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Review Paper

The improve sharp Poincaré–Sobolev type inequalities on the hyperbolic spaces $\mathbb{H}^{3+\epsilon}$

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Abstract

Following the method of V. H. Nguyen [34] we establish a smooth $L^{1+\epsilon}$-version of the Poincaré–Sobolev inequalities on the hyperbolic spaces $\mathbb{H}^{3+\epsilon}$. Moreover we relate both the Poincaré (or Hardy) inequality and the Sobolev inequality with the sharp constant on $\mathbb{H}^{3+\epsilon}$. The study is based on the comparison of the $L^{1+\epsilon}$-norm of gradient of the symmetric decreasing rearrangement of a function on both the hyperbolic and the Euclidean spaces, and the sharp Sobolev inequalities on Euclidean spaces. We also give the proof of the Poincaré-Gagliardo-Nirenberg and Poincaré-Morrey-Sobolev inequalities on the hyperbolic spaces $\mathbb{H}^{3+\epsilon}$. We discuss several other Sobolev inequalities on the hyperbolic spaces $\mathbb{H}^{3+\epsilon}$ which generalize the inequalities due to [27] on $\mathbb{H}^2$.
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I. INTRODUCTION

Given $\epsilon \geq 0$, let $\mathbb{H}^{3+\epsilon}$ denote the hyperbolic space of dimension $(3+\epsilon)$. We will use the Poincaré ball model for the hyperbolic space $\mathbb{H}^{3+\epsilon}$, i.e., a unit ball $\mathbb{B}^{3+\epsilon}$ with center at origin of $\mathbb{R}^{3+\epsilon}$ equipped with the metric $g(x) = \frac{4}{(1-|x|^2)^2}\sum_{i=1}^{3+\epsilon} dx_i^2$. The corresponding Riemannian volume element is $dV = \left(\frac{2}{1-|x|^2}\right)^{3+\epsilon} dx$ and for a measurable set $E \subset \mathbb{H}^{3+\epsilon}$, we denote by $V(E) = \int_E dV$. Our result states as follows.

Theorem 1.1 [34]. Let $\epsilon \geq 1$ and $\frac{2(3+\epsilon)}{2+\epsilon} \leq 1+\epsilon < 3+\epsilon$. Then for any $u_j \in W^{1,1+\epsilon}(\mathbb{H}^{3+\epsilon})$ it holds

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j |\nabla_g u_j|_g^{1+\epsilon} dV - \left(\frac{2+\epsilon}{1+\epsilon}\right)^{1+\epsilon} \int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^{1+\epsilon} dV \geq S(3+\epsilon, 1+\epsilon)^{(1+\epsilon)} \left(\int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^{\frac{(3+\epsilon)(1+\epsilon)}{2}} dV\right)^{\frac{2}{3+\epsilon}}, \qquad (1.1)$$

where $\nabla_g = \left(\frac{1-|x|^2}{2}\right)^2 \nabla$ denotes the hyperbolic gradient, $|\nabla_g u_j|_g = \sqrt{g(\nabla_g u_j, \nabla_g u_j)}$ and $S(3+\epsilon, 1+\epsilon)$ is the best constant in the $L^{1+\epsilon}$-Sobolev inequality on $\mathbb{R}^{3+\epsilon}$ (see, e.g., [1,30]). Furthermore, equality holds true in (1.1) if and only if $u_j \equiv 0$.

The most interest of the inequality (1.1) is that it connects both the sharp Poincaré (or Hardy) inequality and the sharp Sobolev inequality on the hyperbolic space $\mathbb{H}^{3+\epsilon}$. Let $\epsilon > 0$, the sharp Poincaré inequality asserts that

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j |\nabla_g u_j|_g^{1+\epsilon} dV \geq \left(\frac{2+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)} \int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^{1+\epsilon} dV, \ u_j \in C_0^\infty(\mathbb{B}^{3+\epsilon}). \qquad (1.2)$$

The constant $\left(\frac{2+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)}$ is sharp and is never attained. This leaves a room for several improvements of the inequality (1.2). Notice that the non achievement of sharp constant does not always imply improvement (e.g., Hardy operator on the Euclidean space $\mathbb{R}^{3+\epsilon}, \epsilon \geq 0$). However, on the hyperbolic space, the operator $-\Delta_{1+\epsilon,\mathbb{H}^{3+\epsilon}} - \left(\frac{2+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)} = \mathrm{div}\left(\left|\nabla_g \cdot\right|_g^{\epsilon-1}\nabla_g \cdot\right) - \left(\frac{2+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)}$ is subcritical, hence improvement is possible. For examples, see [6-8] for the improvements of (1.2) by adding the remainder terms concerning to Hardy weights, i.e., the inequalities of the form

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|\nabla_g u_j\right|_g^{1+\epsilon} dV - \left(\frac{2+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)} \int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^{1+\epsilon} dV \geq (1+\epsilon) \int_{\mathbb{B}^{3+\epsilon}} \sum_j W|u_j|^{1+\epsilon} dV,$$

for some constant $\epsilon \geq 0$ and the weight W satisfying some appropriate conditions. For the case $\epsilon = 1$, the authors in [23] proved the following Poincaré-Sobolev inequalities on $\mathbb{H}^{3+\epsilon}$ with $\epsilon \geq 0$

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|\nabla_g u_j\right|_g^2 dV - \frac{(2+\epsilon)^2}{4} \int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^2 dV$$
$$\geq (1+\epsilon)\left(\int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^{1+\epsilon} dV\right)^{\frac{2}{1+\epsilon}}, \ u_j \in C_0^\infty(\mathbb{H}^{3+\epsilon}), \tag{1.3}$$

where $2 < 1+\epsilon \leq \frac{2(3+\epsilon)}{1+\epsilon}$ and C is constant. The inequality (1.3) is equivalent to the Hardy-Sobolev-Maz'ya inequality on the half spaces (see [26, Section 2.1.6]). Especially, in the case $\epsilon = \sqrt{5}$, we get

