F, we also note

$$\lim_{n\to\infty}(1-|w_n|^2)^{(\epsilon-1)+2}|T_Ff_{w_n}(z_n)|^{1+\epsilon} = 0\,. \quad (3.13)$$

We now consider operators $T_{J_{1+\epsilon}}, \epsilon \geq 0$. Let $j \in J_{1+\epsilon}, \epsilon > 0$. We have $|(\varphi_i)_j(z_n)| \leq |w_n|$by (ii) and thus

$$\left(\frac{1-|w_n|^2}{|1-\sum_i(\varphi_i)_j(z_n)\overline{w_n}|}\right)^2 \leq \sum_i \frac{(1-|w_n|^2)\left(1-|(\varphi_i)_j(z_n)|^2\right)}{|1-(\varphi_i)_j(z_n)\overline{w_n}|^2} = 1-\rho^2_{1,j}(z_n)$$

for all n. It follows from (3.11)

$$(1-|w_n|^2)^{(\epsilon-1)+2}|T_jf_{w_n}(z_n)|^{1+\epsilon} \leq \left[1-\rho^2_{1,j}(z_n)\right]^{\frac{(1+\epsilon)(1+\epsilon)+(\epsilon-1)+2}{2}}$$

for all n. Thus, setting

$$\eta_1 := \min\left[\lim_{n\to\infty}\rho_{(1+\epsilon)_1,j}(z_n)\right] > 0$$

where the minimum is taken over all $j \in \cup_{\epsilon>0}J_{1+\epsilon}$, we obtain

$$\limsup_{n\to\infty}(1-|w_n|^2)^{(\epsilon-1)+2}\sum_{\epsilon>0}|T_{J_{1+\epsilon}}f_{w_n}(z_n)|^{1+\epsilon} \leq (1+\epsilon)_1(1-\eta_1^2)^{\frac{(1+\epsilon)(1+\epsilon)+(\epsilon-1)+2}{2}}; \quad (3.14)$$

for some constant $(1+\epsilon)_1 = (1+\epsilon)_1(1+\epsilon, N, a^j_1, \dots, a^j_N) > 0$. Now, since we have by (i)

$$T_{J_1} = T - T_F - \sum_{\epsilon>0}T_{J_{1+\epsilon}},$$

we deduce from (3.9), (3.12), (3.13) and (3.14) that

$$\left|\sum_{j\in J_1}\sum_i a^j_j\right| \leq (1+\epsilon)_1(1-\eta_1^2)^{\frac{(1+\epsilon)(1+\epsilon)+(\epsilon-1)+2}{2}}.$$

Recall that $\epsilon \geq 0$is arbitrary. So, taking the limit $\epsilon \to \infty$, we conclude

$$\sum_{j\in J_1}\sum_i a^j_j = 0,$$

as required. This completes the proof for the case $\epsilon = 0$.

We now proceed by induction on $1+\epsilon$. So, let $\epsilon \geq 0$and assume

$$\sum_{j\in J_1}\sum_i a^j_j = \cdots = \sum_{j\in J_{1+\epsilon-1}}\sum_i a^j_j = 0\,. \quad (3.15)$$

We will show

$$\sum_{j\in J_{1+\epsilon}}\sum_i a_j^j = 0\,.\,(3.16)$$

This time our test functions are f_{ξ_n} where $\xi_n = (\varphi_i)_{(1+\epsilon)_{1+\epsilon}}(z_n)$ for each n. Using (3.1), (3.5) and the induction hypothesis (3.15), we have as in the proof of (3.10)

$$\lim_{n\to\infty}(1-|\xi_n|^2)^{(\epsilon-1)+2}\sum_{\epsilon\geq 0}\left|T_{J_{1+\epsilon}}f_{\xi_n}(z_n)\right|^{1+\epsilon} = 0.$$

Also, as in the case of $\epsilon = 1$, we obtain

$$\lim_{n\to\infty}(1-|\xi_n|^2)^{(\epsilon-1)+2}\left|T_F f_{\xi_n}(z_n)\right|^{1+\epsilon} = 0$$

and

$$\lim_{n\to\infty}(1-|\xi_n|^2)^{\frac{(\epsilon-1)+2}{1+\epsilon}}\left|T_{J_{1+\epsilon}}f_{\xi_n}(z_n)\right| = \left|\sum_{j\in J_{1+\epsilon}}\sum_i a_j^j\right|.$$

Setting

$$\eta_{1+\epsilon} := \min\left[\lim_{n\to\infty}\rho_{(1+\epsilon)_{1+\epsilon,j}}(z_n)\right] > 0$$

where the minimum is taken over all $j\in\bigcup_{\epsilon>0}J_{1+\epsilon}$, we also have

$$\limsup_{n\to\infty}(1-|\xi_n|^2)^{(\epsilon-1)+2}\sum_{\epsilon>0}\left|T_{J_{1+\epsilon}}f_{\xi_n}(z_n)\right|^{1+\epsilon} \leq (1+\epsilon)_1(1-\eta_{1+\epsilon}^2)^{\frac{(1+\epsilon)(1+\epsilon)+(\epsilon-1)+2}{2}}.$$

Now, as in the case of $\epsilon = 1$, we conclude (3.16), which completes the induction. So, (iv) holds. The proof of the claim is complete.

Having proved the claim above, we now continue the proof of the implication (b) $\Longrightarrow$ (c). We may assume that the sets $J_1,\ldots,J_{1+2\epsilon},F$ are all nonempty; otherwise the proof issimpler. Let $n_{1+\epsilon} := \#J_{1+\epsilon}$, the number of elements in $J_{1+\epsilon}$, and put

$$\mathfrak{j}_{1+\epsilon} := \left(j_{1+\epsilon,1},\ldots,j_{1+\epsilon,n_{1+\epsilon}}\right),\ \ 1+\epsilon = 1,\ldots,1+2\epsilon$$

where $j_{1+\epsilon,1},\ldots,j_{1+\epsilon,n_{1+\epsilon}}$ are the distinct elements of $J_{1+\epsilon}$. Also, let $d := \#F$ and put

$$\mathfrak{f} := (j_1,\ldots,j_d)$$

where $j_1,\ldots,j_d$ are the distinct elements of F. In these definitions of the vectors $\mathfrak{j}_{1+\epsilon}$ and $\mathfrak{f}$ we simply choose arbitrary but fixed permutations of components.