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|\nabla_g u_j\right|_g^2 dV - \frac{(2+\epsilon)^2}{4} \int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^2 dV$$
$$\geq C_{(3+\epsilon)}\left(\int_{\mathbb{B}^{3+\epsilon}} \sum_j |u_j|^{\frac{2(3+\epsilon)}{1+\epsilon}} dV\right)^{\frac{1+\epsilon}{3+\epsilon}}, \ u_j \in C_0^\infty(\mathbb{H}^{3+\epsilon}), \tag{1.4}$$

where $C_{(3+\epsilon)}$ denotes the sharp constant for which (1.4) holds. It was shown by [32] that if $\epsilon \geq 0$ then $C_{(4+\epsilon)}$ is attained. Using test function, they show that $C_{(4+\epsilon)} < S(4+\epsilon, 2)$ where $S(4+\epsilon, 2)$ denotes the sharp constant in the L^2-Sobolev inequality in $\mathbb{R}^{4+\epsilon}$. More surprisingly, [5] proved that $C_3 = S(3,2)$ and C_3 is not attained. The non achievement of C_3 was also proved by [23] by a different method. See [21] for the Hardy-Sobolev-Maz'ya inequalities of kind (1.4) for higher order derivatives. Therefore, the inequality (1.1) can be seen as a $L^{1+\epsilon}$ analogue of the result of Benguria, Frank and Loss on the Hardy-Sobolev-Maz'ya inequality on $\mathbb{H}^3$.

On the other hand, the inequality (1.1) can be seen as a concrete example on the hyperbolic space of the AB program on the sharp Sobolev inequality in Riemannian manifolds [17]. Let (M, g) be a complete Riemannian manifold of dimension $\epsilon \geq 0$. We denote by $H^{1,1+\epsilon}(M)$ the completion of $C_0^\infty(M)$ under the norm $\| u_j \|_{H^{1,1+\epsilon}} = \left(\| \nabla u_j \|_{L^{1+\epsilon}(M)}^{1+\epsilon} + \| u_j \|_{L^{1+\epsilon}(M)}^{1+\epsilon}\right)^{1/1+\epsilon}$. We wonder to know that for $\epsilon \geq 0$, is there a constant B such that

$$S(4+\epsilon, 1+2\epsilon)^{1+\epsilon} \| \sum_j u_j \|_{L^{(1+2\epsilon)^*}(M)}^{1+\epsilon} \leq \sum_j \| \nabla u_j \|_{L^{1+2\epsilon}(M)}^{1+\epsilon}$$
$$+ \sum_j B \| u_j \|_{L^{1+2\epsilon}(M)}^{1+\epsilon} \tag{$I_{1+2\epsilon,opt}^{1+\epsilon}$}$$

for any $u_j \in H^{1,1+2\epsilon}(M)$? In the case of complete compact Riemannian manifolds, it was proved by [19, 20], by [16] and by [2] that $\left(I_{1+2\epsilon,opt}^{1+\epsilon}\right)$ holds for $(1+\epsilon) = \min\{2, 1+2\epsilon\}$. This solves a long standing conjecture due to [1]. See the original article by [1] or to [18] or by [17] for a complete survey on the compact Riemannian manifolds. In the case of complete non-compact Riemannian manifolds, there is several results in which $\left(I_{1+2\epsilon,opt}^{1+\epsilon}\right)$ is valid. For example, [3] proved that $\left(I_{1+2\epsilon,opt}^{1+\epsilon}\right)$ holds for any $\epsilon \geq 0$ with $B = 0$ on the Cartan-Hadamard manifolds (i.e., complete simply connected Riemannian manifold) satisfying Cartan-Hadamard conjecture. In particular, $\left(I_{1+2\epsilon,opt}^{1+\epsilon}\right)$ is valid on the hyperbolic spaces for any $\epsilon \geq 0$. Since the inequality (1.1) relates both the sharp Poincaré and sharp Sobolev inequalities, then the constants in (1.1) are sharp and can not be improved. Hence, the (1.1) gives an example in which the sharp second constant B can be explicitly computed. See [18, Theorem 7.7] for some other examples in the case $\epsilon = 1$. Note that, on the hyperbolic space $\mathbb{H}^{4+\epsilon}$, the following inequality holds

$$S(4+\epsilon, 2)^2\left(\int_{\mathbb{H}^{4+\epsilon}} \sum_j |u_j|^{\frac{2(4+\epsilon)}{2+\epsilon}} dV\right)^{\frac{2+\epsilon}{4+\epsilon}} \leq \int_{\mathbb{H}^{4+\epsilon}} \sum_j \left|\nabla_g u_j\right|_g^2 dV$$
$$- \frac{(4+\epsilon)(2+\epsilon)}{4} \int_{\mathbb{H}^{4+\epsilon}} \sum_j |u_j|^2 dV. \tag{1.5}$$

The constant $(4+\epsilon)(2+\epsilon)/4$ is sharp when $\epsilon \geq 0$. By the result of Benguria, Frank and Loss, this constant is not sharp when $\epsilon = 1$. In this case, the sharp constant is $\epsilon = -1, -5$. By this observation, we can not hope the valid of (1.1) for any $\epsilon \geq 0$. We will see below that (1.1) follows by a pointwise estimate for which the condition $\frac{2(4+\epsilon)}{3+\epsilon} \leq 1+\epsilon < 4+\epsilon$ is sharp. However, in the case $\epsilon = 1$, we have $\frac{2(4+\epsilon)}{3+\epsilon} = 3 > 2$. Hence, the condition $1+\epsilon \geq \frac{2(4+\epsilon)}{3+\epsilon}$ maybe is not optimal for the valid of (1.1). So, it is more interesting if we can find the sharp $p_0 \in [1,4+\epsilon)$ such that (1.1) holds for $(1+\epsilon) \in [p_0, 4+\epsilon)$.

We explain briefly the method used in the proof of Theorem 1.1. Our proof lies heavily on the symmetric non-increasing rearrangement arguments. For any function $u_j \in W^{1,1+\epsilon}(\mathbb{H}^{4+\epsilon})$ we define a function u_j^* which is non-increasing rearrangement function of u_j. From this u_j^* we define two new functions $(u_j)_g^\sharp$ on $\mathbb{H}^{4+\epsilon}$ and $(u_j)_e^\sharp$ on $\mathbb{R}^{4+\epsilon}$ by $(u_j)_g^\sharp(x) = u_j^*\left(V\left(B_g(0,\rho(x))\right)\right), x \in \mathbb{B}^{4+\epsilon}$ where $\rho(x) = \ln\frac{1+|x|}{1-|x|}$ denotes the geodesic distance from x to 0, and $B_g(0,1+\epsilon)$ denotes the open geodesic ball center at 0 and radius $\epsilon \geq 0$ on $\mathbb{H}^{4+\epsilon}$, and $(u_j)_e^\sharp(x) = u_j^*(\sigma_{(4+\epsilon)}|x|^{4+\epsilon}), x \in \mathbb{R}^{4+\epsilon}$ where $\sigma_{(4+\epsilon)}$ denotes the volume of unit ball on $\mathbb{R}^{4+\epsilon}$, respectively. The functions $(u_j)_g^\sharp$ and $(u_j)_e^\sharp$ has the same decreasing rearrangement function (which is u_j^*), then $\|(u_j)_g^\sharp\|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})} = \|(u_j)_e^\sharp\|_{L^{1+\epsilon}(\mathbb{R}^{4+\epsilon})} = \| u_j \|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}$ for any $\epsilon \geq 0$. The key in our proof is a result which compares $\|\nabla_g (u_j)_g^\sharp\|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}^{1+\epsilon}$ and $\|\nabla (u_j)_e^\sharp\|_{L^{1+\epsilon}(\mathbb{R}^{4+\epsilon})}^{1+\epsilon}$. Indeed, we will show that

$$\left\|\sum_j \nabla_g (u_j)_g^\sharp\right\|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}^{1+\epsilon} - \left\|\sum_j \nabla (u_j)_e^\sharp\right\|_{L^{1+\epsilon}(\mathbb{R}^{4+\epsilon})}^{1+\epsilon} \geq \left(\frac{3+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)} \left\|\sum_j (u_j)_g^\sharp\right\|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}^{1+\epsilon}.$$

Using the sharp Sobolev inequality on $\mathbb{R}^{4+\epsilon}$ and the Pólya-Szegö principle on $\mathbb{H}^{4+\epsilon}$, we obtain the inequality (1.1).

The approach to prove Theorem 1.1 above also yields the proofs for the following Poincaré-Gagliardo-Nirenberg and Poincaré-Morrey-Sobolev inequalities on the hyperbolic space $\mathbb{H}^{4+\epsilon}$,

Theorem 1.2 (see [34]). Let $\epsilon \geq 0, \frac{2(4+\epsilon)}{3+\epsilon} \leq 1+\epsilon < 4+\epsilon$ and $(1+\epsilon) \in \left(0, \frac{4+\epsilon}{3}\right], \epsilon \neq 0$. Then for any $u_j \in C_0^\infty(\mathbb{H}^{4+\epsilon})$, the following inequalities holds.

(i) If $\epsilon > 0$, then we have

$$\| \sum_j u_j \|_{L^{(1+\epsilon)^2}(\mathbb{H}^{4+\epsilon})} \leq GN(4+\epsilon, 1+\epsilon, 1+\epsilon)$$
$$\sum_j \left(\|\nabla_g u_j\|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}^{1+\epsilon} - \left(\frac{3+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)} \| u_j \|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}^{1+\epsilon}\right) \| u_j \|_{L^{(1+\epsilon)(\epsilon)+1}(\mathbb{H}^{4+\epsilon})}^{-\epsilon}, \quad (1.6)$$

with $(1+\epsilon) = \frac{(4+\epsilon)(\epsilon)}{(1+\epsilon)((4+\epsilon)(1+\epsilon)-((1+\epsilon)^2-\epsilon))(3)))}$.

(ii) If $0 < \epsilon < 1$, then we have

$$\| \sum_j u_j \|_{L^{(1-\epsilon)(\epsilon)+1}(\mathbb{H}^{4+\epsilon})} \leq GN(4+\epsilon, 1+\epsilon, 1-\epsilon)$$
$$\sum_j \left(\|\nabla_g u_j\|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}^{1+\epsilon} - \left(\frac{3+\epsilon}{1+\epsilon}\right)^{(1+\epsilon)} \| u_j \|_{L^{1+\epsilon}(\mathbb{H}^{4+\epsilon})}^{1+\epsilon}\right) \| u_j \|_{L^{(1-\epsilon)(1+\epsilon)}(\mathbb{H}^{4+\epsilon})}^{-\epsilon}, \quad (1.7)$$

with $(1+\epsilon) = \frac{(4+\epsilon)(+\epsilon)}{(1-\epsilon^2+\epsilon))((4+\epsilon)-(1-\epsilon)(3)}$.

The constant $G(4+\epsilon, 1+\epsilon, 1-\epsilon)$ which appears in (1.6) and (1.7) denotes the sharp constant in the Gagliardo-Nirenberg inequality on $\mathbb{R}^{4+\epsilon}$ (see, e.g., [13-15]).

Suppose that $\epsilon \geq 0$. Then for any function $u_j \in C_0^\infty(\mathbb{H}^{2+\epsilon})$, it holds

$$\| \sum_j u_j \|_\infty^{3+\epsilon} \leq b_{2+\epsilon,3+\epsilon}^{3+\epsilon} \sum_j V(\operatorname{supp} u_j)^{\frac{1}{2+\epsilon}} \left(\int_{\mathbb{B}^{2+\epsilon}} |\nabla_g u_j|_g^{3+\epsilon} dV - \left(\frac{1+\epsilon}{3+\epsilon}\right)^{(3+\epsilon)} \int_{\mathbb{B}^{2+\epsilon}} |u_j|^{3+\epsilon} dV\right) \quad (1.8)$$

where $\operatorname{supp} u_j$ denotes the support of the function u_j, and $b_{2+\epsilon,3+\epsilon}$ is the sharp constant in the Morrey-Sobolev inequality on $\mathbb{R}^{2+\epsilon}$ (see, e.g., [31]).