Now, consider $\tau\in\wp_N$ given by

$$\tau := (\mathfrak{j}_1,\ldots,\mathfrak{j}_{1+2\epsilon},\mathfrak{f}).$$

Using this, we may rephrase Claim (iv) as

$$\sum_{j=1}^{n_1+\cdots+n_{1+\epsilon}}\sum_i (a^j)_{\tau_j} = 0\ (3.17)$$

for $1+\epsilon = 1,\ldots,1+2\epsilon$. This yields

$$\sum_{j=1}^{N-d}\left|(1+\epsilon)_j^\tau\right|M_{\tau_j,\tau_{j+1}} = \sum_{1+\epsilon=1}^{1+2\epsilon}\sum_{j,1+\epsilon\in J_{1+\epsilon}}\sum_i c_{j,1+\epsilon}^{(1+\epsilon)}M_{j,1+\epsilon}$$

where $c_{j,1+\epsilon}^{(1+\epsilon)}$ are nonnegative coefficients depending on $\{a_\ell^j : \ell\in J_{1+\epsilon}\}$ for each m. On the other hand, note

$$\sum_{j=N-d+1}^{N-d}\left|(1+\epsilon)_j^\tau\right|M_{\tau_j,\tau_{j+1}} + |(1+\epsilon)_N^\tau|M_{\tau_N} \leq \sum_{j\in F}c_j\ \ R_j$$

where c_j are nonnegative coefficients depending on $\left\{a_{j_1}^j,\ldots,a_{j_d}^j\right\}$. It follows that

$$Q_\tau = \sum_{j=1}^{N-1}\left|(1+\epsilon)_j^\tau\right|M_{\tau_j,\tau_{j+1}} + |(1+\epsilon)_N^\tau|M_{\tau_N}$$
$$\leq \sum_{1+\epsilon=1}^{1+2\epsilon}\sum_{j,1+\epsilon\in J_{1+\epsilon}}c_{j,1+\epsilon}^{(1+\epsilon)}M_{j,1+\epsilon} + \sum_{j\in F}\sum_i c_j\ \ R_j.$$

Note that the first term of the above tends to 0 along the sequence $\{z_n\}$ by Claim (iii). The second term also tends to 0 along the sequence $\{z_n\}$ by definition (3.4) of the set F. Consequently, we obtain

$$\lim_{n\to\infty}Q_\tau(z_n) = 0,$$

which is a contradiction to (3.3). The proof is complete.

We now proceed to the proof of the implication (c) $\Longrightarrow$ (a). We recall a couple ofknown estimates. First, we recall the following estimate which is implicit in the proof of[19, Lemma 1] or [4, Lemma 4.3].

Lemma 3.2.*Let $\epsilon > 0$and $0 \leq \epsilon < \infty$. Put $1 + \epsilon := \min\left\{\frac{(\epsilon-1)+1}{2}, 1\right\}$. Let $\varphi_i \in \mathcal{S}, \epsilon > 0andW : \mathbf{D} \to [0,1]$be a Borel function. If*

$$\sup_{z\in\mathbf{D}} \sum_i \left[W(z) R_{\varphi_i}(z)\right] \leq \epsilon \,, (3.18)$$

then there is a constant $1 + \epsilon = (1 + \epsilon)(\epsilon - 1) > 0$such that

$$\int_{\mathbf{D}} \sum_i |f \circ \varphi_i|^{1+\epsilon} W dA_{\epsilon-1} \leq (1 + \epsilon)\epsilon^{1+\epsilon} \|f\|^{1+\epsilon}_{A^{1+\epsilon}_{\epsilon-1}}$$

for $f \in A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

Next, the following lemma is taken from [6, Lemma 3.3].

Lemma 3.3.*Let $\epsilon > 0$and $0 \leq \epsilon < \infty$. Let $\varphi_i, \psi_i \in \mathcal{S}, \epsilon > 0$and $K \subset \mathbf{D}$be a Borel set. If*

$$\sup_{z\in K} \sum_i M_{\varphi_i,\psi_i} \leq \epsilon \,, (3.19)$$

then there is a constant $h(\epsilon) > 0$such that

$$\lim_{\epsilon\to 0} h(\epsilon) = 0$$

and

$$\int_K \sum_i |f \circ \varphi_i - f \circ \psi_i|^{1+\epsilon} W dA_{\epsilon-1} \leq h(\epsilon) \|f\|^{1+\epsilon}_{A^{1+\epsilon}_{\epsilon-1}}$$

for $f \in A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

We are now ready to prove the implication (c) $\Longrightarrow$ (a). In the proof below we use the notation

$$\Omega_\delta(\zeta) := \left\{z \in \mathbf{D} : 1 - \delta < |z| < 1, \left|\zeta - \frac{z}{|z|}\right| < \delta\right\}.$$

for$\zeta \in \mathbf{T}$and $0 < \delta < 1$.

Proof of (c) $\Longrightarrow$ (a). Assume (c). Our proof relies on Lemma 2.1. So, consider an arbitrary sequence $\{f_n\}$in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$such that $\|f_n\|_{A^{1+\epsilon}_{\epsilon-1}} \leq 1$and $f_n \to 0$uniformly on compact sets of $\mathbf{D}$. According to Lemma 2.1, it suffices to show that $Tf_n \to 0$in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

Let $\epsilon > 0$and put

$$U_{\sigma^i,\epsilon} := \left\{z \in \mathbf{D} : Q_{\sigma^i}(z) \leq \epsilon\right\}$$

for $\sigma^i \in \wp_N$. Note from (c) that to each $\zeta \in \mathbf{T}$corresponds $\delta_\zeta = \delta_\zeta(\epsilon) \in (0,1)$such that

$$\Omega_{\delta_\zeta}(\zeta) \subset \bigcup_{\sigma^i\in\wp_N} U_{\sigma^i,\epsilon} \,. (3.20)$$

Indeed, if this does not hold, then there would be some $\zeta_0 \in \mathbf{T}$with the following property: For any $\delta \in (0,1)$the set $\Omega_\delta(\zeta_0)$contains a point $z_\delta \in \mathbf{D}$such that $Q_{\sigma^i}(z_\delta) > \epsilon$for all $\sigma^i \in \wp_N$and hence $Q(z_\delta) > \epsilon^{N!}$, which is impossible by (c).

Now, by compactness of $\mathbf{T}$, we can find finitely many points $\zeta_1, \dots, \zeta_v$in $\mathbf{T}$such that

$$\mathbf{D} \setminus (1 - \epsilon)\mathbf{D} \subset \bigcup_{j=1}^{v} \Omega_{\delta_j}(\zeta_j) \subset \bigcup_{\sigma^i\in\wp_N} U_{\sigma^i,\epsilon}$$

where$\delta_j := \delta_{\zeta_j}$and $1 - \epsilon := \max(1 - \delta_j)$; the second inclusion above holds by (3.20). It follows that

$$\int_{\mathbf{D}} |Tf_n|^{1+\epsilon} dA_{\epsilon-1} \leq \int_{(1-\epsilon)\mathbf{D}} + \sum_{\sigma^i\in\wp_N} \sum_i \int_{U_{\sigma^i,\epsilon}}$$
$$=: I_n + \sum_{\sigma^i\in\wp_N} \sum_i II_n^{\sigma^i,\epsilon}$$

for each n. For the first term of the above, note that $f_n \to 0$uniformly on $\bigcup_{j=1}^{N} (\varphi_i)_j\big((1 - \epsilon)\mathbf{D}\big)$, which is contained in a compact set of $\mathbf{D}$. It follows that

$$I_n \to 0 (3.21)$$

as $n \to \infty$. To estimate the second term, let $\sigma^i \in \wp_N$. Recall that $Q_{\sigma^i} \leq \epsilon$on $U_{\sigma^i,\epsilon}$. Thus, in conjunction with the representation (1.4) of T, we see from Lemma 3.3 that there is a constant $h_{\sigma^i}(\epsilon) > 0$such that $\lim_{\epsilon\to 0} h_{\sigma^i}(\epsilon) = 0$and

$$\sum_{j=1}^{N-1} \sum_i \left|(1 + \epsilon)^{\sigma^i}_j\right|^{1+\epsilon} \int_{U_{\sigma^i,\epsilon}} \left|T_{\sigma^i_j,\sigma^i_{j+1}} f_n\right|^{1+\epsilon} dA_{\epsilon-1} \leq h_{\sigma^i}(\epsilon)$$

for all n. In addition, we see from Lemma 3.2 that there are constants $\epsilon \geq 0$, independent of ϵ, such that

$$\sum_i \left|(1+\epsilon)_N^{\sigma^i}\right|^{1+\epsilon} \int_{U_{\sigma^i,\epsilon}} \left|T_{\sigma_N^i} f_n\right|^{1+\epsilon} dA_{\epsilon-1} \leq (1+\epsilon)\epsilon^{1+\epsilon}$$

for all n. Accordingly, there is a constant $\epsilon \geq 0$, independent of ϵ, such that

$$\sum_i II_n^{\sigma^i,\epsilon} \leq (1+\epsilon)\left[h_{\sigma^i}(\epsilon) + \epsilon^{1+\epsilon}\right] \quad (3.22)$$

for all n. We now conclude by (3.21) and (3.22) that

$$\limsup_{n\to\infty} \int_{\mathbf{D}} |Tf_n|^{1+\epsilon} dA_{\epsilon-1} \leq h(\epsilon)$$

for some constant $h(\epsilon) > 0$satisfying $h(\epsilon) \to 0$as $\epsilon \to 0$. Finally, taking the limit $\epsilon \to 0$in the right-hand side of the above, we conclude, as required, that $Tf_n \to 0$in $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$.The proof is complete.

3. Carleson Measure Characterization:

This section is devoted to the proof of Theorem 1.3. Before proceeding, we observe acouple of basic properties of the joint pullback measures defined in (1.8). Let$\epsilon > 0$and$0 \leq \epsilon < \infty$.

Let $\sigma^i \in \wp_N$. We note from definitions of $Q_{\sigma^i}$and $\mu_{\epsilon-1,1+\epsilon}^{\sigma^i}$

$$\mu_{\epsilon-1,1+\epsilon}^{\sigma^i}(E) \leq 2\sum_{j=1}^{N}\sum_i \int_{(\varphi_i)_j^{-1}(E)\cap G_{\sigma^i}} Q_{\sigma^i}^{1+\epsilon} dA_{\epsilon-1} \quad (4.1)$$

forBorel sets $E \subset \mathbf{D}$. We also note via the standard argument that led to (2.12) the change-of-variable formula

$$\int_D \sum_i g\, d\mu_{\epsilon-1,1+\epsilon}^{\sigma^i} = \sum_{j=1}^{N-1}\sum_i \left|(1+\epsilon)_j^{\sigma^i}\right|^{1+\epsilon} \int_{G_{\sigma^i}} \left[g\circ(\varphi_i)_{\sigma_j^i} + g\circ(\varphi_i)_{\sigma_{j+1}^i}\right] M_{\sigma_j^i,\sigma_{j+1}^i}^{1+\epsilon} dA_{\epsilon-1}$$

$$+\sum_i \left|(1+\epsilon)_N^{\sigma^i}\right|^{1+\epsilon} \int_{G_{\sigma^i}} \left(g\circ(\varphi_i)_{\sigma_N^i}\right) R_{\sigma_N^i}^{1+\epsilon} dA_{\epsilon-1} \quad (4.2)$$

valid for positive Borel functions $g$on $\mathbf{D}$.

Lemma 4.1.*Let $\epsilon > 0$and $0 \leq \epsilon < \infty$. Let $\varphi_i, \psi_i \in \mathcal{S}$and $W : \mathbf{D} \to [0,1]$be a Borel function. Let μ and v be the measures defined by*

$$\mu(E) \coloneqq \int_{\varphi_i^{-1}(E)} \sum_i \left(R_{\varphi_i} + R_{\psi_i}\right)^{1+\epsilon} W dA_{\epsilon-1} + \int_{\psi_i^{-1}(E)} \sum_i \left(R_{\varphi_i} + R_{\psi_i}\right)^{1+\epsilon} W dA_{\epsilon-1}$$

and

$$v(E) \coloneqq \int_{\varphi_i^{-1}(E)} \sum_i W dA_{\epsilon-1} + \int_{\psi_i^{-1}(E)} \sum_i W dA_{\epsilon-1}$$

forBorel sets $E \subset \mathbf{D}$. If $\mu$is a compact $(\epsilon-1)$-Carleson measure on $\mathbf{D}$, then so is v.

Proof. Let $\epsilon \in (0,1)$and put

$$K_\epsilon \coloneqq \left\{z \in \mathbf{D} : R_{\varphi_i}(z) + R_{\psi_i}(z) \leq \epsilon\right\}.$$

Let $E \subset \mathbf{D}$be a Borel set. Note

$$\int_{\varphi_i^{-1}(E)} \sum_i W dA_{\epsilon-1} = \int_{\varphi_i^{-1}(E)\cap K_\epsilon} \sum_i + \int_{\varphi_i^{-1}(E)\setminus K_\epsilon} \sum_i W dA_{\epsilon-1}$$

$$\leq \int_{\varphi_i^{-1}(E)} \sum_i \chi_{K_\epsilon} W dA_{\epsilon-1} + \frac{1}{\epsilon^{1+\epsilon}} \int_{\varphi_i^{-1}(E)} \sum_i \left(R_{\varphi_i} + R_{\psi_i}\right)^{1+\epsilon} W dA_{\epsilon-1}$$

where$\chi_{K_\epsilon}$is the characteristic function of the set K_ϵ. The same estimate holds for ψ_i. Thus, setting the measure

$$v_\epsilon(E) := \int_{\varphi_i^{-1}(E)} \sum_i \chi_{K_\epsilon} W dA_{\epsilon-1} + \int_{\psi_i^{-1}(E)} \sum_i \chi_{K_\epsilon} W dA_{\epsilon-1},$$

we deduce

$$v \leq v_\epsilon + \frac{1}{\epsilon^{1+\epsilon}}\mu.$$

Accordingly, using the notation introduced in (2.11), we obtain

$$\frac{v[E(z)]}{A_{\epsilon-1}[E(z)]} \leq \|v_\epsilon\|_{\epsilon-1} + \frac{1}{\epsilon^{1+\epsilon}} \frac{\mu[E(z)]}{A_{\epsilon-1}[E(z)]}$$

for all $z \in \mathbf{D}$. Now, assuming that $\mu$is a compact $(\epsilon-1)$-Carleson measure on $\mathbf{D}$and letting $|z| \to 1$, we obtain

$$\limsup_{|z|\to 1} \frac{v[E(z)]}{A_{\epsilon-1}[E(z)]} \leq \|v_\epsilon\|_{\epsilon-1}$$

for each $\epsilon > 0$. Note from Lemma 3.2 and (2.11) that $\|v_\epsilon\|_{\epsilon-1} \to 0$as $\epsilon \to 0$. Thus, taking the limit $\epsilon \to 0$, we conclude the lemma. The proof is complete.

We will complete the proof of Theorem 1.3 by establishing the equivalences

$$(a) \Leftrightarrow (b) \text{and} \quad (b) \Leftrightarrow (c).$$

Note that the implication (b) ⟺ (c) holds by Lemma 4.1. The implication (c) ⟺ (b)is clear, because functions $R_j$are all bounded. It remains to establish the equivalence (a) ⟺ (b).

In what follows we put

$$E(z) := E_{\frac{1}{2}}(z), \quad z \in \mathbf{D}$$

for short. Also, we put

$$\mathbf{D}^{1-\epsilon} := \mathbf{D} \setminus (1-\epsilon)\mathbf{D}$$

for $0 < \epsilon < 1$.