Similar to (1.1), the inequalities (1.6), (1.7) and (1.8) relate both the sharp Poincaré inequality and the sharp Gagliardo-Nirenberg and the sharp Morrey-Sobolev inequalities on the hyperbolic spaces $\mathbb{H}^{2+\epsilon}$, so they can not be improved on the constants. The inequality (1.1) is a special case of (1.6). The case $\epsilon = 1$ is not included in Theorems 1.1 and 1.2. In this situation, there are some Hardy-Moser-Trudinger type inequalities (see, e.g., [22, 24, 25, 28, 29, 33]). See [9-12] for more information about the Gagliardo-Nirenberg inequality in the compact Riemannian manifolds.

Here, we recall some basic facts about the symmetric decreasing rearrangement of function on the hyperbolic space $\mathbb{H}^{2+\epsilon}$ and prove an important result relating the symmetric decreasing rearrangement of function both on hyperbolic space and Euclidean space. We devoted to prove Theorems 1.1 and 1.2. We discuss some related Sobolev inequalities on hyperbolic space which generalize the inequalities due to [27] on $\mathbb{H}^2$ to higher dimension.

2. Symmetric Decreasing Rearrangements

It is now known that the symmetrization argument works well in the setting of the hyperbolic spaces $\mathbb{H}^{2+\epsilon}$ (see, e.g., [4]). We recall some facts about the rearrangement on the hyperbolic spaces. Let $u_j: \mathbb{H}^{2+\epsilon} \to \mathbb{R}$ be a function such that

$$\mathrm{Vol}_g\left(\{x \in \mathbb{H}^{2+\epsilon}: |u_j(x)| > 1+\epsilon\}\right) = \int_{\{x\in\mathbb{H}^{2+\epsilon}:|u_j(x)|>1+\epsilon\}} \sum_j dV < \infty,\ \forall \epsilon \geq 0.$$

For such a function u_j, its distribution function, denoted by μ_{u_j}, is defined by

$$\mu_{u_j}(1+\epsilon) = V\{x \in \mathbb{H}^{2+\epsilon}: |u_j(x)| > 1+\epsilon\},\ \epsilon \geq 0.$$

The function $(0,\infty) \ni 1+\epsilon \mapsto \mu_{u_j}(1+\epsilon)$ is non-increasing and right-continuous. Then the decreasing rearrangement function u_j^* of u_j is defined by

$$u_j^*(1+\epsilon) = \sup\left\{\epsilon \geq 0: \mu_{u_j}(1+\epsilon) > 1+\epsilon\right\}.$$

Note that the function $(0,\infty) \ni 1+\epsilon \to u_j^*(1+\epsilon)$ is non-increasing. We now define the symmetric decreasing rearrangement function $(u_j)_g^\sharp$ of u_j by

$$(u_j)_g^\sharp(x) = u_j^*\left(V\left(B_g(0,\rho(x))\right)\right),\ x \in \mathbb{B}^{2+\epsilon}. \tag{2.1}$$

We also define a function $(u_j)_e^\sharp$ on $\mathbb{R}^{2+\epsilon}$ by

$$(u_j)_e^\sharp(x) = u_j^*\left(\sigma_{(2+\epsilon)}|x|^{2+\epsilon}\right),\ x \in \mathbb{R}^{2+\epsilon}, \tag{2.2}$$

where $\sigma_{(2+\epsilon)}$ denotes the volume of unit ball on $\mathbb{R}^{2+\epsilon}$. Since u_j, $(u_j)_g^\sharp$ and $(u_j)_e^\sharp$ has the same non-increasing rearrangement function (which is u_j^*), then we have

$$\int_{\mathbb{B}^{2+\epsilon}} \sum_j \Phi(|u_j|)dV = \int_{\mathbb{B}^{2+\epsilon}} \sum_j \Phi\left(\left(u_j\right)_g^\sharp\right)dV$$
$$= \int_{\mathbb{R}^{2+\epsilon}} \sum_j \Phi\left((u_j)_e^\sharp\right)dx = \int_0^\infty \sum_j \Phi\left(u_j^*(1+\epsilon)\right)d(1+\epsilon), \tag{2.3}$$

for any increasing function $\Phi: [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$. This equality is a consequence of layer cake representation. Moreover, by Pólya-Szegö principle, we have

$$\int_{\mathbb{B}^{2+\epsilon}} \sum_j \left|\nabla_g (u_j)_g^\sharp\right|_g^{3+\epsilon} dV \leq \int_{\mathbb{B}^{2+\epsilon}} \sum_j \left|\nabla_g u_j\right|_g^{3+\epsilon} dV. \tag{2.4}$$

We compare $\|\nabla_g (u_j)_g^\sharp\|_{L^{3+\epsilon}(\mathbb{H}^{2+\epsilon})}^{3+\epsilon}$ and $\|\nabla (u_j)_e^\sharp\|_{L^{3+\epsilon}(\mathbb{R}^{2+\epsilon})}^{3+\epsilon}$. For simplifying notation, we denote $v_j = u_j^*$. By a straightforward computation, we have

$$\int_{\mathbb{R}^{2+\epsilon}} \sum_j \left|\nabla (u_j)_e^\sharp\right|^{3+\epsilon} dx = \left((2+\epsilon)\sigma_{(2+\epsilon)}\right)^{3+\epsilon} \int_0^\infty \sum_j \left|v_j'(1+\epsilon)\right|^{3+\epsilon} \left(\frac{1+\epsilon}{\sigma_{(2+\epsilon)}}\right)^{\frac{(1+\epsilon)(3+\epsilon)}{2+\epsilon}} d(1+\epsilon). \tag{2.5}$$