Proof of (a) ⟹ (b). Assume that $T$is compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. Let $\sigma^i \in \wp_N$. Note from definition (1.7) of the set G_{σ^i}

$$\left[Q_{\sigma^i}(w)\right]^{N!} \leq Q(w), \quad w \in G_{\sigma^i}. \quad (4.3)$$

Thus, given $\epsilon > 0$, we see from Theorem 1.1 and (4.3) that there is $0 < \epsilon \in 1$such that

$$Q_{\sigma^i}(w) \leq [Q(w)]^{\frac{1}{N!}} < \epsilon, \quad w \in G_{\sigma^i} \cap \mathbf{D}^{1-\epsilon}.$$

Choose $0 < \epsilon \in 1$such that

$$(\varphi_i)_j\big((1-\epsilon)\mathbf{D}\big) \subset (1-\epsilon)\mathbf{D}, \quad \text{or equivalentlym,} \qquad (\varphi_i)_j^{-1}\big(\mathbf{D}^{(1-\epsilon)}\big) \subset \mathbf{D}^{1-\epsilon}$$

for each j.

Note from (2.2) that there is $0 < \epsilon \in 1$such that $E(z) \subset \mathbf{D}^{(1-\epsilon)}$for $\in D^{(1-\epsilon)}$. It follows from the observations in the preceding paragraph that

$$Q_{\sigma^i} < \epsilon \quad \text{on} \quad (\varphi_i)_j^{-1}[E(z)] \cap G_{\sigma^i}, z \in \mathbf{D}^{(1-\epsilon)}$$

for each j. This, together with (4.1), yields

$$\mu^{\sigma^i}_{\epsilon-1,1+\epsilon}[E(z)] \leq 2\epsilon^{1+\epsilon} \sum_{j=1}^{N} \sum_i \big(A_{\epsilon-1} \circ (\varphi_i)_j^{-1}\big)[E(z)], \quad z \in \mathbf{D}^{(1-\epsilon)}$$

so that

$$\sup_{z\in\mathbf{D}^{(1-\epsilon)}} \sum_i \frac{\mu^{\sigma^i}_{\epsilon-1,1+\epsilon}[E(z)]}{A_{\epsilon-1}[E(z)]} \leq 2\epsilon^{1+\epsilon} \sum_{j=1}^{N} \sum_i \left\|A_{\epsilon-1} \circ (\varphi_i)_j^{-1}\right\|_{\epsilon-1}.$$

Note $\left\|A_{\epsilon-1} \circ (\varphi_i)_j^{-1}\right\|_{\epsilon-1} < \infty$for each j; see the remark after (2.12). It follows from the above that

$$\lim_{|z|\to 1} \sum_i \frac{\mu^{\sigma^i}_{\epsilon-1,1+\epsilon}[E(z)]}{A_{\epsilon-1}[E(z)]} = 0,$$

or said differently, that $\mu^{\sigma^i}_{\epsilon-1,1+\epsilon}$is a compact $(\epsilon-1)$-Carleson measure. Since $\sigma^i \in \wp_N$is arbitrary, we finally conclude that $\mu_{\epsilon-1,1+\epsilon}$is a compact $(\epsilon-1)$-Carleson measure. The proof is complete.

Proof of (b) ⟹ (a). Assume that $\mu_{\epsilon-1,1+\epsilon}$is a compact $(\epsilon-1)$-Carleson measure. As in the proof of Theorem 1.1, our proof relies on Lemma 2.1. So, consider an arbitrary sequence $\{f_n\}$in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$such that $\sup_n \|f_n\|_{A^{1+\epsilon}_{\epsilon-1}} \leq 1$and $f_n \to 0$uniformly on compact sets of $\mathbf{D}$. We need to show that $Tf_n \to 0$in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$.

Fix $0 < \epsilon \in 1$. Since $f_n \to 0$uniformly on $(1-\epsilon)\mathbf{D}$and

$$\sum_i \bigcup_{\sigma^i \in \wp_N} G_{\sigma^i} = \mathbf{D},$$

we have

$$\limsup_{n\to\infty} \int_{\mathbf{D}} |Tf_n|^{1+\epsilon} dA_{\epsilon-1} = \limsup_{n\to\infty} \int_{\mathbf{D}^{1-\epsilon}} |Tf_n|^{1+\epsilon} dA_{\epsilon-1}$$

$$\leq \limsup_{n\to\infty} \sum_{\sigma^i \in \wp_N} \sum_i \int_{G^r_{\sigma^i}} |Tf_n|^{1+\epsilon} dA_{\epsilon-1} \quad (4.4)$$

where $G^r_{\sigma^i} := G_{\sigma^i} \cap \mathbf{D}^{1-\epsilon}$.

Fix $\sigma^i \in \wp_N$. Let $\epsilon > 0$. With the representation (1.4) in mind, put

$$I_n^{\sigma^i,\epsilon,j} := \int_{G^r_{\sigma^i} \cap \left\{M_{\sigma^i_j,\sigma^i_{j+1}} \geq \epsilon\right\}} \sum_{\boldsymbol{i}} \left|T_{\sigma^i_j,\sigma^i_{j+1}} f_n\right|^{1+\epsilon} dA_{\epsilon-1} \quad (4.5)$$

and

$$II_n^{\sigma^i,\epsilon,j} := \int_{G^r_{\sigma^i}\cap\left\{M_{\sigma^i_j,\sigma^i_{j+1}}<\epsilon\right\}} \sum_{i} \left|T_{\sigma^i_j,\sigma^i_{j+1}} f_n\right|^{1+\epsilon} dA_{\epsilon-1} \quad (4.6)$$

for each $1 \leq j < N$and n. Similarly, put

$$I_n^{\sigma^i,\epsilon,N} := \int_{G^r_{\sigma^i}\cap\left\{R_{\sigma^i_N}\geq\epsilon\right\}} \sum_{i} \left|T_{\sigma^i(N)} f_n\right|^{1+\epsilon} dA_{\epsilon-1} \quad (4.7)$$

and

$$II_n^{\sigma^i,\epsilon,N} := \int_{G^r_{\sigma^i}\cap\left\{R_{\sigma^i_N}<\epsilon\right\}} \sum_{i} \left|T_{\sigma^i(N)} f_n\right|^{1+\epsilon} dA_{\epsilon-1} \quad (4.8)$$

for each n. Using these notation and the representation (1.4) of T, we have

$$\int_{G^{1-\epsilon}_{\sigma^i}} \sum_{i} |Tf_n|^{1+\epsilon} dA_{\epsilon-1} \lesssim \sum_{j=1}^{N} \sum_{i} \left|(1+\epsilon)^{\sigma^i}_j\right|^{1+\epsilon} \left(I_n^{\sigma^i,\epsilon,j} + II_n^{\sigma^i,\epsilon,j}\right) \quad (4.9)$$

for each n.