Note that

$$V(B(0,\rho(x))) = (2+\epsilon)\sigma_{(2+\epsilon)} \int_0^{\rho(x)} (\sinh(1+\epsilon))^{1+\epsilon} d(1+\epsilon) = \sigma_{(2+\epsilon)}\Phi(\rho(x)),$$

where

$$\Phi(1+\epsilon) = (2+\epsilon)\int_0^{1+\epsilon} (\sinh(1+\epsilon))^{1+\epsilon} d(1+\epsilon). \tag{2.6}$$

Note that the function $\Phi: [0,\infty) \to [0,\infty)$ is a diffeomorphism, strictly increasing with $\Phi(0) = 0$ and $\lim_{\epsilon\to\infty}\Phi(1+\epsilon) = \infty$. The gradient of $V\left(B_g(0,\rho(x))\right)$ is then given by

$$\nabla_g V\left(B(x_0,\rho(x))\right) = (2+\epsilon)\sigma_{(2+\epsilon)}(\sinh\rho(x))^{1+\epsilon}\nabla_g\rho(x).$$

Since $\left|\nabla_g\rho(x)\right|_g = 1$ for $x \neq 0$, then we get

$$\int_{\mathbb{B}^{2+\epsilon}} \sum_j \left|\nabla_g (u_j)_g^\sharp(x)\right|_g^{3+\epsilon} dV = \int_{\mathbb{B}^{2+\epsilon}} \sum_j \left|v_j'\left(V\left(B_g(0,\rho(x))\right)\right)\right|^{3+\epsilon} \left((2+\epsilon)(\sinh(\rho(x))^{1+\epsilon}\right)^{3+\epsilon} dV$$

$$= (2+\epsilon)\sigma_{(2+\epsilon)} \int_0^\infty \sum_j \left|v_j'\left(V\left(B_g(0,1+\epsilon)\right)\right)\right|^{3+\epsilon} \left((2+\epsilon)\sigma_{(2+\epsilon)}(\sinh(1+\epsilon))^{1+\epsilon}\right)^{3+\epsilon}(\sinh(1+\epsilon))^{1+\epsilon} d(1+\epsilon)$$

$$= \left((2+\epsilon)\sigma_{(2+\epsilon)}\right)^{3+\epsilon} \int_0^\infty \sum_j \left|v_j'\left(V\left(B_g(0,1+\epsilon)\right)\right)\right|^{3+\epsilon} (\sinh(1+\epsilon))^{(3+\epsilon)(1+\epsilon)}(2+\epsilon)\sigma_{(2+\epsilon)}(\sinh(1+\epsilon))^{1+\epsilon} d(1+\epsilon).$$

Making the change of variable $(1+\epsilon) = V\left(B_g(0,1+\epsilon)\right) = \sigma_{(2+\epsilon)}\Phi(1+\epsilon)$ or $(1+\epsilon) = \Phi^{-1}\left(\frac{1+\epsilon}{\sigma_{(2+\epsilon)}}\right)$, we have $d(1+\epsilon) = (2+\epsilon)\sigma_{(2+\epsilon)}(\sinh(1+\epsilon))^{1+\epsilon} d(1+\epsilon)$ and

$$\int_{\mathbb{B}^{2+\epsilon}} \sum_j \left|\nabla_g (u_j)_g^\sharp\right|_g^{3+\epsilon} dV = \left((2+\epsilon)\sigma_{(2+\epsilon)}\right)^{3+\epsilon} \int_0^\infty \sum_j \left|v_j'(1+\epsilon)\right|^{3+\epsilon} \left(\sinh \Phi^{-1}\left(\frac{1+\epsilon}{\sigma_{(2+\epsilon)}}\right)\right)^{(3+\epsilon)(1+\epsilon)} d(1+\epsilon). \quad (2.7)$$

We define the function $k_{2+\epsilon,3+\epsilon}$ on $[0,\infty)$ by

$$k_{2+\epsilon,3+\epsilon}(1+\epsilon) = (\sinh \Phi^{-1}(1+\epsilon))^{(3+\epsilon)(1+\epsilon)} - (1+\epsilon)^{\frac{(3+\epsilon)(1+\epsilon)}{2+\epsilon}}.$$

We then obtain from (2.5) and (2.7) that

$$\int_{\mathbb{B}^{2+\epsilon}} \sum_j \left|\nabla_g (u_j)_g^\sharp\right|_g^{3+\epsilon} dV = \int_{\mathbb{R}^{2+\epsilon}} \sum_j \left|\nabla (u_j)_e^\sharp\right|^{3+\epsilon} dx$$
$$+ \left((2+\epsilon)\sigma_{(2+\epsilon)}\right)^{3+\epsilon} \int_0^\infty \sum_j \left|v_j'(1+\epsilon)\right|^{3+\epsilon} k_{2+\epsilon,3+\epsilon}\left(\frac{1+\epsilon}{\sigma_{(2+\epsilon)}}\right) d(1+\epsilon). \quad (2.8)$$

To proceed, we next find an estimate for $k_{2+\epsilon,3+\epsilon}$ from below. In fact, we have the following results.

Lemma 2.1 (see [34]). It holds

$$k_{2+\epsilon,2+\epsilon}(1+\epsilon) \geq \left(\frac{1+\epsilon}{2+\epsilon}\right)^{2+\epsilon} (1+\epsilon)^{2+\epsilon},\ \epsilon \geq 0, \quad (2.9)$$

for any $\epsilon \geq 0$ if $\epsilon = 0$, and for any $\epsilon \geq 1$ if $\epsilon \geq 0$.

Proof. It is enough to prove that

$$F_{2+\epsilon,2+\epsilon}(1+\epsilon) = k_{2+\epsilon,2+\epsilon}(\Phi(1+\epsilon)) - \left(\frac{1+\epsilon}{2+\epsilon}\right)^{(2+\epsilon)} (\Phi(1+\epsilon))^{2+\epsilon} \geq 0,\ \epsilon \geq -1, \quad (2.10)$$

for any $\epsilon \geq 0$ if $\epsilon = 0$, and for any $\epsilon \geq 1$ if $\epsilon \geq 0$.