For the integrals in (4.5) and (4.7), we have

$$I_n^{\sigma^i,\epsilon,j} \leq \frac{1}{\epsilon^{1+\epsilon}} \int_{G_{\sigma^i}} \sum_{i} \left|T_{\sigma^i_j,\sigma^i_{j+1}} f_n\right|^{1+\epsilon} M^{1+\epsilon}_{\sigma^i_j,\sigma^i_{j+1}} dA_{\epsilon-1}, \quad 1 \leq j < N$$

and

$$I_n^{\sigma^i,\epsilon,j} \leq \frac{1}{\epsilon^{1+\epsilon}} \int_{G_{\sigma^i}} \sum_{i} \left|T_{\sigma^i_N} f_n\right|^{1+\epsilon} R^{1+\epsilon}_{\sigma^i_N} dA_{\epsilon-1}$$

for all n. Now, since

$$\sum_{i} \left|T_{\sigma^i_j,\sigma^i_{j+1}} f_n\right|^{1+\epsilon} \lesssim \sum_{i} \left|T_{\sigma^i_j} f_n\right|^{1+\epsilon} + \sum_{i} \left|T_{\sigma^i_{j+1}} f_n\right|^{1+\epsilon}$$

it follows from (4.2) that

$$\sum_{j=1}^{N} \sum_{i} \left|(1+\epsilon)^{\sigma^i}_j\right|^{1+\epsilon} I_n^{\sigma^i,\epsilon,j} \lesssim \frac{1}{\epsilon^{1+\epsilon}} \int_{\mathbf{D}} \sum_{i} |f_n|^{1+\epsilon} d\mu^{\sigma^i}_{\epsilon-1,1+\epsilon}$$

for all n; the constant suppressed above depends only on $1+\epsilon$. Note that $\mu^{\sigma^i}_{\epsilon-1,1+\epsilon}$is a compact $(\epsilon-1)$-Carleson measure, because $\mu_{\epsilon-1,1+\epsilon}$is by assumption. Also, recall that $\sup_n \|f_n\|_{A^{1+\epsilon}_{\epsilon-1}} \leq 1$and $f_n \to 0$uniformly on compact sets of $\mathbf{D}$. Thus the above estimate implies

$$\lim_{n\to\infty} \sum_{j=1}^{N} \sum_{i} \left|(1+\epsilon)^{\sigma^i}_j\right|^{1+\epsilon} I_n^{\sigma^i,\epsilon,j} = 0\,. \quad (4.10)$$

Meanwhile, for the integrals in (4.6) we have by Lemma 3.3 a constant $h_{\sigma^i}(\epsilon) > 0$such that $\lim_{n\to\infty} h_{\sigma^i}(\epsilon) = 0$and

$$\sum_{j=1}^{N-1} \sum_{i} \left|(1+\epsilon)^{\sigma^i}_j\right|^{1+\epsilon} II_n^{\sigma^i,\epsilon,j} \leq h_{\sigma^i}(\epsilon) \quad (4.11)$$

for all n. For the integrals in (4.8) we have by Lemma 3.2 constants $\epsilon \geq 0$, independent of ϵ, such that

$$\sum_{i} \left|(1+\epsilon)^{\sigma^i}_N\right|^{1+\epsilon} II_n^{\sigma^i,\epsilon,N} \leq (1+\epsilon)\epsilon^{1+\epsilon} \quad (4.12)$$

for all n. Now, we see from (4.9), (4.10), (4.11) and (4.12)

$$\limsup_{n\to\infty} \int_{G^{1-\epsilon}_{\sigma^i}} \sum_{i} |Tf_n|^{1+\epsilon} dA_{\epsilon-1} \leq h_{\sigma^i}(\epsilon) + (1+\epsilon)\epsilon^{1+\epsilon}.$$

So, taking the limit $\epsilon \to 0$, we obtain

$$\lim_{n\to\infty} \int_{G^{1-\epsilon}_{\sigma^i}} \sum_{i} |Tf_n|^{1+\epsilon} dA_{\epsilon-1} = 0$$

and thus conclude by (4.4) that $Tf_n \to 0$in $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$, as required. The proof is complete.

4. Applications

In this section we first prove Theorem 1.2 as an application of Theorem 1.1. Also, applying our results, we will recover some known results about compactness of linear combinations of composition operators.

We begin with the simple lemma below, which will be used repeatedly in our proofs later. Recall that R_{φ_i} and M_{φ_i,ψ_i} are the functions defined in (1.1) and (1.3).

Lemma 5.1. *Let $\varphi_i, \psi_i \in \mathcal{S}$. Let $\{z_n\}$ be a sequence in* **D** *such that*

$$\inf_n \sum_i R_{\varphi_i}(z_n) > 0 \quad \textit{and} \quad \lim_{n\to\infty} \sum_i R_{\psi_i}(z_n) = 0\,. \tag{5.1}$$

Then

$$\lim_{n\to\infty} \sum_i \rho\big(\varphi_i(z_n), \psi_i(z_n)\big) = 1$$

and

$$\limsup_{n\to\infty} \sum_i M_{\varphi_i,\psi_i}(z_n) > 0.$$

Proof. By (2.1) we have

$$1 - \rho^2(\varphi_i, \psi_i) \le 4\frac{1 - |\varphi_i|^2}{1 - |\psi_i|^2}.$$

Note that the right-hand side of the above tends to 0 along the sequence $\{z_n\}$ by (5.1). So, the first part of the lemma holds. The second part is also clear by (5.1) and the first part.

We now prove Theorem 1.2.

Proof of Theorem 1.2. The sufficiency is obvious by (1.4). For the necessity, suppose that T is compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. We first prove (a). Due to the compactness of T, we have by Theorem 1.1

$$\lim_{|z|\to 1} \sum_i \prod_{\sigma^i \in \wp_N} Q_{\sigma^i}(z) = 0.$$

Consider an arbitrary sequence $\{z_n\}$ such that $|z_n| \to 1$. We see from the above that there exists $\tau \in \wp_N$ and a subsequence $\{z_{n_\ell}\}$ such that

$$\lim_{\ell\to\infty} Q_\tau\big(z_{n_\ell}\big) = 0.$$

This, together with (CNC), yields

$$\lim_{\ell\to\infty} M_{\tau_j,\tau_{j+1}}\big(z_{n_\ell}\big) = 0 \tag{5.2}$$

for $1 \le j < N$.

We now assume that (a) fails and will derive a contradiction to complete the proof of (a). Since (a) fails, we have in addition to (5.2)

$$\lim_{\ell\to\infty} R_{\tau_N}\big(z_{n_\ell}\big) = 0.$$

This, together with Lemma 5.1 and (5.2) with $j = N - 1$, yields $\lim_{\ell\to\infty} R_{\tau_N}\big(z_{n_\ell}\big) = 0$.

Repeating the same argument, we have

$$\lim_{\ell\to\infty} R_j\big(z_{n_\ell}\big) = 0$$

for each $j = 1, \dots, N$. This subsequential property implies

$$\lim_{|z|\to 1} R_j(z) = 0,$$

for each $j = 1, \dots, N$. Thus operators $T_1, \dots, T_N$ are all compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ by the characterization (1.1) due to Shapiro and Taylor, which is impossible by assumption that at least one of the operators is not compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$. This completes the proof of (a).

We now turn to the proof of (b). Having proved (a), we note that (b) is trivial for $N = 2$.
So, we assume $N \ge 3$. We introduce some temporary notation. For $1 + \epsilon \in \Lambda_N$ with $1 + \epsilon \ne 1$, let $B_{1+\epsilon} (\ne \emptyset)$ be the set of all $\sigma^i \in \wp_N$ such that

$$\{\sigma^i_j, \sigma^i_{j+1}\} = \{1, 1+\epsilon\} \text{ for some } j \ne N$$

and put

$$Q_{B_{1+\epsilon}} := \prod_{\sigma^i \in B_{1+\epsilon}} Q_{\sigma^i}.$$

Note that each $\sigma^i \in \wp_N$ belongs to at least one and at most two of the sets $B_2, \dots, B_N$. Since functions Q_{σ^i} are all bounded, it follows that

$$\prod_{1+\epsilon=2}^{N} Q_{B_{1+\epsilon}} \le (1+\epsilon) Q \tag{5.3}$$

for some constant $\epsilon \ge 0$.