If $\epsilon = 0$, we have $\Phi(1+\epsilon) = 2(\cosh \epsilon)$, and

$$F_{2,2+\epsilon}(1+\epsilon) = \left(\frac{\Phi(1+\epsilon)^2}{4} + \Phi(1+\epsilon)\right)^{\frac{2+\epsilon}{2}} - \Phi(1+\epsilon)^{\frac{2+\epsilon}{2}} - \frac{1}{2^{(2+\epsilon)}}\Phi(1+\epsilon)^{2+\epsilon} \geq 0,$$

for any $\epsilon \geq -1$ if $\epsilon \geq 0$.

Suppose that $\epsilon \geq 0$. Differentiating the function $F_{3+\epsilon,2+\epsilon}$ we get

$$\begin{aligned} F_{3+\epsilon,2+\epsilon}'(1+\epsilon) = {} & (2+\epsilon)^2(\sinh(1+\epsilon))^{(2+\epsilon)^2-1}\cosh(1+\epsilon) - (2+\epsilon)^2(\sinh(1+\epsilon))^{2+\epsilon}\Phi(1+\epsilon)^{\frac{(2+\epsilon)^2}{3+\epsilon}-1} \\ & -\left(\frac{2+\epsilon}{3+\epsilon}\right)^{(2+\epsilon)}(2+\epsilon)(3+\epsilon)(\sinh(1+\epsilon))^{2+\epsilon}\Phi(1+\epsilon)^{1+\epsilon} \\ = {} & (2+\epsilon)^2(\sinh(1+\epsilon))^{2+\epsilon}\left((\sinh(1+\epsilon))^{(2+\epsilon)^2-(3+\epsilon)}\cosh(1+\epsilon) - \Phi(1+\epsilon)^{\frac{(2+\epsilon)^2}{3+\epsilon}-1} - \left(\frac{2+\epsilon}{3+\epsilon}\right)^{(2+\epsilon)}\Phi(1+\epsilon)^{1+\epsilon}\right) \\ = {} & :(2+\epsilon)^2(\sinh(1+\epsilon))^{2+\epsilon}G_{3+\epsilon,2+\epsilon}(1+\epsilon). \end{aligned}$$

We continue differentiating the function $G_{3+\epsilon,2+\epsilon}$ to obtain

$$\begin{aligned} G_{3+\epsilon,2+\epsilon}'(1+\epsilon) = {} & ((2+\epsilon)^2-(3+\epsilon))(\sinh(1+\epsilon))^{(2+\epsilon)^2-2+\epsilon}(\cosh(1+\epsilon))^2 + (\sinh(1+\epsilon))^{(1+\epsilon)(2+\epsilon)} \\ & -((2+\epsilon)^2-(3+\epsilon))(\sinh(1+\epsilon))^{2+\epsilon}\Phi(1+\epsilon)^{\frac{(2+\epsilon)^2}{3+\epsilon}-2} - \left(\frac{2+\epsilon}{3+\epsilon}\right)^{1+\epsilon}(1+\epsilon)(3+\epsilon)(\sinh(1+\epsilon))^{2+\epsilon}\Phi(1+\epsilon)^{\epsilon}. \end{aligned}$$

Replacing $(\cosh(1+\epsilon))^2$ by $1+(\sinh(1+\epsilon))^2$, we simplify the expression of $G_{3+\epsilon,2+\epsilon}'$ as

$$\begin{aligned} G_{3+\epsilon,2+\epsilon}'(1+\epsilon) = {} & (1+\epsilon)(2+\epsilon)(\sinh(1+\epsilon))^{(1+\epsilon)(2+\epsilon)} + ((2+\epsilon)^2-(3+\epsilon))(\sinh(1+\epsilon))^{(2+\epsilon)^2-4-\epsilon} \\ & -((2+\epsilon)^2-(3+\epsilon))(\sinh(1+\epsilon))^{2+\epsilon}\Phi(1+\epsilon)^{\frac{(2+\epsilon)^2}{3+\epsilon}-2} - \left(\frac{2+\epsilon}{3+\epsilon}\right)^{1+\epsilon}(1+\epsilon)(3+\epsilon)(\sinh(1+\epsilon))^{2+\epsilon}\Phi(1+\epsilon)^{\epsilon} \\ = {} & (1+\epsilon)(2+\epsilon)(\sinh(1+\epsilon))^{2+\epsilon}\left((\sinh(1+\epsilon))^{(\epsilon)(2+\epsilon)} - \left(\frac{2+\epsilon}{3+\epsilon}\right)^{\epsilon}\Phi(1+\epsilon)^{\epsilon}\right) \\ & +((2+\epsilon)^2-(3+\epsilon))(\sinh(1+\epsilon))^{2+\epsilon}\left((\sinh(1+\epsilon))^{(2+\epsilon)^2-2(3+\epsilon)} - \Phi(1+\epsilon)^{\frac{(2+\epsilon)^2}{3+\epsilon}-2}\right). \end{aligned}$$

It is easy to see that

$$\Phi(1+\epsilon) = (3+\epsilon)\int_0^{1+\epsilon} (\sinh(1+\epsilon))^{2+\epsilon} d(1+\epsilon)$$
$$< (3+\epsilon)\int_0^{1+\epsilon} (\sinh(1+\epsilon))^{2+\epsilon}\cosh(1+\epsilon) d(1+\epsilon) = (\sinh(1+\epsilon))^{3+\epsilon},\ \epsilon \geq 0,$$

and

$$\Phi(1+\epsilon) = (3+\epsilon)\int_0^{1+\epsilon} (\sinh(1+\epsilon))^{2+\epsilon} d(1+\epsilon)$$
$$< (3+\epsilon)\int_0^{1+\epsilon} (\sinh(1+\epsilon))^{1+\epsilon}\cosh(1+\epsilon) d(1+\epsilon)$$
$$= \frac{3+\epsilon}{2+\epsilon}(\sinh(1+\epsilon))^{2+\epsilon},\ \epsilon \geq 0.$$

Plugging these previous estimates into the expression of $G'_{3+\epsilon,2+\epsilon}$, we get that $G'_{3+\epsilon,2+\epsilon}(1+\epsilon) > 0$ for any $\epsilon \geq 0$. This implies $G_{3+\epsilon,2+\epsilon}(1+\epsilon) > G_{3+\epsilon,2+\epsilon}(0) = 0$ for any $\epsilon \geq 0$, or equivalently $F'_{3+\epsilon,2+\epsilon}(1+\epsilon) > 0$ for any $\epsilon \geq 0$. Consequently, $F_{3+\epsilon,2+\epsilon}(1+\epsilon) > F(0) = 0$ for any $\epsilon \geq 0$. This completes our proof.
It is remarkable that the pointwise estimate (2.9) is sharp in $(1+\epsilon)$. Indeed, if $\epsilon = -1$ and $0 < \epsilon < 1$ then a reversed estimate of (2.9) holds. Suppose that $\epsilon \geq 0$ and $1+\epsilon < \frac{2(3+\epsilon)}{2+\epsilon}$ we next show that a reversed estimate of (2.9) holds for $(1+\epsilon)$ large enough. Indeed, suppose that $\epsilon \geq 0$, we have

$$\Phi(1+\epsilon) = (4+\epsilon)2^{3-\epsilon}\int_0^{1+\epsilon} \left(e^{(1+\epsilon)} - e^{-(1+\epsilon)}\right)^{3+\epsilon} d(1+\epsilon)$$
$$= \frac{(4+\epsilon)2^{3-\epsilon}}{3+\epsilon} e^{(3+\epsilon)(1+\epsilon)}\left(1 - \frac{(3+\epsilon)^2}{1+\epsilon} e^{-2(1+\epsilon)} + o\left(e^{-2(1+\epsilon)}\right)\right),$$

as $\epsilon \to \infty$. Consequently

$$\Phi(1+\epsilon)^{\frac{(1+\epsilon)(3+\epsilon)}{4+\epsilon}} = \left(\frac{(4+\epsilon)2^{3-\epsilon}}{3+\epsilon}\right)^{\frac{(1+\epsilon)(3+\epsilon)}{4+\epsilon}} e^{\frac{(1+\epsilon)(3+\epsilon)^2}{4+\epsilon}(1+\epsilon)}\left(1 - \frac{(3+\epsilon)^3}{(4+\epsilon)} e^{-2(1+\epsilon)} + o\left(e^{-2(1+\epsilon)}\right)\right),$$

and

$$\Phi(1+\epsilon)^{1+\epsilon} = \left(\frac{(4+\epsilon)2^{3-\epsilon}}{3+\epsilon}\right)^{(1+\epsilon)} e^{(1+\epsilon)^2(3+\epsilon)}\left(1 - (3+\epsilon)^2 e^{-2(1+\epsilon)} + o\left(e^{-2(1+\epsilon)}\right)\right)$$

as $\epsilon \to \infty$. Note that

$$(\sinh(1+\epsilon))^{(1+\epsilon)(3+\epsilon)} = 2^{(1+\epsilon)(-3-\epsilon)} e^{(1+\epsilon)^2(3+\epsilon)}\left(1 - (1+\epsilon)(3+\epsilon)e^{-2(1+\epsilon)} + o\left(e^{-2(1+\epsilon)}\right)\right),$$
$$(\sinh(1+\epsilon))^{(1+\epsilon)(3+\epsilon)} = 2^{(1+\epsilon)(-3-\epsilon)} e^{(1+\epsilon)^2(3+\epsilon)}\left(1 - (1+\epsilon)(3+\epsilon)e^{-2(1+\epsilon)} + o\left(e^{-2(1+\epsilon)}\right)\right),$$

as $\epsilon \to \infty$. Therefore

$$F_{4+\epsilon,1+\epsilon}(1+\epsilon) = (\sinh(1+\epsilon))^{(1+\epsilon)(3+\epsilon)} - \Phi(1+\epsilon)^{\frac{(1+\epsilon)(3+\epsilon)}{4+\epsilon}} - \left(\frac{3+\epsilon}{4+\epsilon}\right)^{(1+\epsilon)} \Phi(1+\epsilon)^{1+\epsilon}$$
$$= 2^{(1+\epsilon)(-3-\epsilon)} e^{(1+\epsilon)^2(3+\epsilon)-2(1+\epsilon)}\left(2(3+\epsilon) - \left(\frac{(4+\epsilon)2^{-3-\epsilon}}{3+\epsilon}\right)^{\frac{(1+\epsilon)(3+\epsilon)}{4+\epsilon}} e^{2(1+\epsilon) - \frac{(1+\epsilon)^2(3+\epsilon)}{4+\epsilon}} + o(1)\right)$$

as $\epsilon \to \infty$. If $1+\epsilon < \frac{2(4+\epsilon)}{3+\epsilon}$ we then have $\frac{(1+\epsilon)(3+\epsilon)}{4+\epsilon} < 2$, and $F_{4+\epsilon,1+\epsilon}(1+\epsilon) < 0$ for $\epsilon \geq 0$ large enough.
Suppose $\epsilon = 0$, we have

$$\Phi(1+\epsilon) = \frac{3}{8}\left(e^{2(1+\epsilon)} - e^{-2(1+\epsilon)} - 4(1+\epsilon)\right)$$
$$= \frac{3}{8} e^{2(1+\epsilon)}\left(1 - 4(1+\epsilon)e^{-2(1+\epsilon)} + o\left((1+\epsilon)e^{-2(1+\epsilon)}\right)\right),$$

as $\epsilon \to \infty$. Hence

$$\Phi(1+\epsilon)^{1+\epsilon} = \frac{3^{1+\epsilon}}{8^{1+\epsilon}} e^{2(1+\epsilon)^2}\left(1 - 4(1+\epsilon)^2 e^{-2(1+\epsilon)} + o\left(e^{-2(1+\epsilon)}(1+\epsilon)\right)\right),$$