To prove (b), it suffices to show that $T_{1,j}$ is compact on $A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$ for each $j = 2, \dots, N$.

Suppose not. For simplicity we may suppose that $T_{1,2}$is not compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. By the characterization (1.3) due to Moorhouse, there is a sequence $\{z_n\}$in $\mathbf{D}$such that $|z_n| \to 1$and

$$\inf_n\{[R_1(z_n) + R_2(z_n)]\rho_{1,2}(z_n)\} > 0 \,. \quad (5.4)$$

In addition, since $R_1$and $R_2$are bounded on $\mathbf{D}$, we may also assume

$$\inf_n \rho_{1,2}(z_n) > 0 \,. \quad (5.5)$$

Note also from (5.4) that Q_{B_2} stays away from 0 along the sequence $\{z_n\}$. It follows from (5.3) and Theorem 1.1 that

$$\lim_{n\to\infty} \prod_{1+\epsilon=3}^{N} Q_{B_{1+\epsilon}}(z_n) = 0.$$

This implies that one of the factors tend to 0 along a subsequence, still denoted by $\{z_n\}$. So, we may assume $Q_{B_3}(z_n) \to 0$. By the same argument we may further assume

$$Q_\eta(z_n) \to 0 \quad (5.6)$$

for some $\eta \in B_3$. Since

$$Q_\eta = \sum_{j=1}^{N-1} \left|(1+\epsilon)_j^\eta\right| M_{\eta_j,\eta_{j+1}}$$

by (a), it follows from (CNC) that (5.6) is equivalent to

$$\lim_{n\to\infty} M_{\eta_j,\eta_{j+1}}(z_n) = 0 \quad (5.7)$$

for each $1 \le j < N$.

Note from (5.4) that (after passing to a subsequence if necessary)

$$\inf_n R_{j_0}(z_n) > 0$$

for some $j_0 \in \{1, 2\}$. It follows from Lemma 5.1 and (5.7) that (after passing to a subsequence if necessary)

$$\inf_n R_{j_1}(z_n) > 0,$$

when$j_1$is adjacent to j_0, i.e., when $\{j_0, j_1\} = \{\eta_j, \eta_{j+1}\}$for some $1 \le j < N$. Starting with the above, we see by the same argument that (after passing to a subsequence if necessary)

$$\inf_n R_{j_2}(z_n) > 0$$

when$j_2$is adjacent to j_1. Repeating the same arguments, we can find a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that

$$\inf_n R_j(z_n) > 0$$

for each $j \in \Lambda_N$. This, together with (5.7), implies

$$\rho_{\eta_j,\eta_{j+1}}(z_n) \to 0$$

for each $j \in \Lambda_N$. It follows that $\rho_{1,2}(z_n) \to 0$, which is a contradiction to (5.5). The proof is complete.

We now notice an easy special case of Theorem 1.2, which might be of independent interests.

Corollary 5.2.*Let $\epsilon > 0$and $0 \le \epsilon < \infty$. Let $(\varphi_i)_1, (\varphi_i)_2, \ldots, (\varphi_i)_N \in \mathcal{S}$. Assume $c_j > 0$for each $j = 2, \ldots, N$. Then*

$$\sum_{j=2}^{N} c_j T_{1,j} \quad \textit{is compact on} \quad A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$$

if and only if

$$T_{j,1+\epsilon} \textit{is compact on} \ A_{\epsilon-1}^{1+\epsilon}(\mathbf{D}) \textit{for each} \ j, 1+\epsilon \in \Lambda_N.$$

Proof. The sufficiency being trivial, we only need to prove the necessity. Before proceeding, we note

$$\sum_{j=2}^{N}\sum_i c_j T_{1,j} = \sum_{j=1}^{N}\sum_i a_j^j T_j$$

where$a_1^j = \sum_{j=2}^N \sum_i c_j$and $a_j^j = -c_j$for $j = 2, \ldots, N$. It is easy to see that (CNC) holdsand$\sum_{j=1}^N \sum_i a_j^j = 0$.

If all the operators $T_m, \ldots, T_N$are compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$, there is nothing to prove. If atleast one of them is not compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$, then we see from Theorem 1.2 that $T_{j,1+\epsilon}$iscompact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$for each $j, 1+\epsilon \in \Lambda_N$. The proof is complete.

Example. Applying Theorem 1.2, we can easily check whether a linear combination ofcomposition operators, for which (CNC) is fulfilled, is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. For example:

- $6T_m - T_{m+1} - 2T_{m+2} - 3T_{m+3}$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$if and only if $T_{j,1+\epsilon}$is compact on$A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$for each $j, 1+\epsilon = 1, 2, 3, 4$.
- $4T_m - 2T_{m+1} - T_{m+2}$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$only when $T_m, T_{m+1}, T_{m+2}$are all compact on$A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$.

- $T_m - iT_{m+1} - 2T_{m+2}$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$only when $T_m, T_{m+1}, T_{m+2}$are all compact on$A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$.

  In [14] the last two of the current authors characterized compactness of 3-combinations.Here, we provide new proofs based on Theorem 1.1 in the next three corollaries. First,when (CNC) is satisfied, we have the next immediate consequence of Theorem 1.2.

Corollary 5.3 ([14]).*Let $\epsilon > 0$and $0 \le \epsilon < \infty$. Let $(\varphi_i)_1, (\varphi_i)_2, (\varphi_i)_3 \in \mathcal{S}$and $a_1^j, a_2^j, a_3^j \in \mathbf{C}\backslash\{0\}$. Assume $a_j^j + a_{1+\epsilon}^j \neq 0$for each $j, 1+\epsilon = 1,2,3$. Also, assume that at least one of theoperators $T_m, T_{m+1}, T_{m+2}$is not compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. Then*

$$a_1^j T_m + a_2^j T_{m+1} + a_3^j T_{m+2} \textit{is compact on } A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$$

if and only if both of the following two conditions are fulfilled:

(i) $a_1^j + a_2^j + a_3^j = 0$;

(ii) $T_m - T_{m+1}$*and*$T_m - T_{m+2}$*are both compact on* $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$.

Next, when (CNC) is not satisfied, note that 3-combination reduces (after re-indexing)to a nontrivial combination of $T_m - T_{m+1}$and T_{m+2}. In conjunction with this observation wehave the next corollary.

Corollary 5.4 ([14]).*Let $\epsilon > 0$and $0 \le \epsilon < \infty$. Let $a^j \in \mathbf{C}\backslash\{0\}$. Let $(\varphi_i)_1, (\varphi_i)_2, (\varphi_i)_3 \in \mathcal{S}$. Suppose that none of $T_m, T_{m+1}, T_{m+2}$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$.*

If $T_m - T_{m+1} + a^j T_{m+2}$is compact on$A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$, then $a^j = 1$or $a^j = -1$.

Proof.We first compute the function Qassociated with the operator under consideration.

Note

$$\begin{aligned} T &:= (T_m - T_{m+1}) + a^j T_{m+2} \\ &= (T_m - T_{m+2}) + (1+a)(T_{m+2} - T_{m+1})a^j T_{m+1} \\ &= -(T_{m+1} - T_m) + a^j T_{m+2} \\ &= -(T_{m+1} - T_{m+2}) + (-1 + a^j)(T_{m+2} - T_m) + a^j T_m \\ &= a^j(T_{m+2} - T_m) + (a^j + 1)(T_m - T_{m+1}) + a^j T_{m+1} \\ &= a^j(T_{m+2} - T_{m+1}) + (a^j - 1)(T_{m+1} - T_m) + a^j T_m \end{aligned}$$

Thus, setting

$$\begin{aligned} S_1 &:= M_{1,2} + |a^j| R_3 \\ S_2 &:= M_{3,1} + |a^j + 1| M_{3,2} + |a^j| R_2 \\ S_3 &:= M_{2,1} + |a^j| R_3 \\ S_4 &:= M_{2,3} + |a^j - 1| M_{3,1} + |a^j| R_1 \\ S_5 &:= |a^j| M_{3,1} + |a^j + 1| M_{1,2} + |a^j| R_2 \\ S_6 &:= |a^j| M_{3,2} + |a^j - 1| M_{2,1} + |a^j| R_1 \end{aligned}$$

we have

$$Q = S_1 S_2 S_3 S_4 S_5 S_6.$$

Now, assume $T$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. Note that $T_m - T_{m+1}$is not compact. Thus there is a sequence $\{z_n\}$in $\mathbf{D}$with $|z_n| \to 1$such that

$$\inf_n M_{1,2}(z_n) > 0\,. \quad (5.8)$$

Assume $a^j \neq \pm 1$. Then $S_1 S_3 S_5 S_6$stays away from 0 along the sequence $\{z_n\}$. Accordingly, we have $\lim_{n\to\infty} S_2(z_n) S_4(z_n) = 0$by Theorem 1.1. Thus, we may assume either $\lim_{n\to\infty} S_2(z_n) = 0$or $\lim_{n\to\infty} S_4(z_n) = 0$after passing a subsequence if necessary.

Suppose $\lim_{n\to\infty} S_2(z_n) = 0$, i.e.,

(i) $\lim_{n\to\infty} R_2(z_n) = 0$,

(ii) $\lim_{n\to\infty} M_{1,3}(z_n) = 0$,

(iii) $\lim_{n\to\infty} M_{2,3}(z_n) = 0$.

By (i), (iii) and Lemma 5.1 we have $\lim_{n\to\infty} R_3(z_n) = 0$. This, together with (ii) andLemma 5.1 again, in turn yields

$$\lim_{n\to\infty} R_1(z_n) = 0,$$

which is impossible by (5.8) and (i). We thus conclude $a^j = 1$or $a^j = -1$, as desired. The proof is complete.

Finally, when (CNC) is not satisfied, note from Corollary 5.4 that 3-combination reduces (after re-indexing) to a constant multiple of $T_m - T_{m+1} - T_{m+2}$. In conjunction with this observation we have the next corollary. Recall that $F(\varphi_i)$denotes the angular derivative set of $\varphi_i$defined in (2.14).

Corollary 5.5 ([14]).*Let $\epsilon > 0$and $0 \le \epsilon < \infty$. Let $(\varphi_i)_1, (\varphi_i)_2, (\varphi_i)_3 \in \mathcal{S}$. Suppose that none of $T_m, T_{m+1}, T_{m+2}$is compact on $A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$. Then*

$$T_m - T_{m+1} - T_{m+2} \textit{ is compact on } A_{\epsilon-1}^{1+\epsilon}(\mathbf{D})$$

if and only if both of the following two conditions are fulfilled:
(a) $F((\varphi_i)_1)$*is the disjoint union of* $F((\varphi_i)_2)$*and* $F((\varphi_i)_3)$*;*
(b) $\lim_{z\to\zeta} M_{1,j}(z) = 0$*for each* $\zeta \in F\big((\varphi_i)_j\big)$*and* $j = 2, 3$.

Proof. By Theorem 1.1 it suffices to prove

$$\lim_{|z|\to 1} Q(z) = 0 \iff \text{(a) and (b) hold.}$$

In conjunction with this assertion, we note that the function Qassociated with $T_m - T_{m+1} - T_{m+2}$is given by

$$Q = \big(M_{1,2} + R_3\big)^2\big(M_{1,3} + R_2\big)^2 \times \big(M_{2,3} + 2M_{1,3} + R_1\big)\big(M_{2,3} + 2M_{1,2} + R_1\big) \quad (5.9)$$

by the proof of Corollary 5.4. Using this explicit expression of Q, one may easily verify that (a) and (b) imply $Q(z) \to 0$as $z \to \zeta$for any $\zeta \in \mathbf{T}$. Thus the implication$\Longleftarrow$holds.
We now prove the implication $\Longrightarrow$. So, assume

$$\lim_{|z|\to 1} Q(z) = 0\,. \quad (5.10)$$

We first prove (a). Let $\zeta_0 \in F((\varphi_i)_1)$. We have

$$\inf_n R_1(z_n) > 0$$

for some sequence $\{z_n\}$with $z_n \to \zeta_0$. We have by (5.9) and (5.10)

$$\big(M_{1,2} + R_3\big)\big(M_{1,3} + R_2\big) \to 0 \ \text{ along the sequence}\{z_n\}.$$

Now, suppose $\zeta_0 \notin F((\varphi_i)_2)$so that $R_2(z_n) \to 0$. Note from Lemma 5.1 that

$$\inf_n M_{1,2}(z_n) > 0$$

after passing to a subsequence if necessary. It follows from the above that $M_{1,3}(z_n) \to 0$. Accordingly, we have

$$R_3(z_n) \nrightarrow 0 \quad (5.11)$$

by Lemma 5.1 and thus $\zeta_0 \in F((\varphi_i)_3)$. We therefore conclude

$$F((\varphi_i)_1) \subset F((\varphi_i)_2) \cup F((\varphi_i)_3). \quad (5.12)$$

To prove the converse inclusion, suppose not and pick $\zeta_1 \in F((\varphi_i)_2) \cup F((\varphi_i)_3)$such that$\zeta_1 \notin F((\varphi_i)_1)$. We may assume $\zeta_1 \in F((\varphi_i)_2)$by symmetry so that

$$\inf_n R_2(w_n) > 0 \quad (5.13)$$

for some sequence $\{w_n\}$with $w_n \to \zeta_1$. Meanwhile, since $\zeta_1 \notin F((\varphi_i)_1)$, we have

$$R_1(w_n) \to 0 \quad (5.14)$$

and thus $\inf_n M_{1,2}(w_n) > 0$by Lemma 5.1 after passing to a subsequence if necessary. Now we have by (5.10) and (5.9)
(i) $\lim_{n\to\infty} M_{2,3}(w_n) = 0$ and (ii) $\lim_{n\to\infty} M_{1,3}(w_n) = 0$.
Applying Lemma 5.1, we note from (5.13) that (i) implies $R_3(z_n) \nrightarrow 0$, and thus (5.14) and (ii) implies $R_3(z_n) \to 0$. This contradiction shows that the inclusion in (5.12) can be reversed, as required.

If $\zeta_2 \in F((\varphi_i)_2) \cap F((\varphi_i)_3) \subset F((\varphi_i)_1)$, then we have by (2.13)

$$\angle \lim_{z\to\zeta_2} (R_1R_2R_3)(z) = \frac{1}{|(\varphi_i)_1'(\zeta_2)||(\varphi_i)_2'(\zeta_2)||(\varphi_i)_3'(\zeta_2)|} > 0,$$

which is impossible by (5.9). So, we conclude $F((\varphi_i)_2) \cap F((\varphi_i)_3) = \emptyset$. This completes the proof of (a).

We now prove (b). Suppose not. By symmetry we may assume that there is $\zeta_3 \in F((\varphi_i)_2)$such that $\limsup M_{1,2}(z) > 0$as $z \to \zeta_3$. Pick a sequence $\{\zeta_n\} \subset \mathbf{D}$with $\zeta_n \to \zeta_3$such that

$$\inf_n M_{1,2}(\zeta_n) > 0\,. \quad (5.15)$$

So, we may further assume
(iii) $\inf_n R_1(\zeta_n) > 0$ or (iv) $\inf_n R_2(\zeta_n) > 0$
after passing to a subsequence if necessary. Since $\zeta_3 \in F((\varphi_i)_2)$, we note $\zeta_3 \notin F((\varphi_i)_3)$and thus $R_3(\zeta_n) \to 0$.

In case of (iii), we see from (5.15) and (5.9) that $M_{13}(\zeta_n) + R_2(\zeta_n) \to 0$. So, we have $\zeta_3 \in F((\varphi_i)_3)$as in the proof of (5.11), which is a contradiction. In case of (iv), since $R_3(\zeta_n) \to 0$, we have by Lemma 5.1

$$\inf_n M_{2,3}(\zeta_n) > 0 \quad (5.16)$$

after passing to a subsequence if necessary. So, we see from (5.15), (iv), (5.16) and (5.9) that $Q(\zeta_n) \nrightarrow 0$, which is also a contradiction to (5.10). We thus conclude (b). The proof is complete.

The current authors have recently characterized compactness of double differences of composition operators. Here, we provide a new proof below, which is much simpler (thanks to Theorem 1.1) than the original one.

Corollary 5.6 ([6]).*Let* $\epsilon > 0$*and* $0 \le \epsilon < \infty$. *Let* $(\varphi_i)_1, (\varphi_i)_2, (\varphi_i)_3, (\varphi_i)_4 \in \mathcal{S}$. *Then*

$$(T_m - T_{m+1}) - (T_{m+2} - T_{m+3}) \ \text{ is compact on } A^{1+\epsilon}_{\epsilon-1}(\mathbf{D})$$

if and only if

$$\lim_{|z|\to 1}\big[M_{1,2}(z)+M_{3,4}(z)\big]\big[M_{1,3}(z)+M_{2,4}(z)\big]=0\,.\ (5.17)$$

Proof. We first identify the function Q associated with the operator under consideration. Note

$$\begin{aligned}T &:= (T_m - T_{m+1}) - (T_{m+2} - T_{m+3})\\ &= (T_m - T_3) - (T_{m+1} - T_{m+3})\\ &= (T_m - T_{m+3}) + 2(T_{m+3} - T_{m+1}) + (T_{m+1} - T_{m+2})\\ &= (T_m - T_{m+3}) + 2(T_{m+3} - T_{m+2}) + (T_{m+2} - T_{m+1})\\ &= -(T_{m+1} - T_{m+2}) - 2(T_{m+2} - T_m) - (T_m - T_{m+3})\\ &= -(T_{m+1} - T_{m+2}) - 2(T_{m+2} - T_{m+3}) - (T_{m+3} - T_m)\\ &= -(T_{m+2} - T_{m+1}) - 2(T_{m+1} - T_m) - (T_m - T_{m+3})\\ &= -(T_{m+2} - T_{m+1}) - 2(T_{m+1} - T_{m+3}) - (T_{m+3} - T_m)\\ &= (T_{m+3} - T_m) + 2(T_m - T_{m+1}) + (T_{m+1} - T_{m+2})\\ &= (T_{m+3} - T_m) + 2(T_m - T_{m+2}) + (T_{m+2} - T_{m+1}).\end{aligned}$$

Associated with these representations, put

$$\begin{aligned}&S_1 := M_{1,2} + M_{3,4}, && S_2 := M_{1,3} + M_{2,4}\\ &S_3 := M_{1,4} + 2M_{4,2} + M_{2,3}, && S_4 := M_{1,4} + 2M_{4,3} + M_{3,2}\\ &S_5 := M_{2,3} + 2M_{3,1} + M_{1,4}, && S_6 := M_{2,3} + 2M_{3,4} + M_{4,1}\\ &S_7 := M_{3,2} + 2M_{2,1} + M_{1,4}, && S_8 := M_{3,2} + 2M_{2,4} + M_{4,1}\\ &S_9 := M_{4,1} + 2M_{1,2} + M_{2,3}, && S_{10} := M_{4,1} + 2M_{1,3} + M_{3,2}.\end{aligned}$$

It is routine to check

$$Q = (S_1S_2)^8(S_3S_4\cdots S_9S_{10}).\ (5.18)$$

Hence, to complete the proof, it is sufficient to prove the equivalence

$$\lim_{|z|\to 1} Q(z) = 0 \iff \lim_{|z|\to 1} S_1(z)S_2(z) = 0\ (5.19)$$

by Theorem 1.1. The implication $\Leftarrow$ is clear.

For the implication $\Rightarrow$, we suppose that it fails and will derive a contradiction. So, assume $Q(z)\to 0$ but $S_1(z)S_2(z) \nrightarrow 0$ as $|z|\to 1$. Then there is a sequence $\{z_n\}$ in **D** with $|z_n|\to 1$ such that

$$\inf_n S_1(z_n) > 0,\ \ \inf_n S_2(z_n) > 0\ (5.20)$$

and $\lim_{n\to\infty} S_j(z_n) = 0$ for some $j \ge 3$. We consider the case $j = 3$; proofs for other cases are similar. So, we have $\lim_{n\to\infty} S_3(z_n) = 0$, or said differently,

(i) $\lim_{n\to\infty} M_{1,4}(z_n) = 0$,

(ii) $\lim_{n\to\infty} M_{2,3}(z_n) = 0$,

(iii) $\lim_{n\to\infty} M_{2,4}(z_n) = 0$.

By (5.20) and (iii) we have

$$\inf_n M_{1,3}(z_n) > 0\ (5.21)$$

after passing to a subsequence if necessary. We now split the proof into two cases as follows.

First, consider the case when $R_1(z_n)\to 0$. In this case we have by (i) and Lemma 5.1

$$R_4(z_n)\to 0.\ (5.22)$$

We also have $R_3(z_n)\nrightarrow 0$ by (5.21). It follows from (ii) that

$$\lim_{n\to\infty}\rho_{2,3}(z_n) = 0\,,\ (5.23)$$

after passing to a subsequence if necessary. Therefore we have $R_2(z_n)\nrightarrow 0$ by Lemma 5.1. By the same argument based on (iii) we have

$$\lim_{n\to\infty}\rho_{2,4}(z_n) = 0\ (5.24)$$

after passing to a subsequence if necessary and thus $R_4(z_n)\nrightarrow 0$, which is a contradiction to (5.22).

Next, consider the case when $R_1(z_n)\nrightarrow 0$. In this case note from (i)-(iii) and Lemma 5.1 that $R_j(z_n)\nrightarrow 0$ for $j = 1, 2, 3, 4$. Accordingly, we see from (i)-(iii) again that

$$\rho_{1,4}(z_n) + \rho_{2,3}(z_n) + \rho_{2,4}(z_n)\to 0,$$

after passing to a subsequence if necessary. This implies $\rho_{1,3}(z_n)\to 0$, which is also a contradiction to (5.21). The proof is complete.

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