and

$$\Phi(1+\epsilon)^{\frac{2(1+\epsilon)}{3}} = \left(\frac{3}{8}\right)^{\frac{2(1+\epsilon)}{3}} e^{\frac{4(1+\epsilon)^2}{3}}\left(1 - \frac{8(1+\epsilon)^2}{3} e^{-2(1+\epsilon)} + o\left((1+\epsilon)e^{-2(1+\epsilon)}\right)\right),$$

as $\epsilon \to \infty$. Evidently,

$$(\sinh(1+\epsilon))^{2(1+\epsilon)} = 2^{-2(1+\epsilon)} e^{2(1+\epsilon)^2}\left(1 + o\left((1+\epsilon)e^{-2(1+\epsilon)}\right)\right),$$

as $\epsilon \to \infty$. Consequently, we get

$$F_{3,1+\epsilon}(1+\epsilon) = \frac{1}{4^{1+\epsilon}} e^{2(\epsilon)(1+\epsilon)}(1+\epsilon)\left(4(1+\epsilon) - 4^{1+\epsilon}\left(\frac{3}{8}\right)^{\frac{2(1+\epsilon)}{3}} (1+\epsilon)^{-1} e^{2(1+\epsilon) - \frac{2(1+\epsilon)^2}{3}} + o(1)\right),$$

as $\epsilon \to \infty$. Since $\epsilon < 0$, then we have $\epsilon > 0$ and hence $F_{3,3-\epsilon}(1+\epsilon) < 0$ for $\epsilon \geq 0$ large enough.

Combining (2.9) and (2.8) together, we arrive

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|\nabla_g (u_j)_g^\sharp\right|_g^{3-\epsilon} dV \geq \int_{\mathbb{R}^{3+\epsilon}} \sum_j \left|\nabla (u_j)_e^\sharp\right|^{3-\epsilon} dx$$

$$+(2+\epsilon)^{3-\epsilon} \int_0^\infty \sum_j \left|v_j'(1+\epsilon)\right|^{3-\epsilon} (1+\epsilon)^{3-\epsilon} d(1+\epsilon). \tag{2.11}$$

Making the change of function $w_j(1+\epsilon) = v_j(1+\epsilon)(1+\epsilon)^{\frac{1}{3-\epsilon}}$ or equivalently $v_j(1+\epsilon) = w_j(1+\epsilon)(1+\epsilon)^{-\frac{1}{3-\epsilon}}$. Differentiating the function v_j, we have

$$v_j'(1+\epsilon) = w_j'(1+\epsilon)(1+\epsilon)^{-\frac{1}{3-\epsilon}} - \frac{1}{3-\epsilon} w_j(1+\epsilon)(1+\epsilon)^{-\frac{1}{3-\epsilon}-1}.$$

We can readily check that if $a-b \leq 0, b \geq 0$ and $\epsilon \geq 0$ then

$$|a-b|^{2+\epsilon} \geq |a|^{2+\epsilon} + |b|^{2+\epsilon} - (2+\epsilon)ab^{1+\epsilon}$$

Since $v_j' \leq 0$, applying the previous inequality we get

$$\int_0^\infty \sum_j |v_j'(1+\epsilon)|^{2+\epsilon}(1+\epsilon)^{2+\epsilon}d(1+\epsilon) \geq \int_0^\infty \sum_j |w_j'(1+\epsilon)|^{2+\epsilon}(1+\epsilon)^{1+\epsilon}d(1+\epsilon) + \frac{1}{p^{(2+\epsilon)}}\int_0^\infty \sum_j w_j(1+\epsilon)^{2+\epsilon}(1+\epsilon)^{-1}d(1+\epsilon) - p^{-\epsilon}\int_0^\infty \sum_j w_j'(1+\epsilon)w_j(1+\epsilon)^{1+\epsilon}d(1+\epsilon)$$

$$= \int_0^\infty \sum_j |w_j'(1+\epsilon)|^{2+\epsilon}(1+\epsilon)^{1+\epsilon}d(1+\epsilon) + \frac{1}{p^{(2+\epsilon)}}\int_0^\infty \sum_j v_j(1+\epsilon)^{2+\epsilon}d(1+\epsilon),$$

here we use integration by parts. Plugging this estimate into (2.11) we get

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|\nabla_g (u_j)_g^\sharp\right|_g^{2+\epsilon} dV \geq \int_{\mathbb{R}^{3+\epsilon}} \sum_j \left|\nabla (u_j)_e^\sharp\right|^{2+\epsilon} dx$$

$$+\frac{(2+\epsilon)^{2+\epsilon}}{p^{(2+\epsilon)}} \int_0^\infty \sum_j |v_j|^{2+\epsilon} d(1+\epsilon) + (2+\epsilon)^{2+\epsilon} \int_0^\infty \sum_j \left|\left(v_j(1+\epsilon)(1+\epsilon)^{\frac{1}{2+\epsilon}}\right)'\right|^{2+\epsilon} (1+\epsilon)^{1+\epsilon} d(1+\epsilon). \tag{2.12}$$

Since $v_j = u_j^*$ is non-increasing rearrangement function of $(u_j)_g^\sharp$, then

$$\int_0^\infty \sum_j |v_j|^{2+\epsilon} d(1+\epsilon) = \int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|(u_j)_g^\sharp\right|^{2+\epsilon} dV. \tag{2.13}$$

Plugging (2.13) into (2.12), we obtain the main result as follows,

Theorem 2.2 (see [34]). Let $\epsilon \geq 0$ if $\epsilon = 1$ and $2+\epsilon \geq \frac{2(3+\epsilon)}{2+\epsilon}$ if $\epsilon \geq 0$. It holds

$$\int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|\nabla_g (u_j)_g^\sharp\right|_g^{2+\epsilon} dV - \int_{\mathbb{B}^{3+\epsilon}} \sum_j \left|(u_j)_g^\sharp\right|^{2+\epsilon} dV$$

$$\geq \int_{\mathbb{R}^{3+\epsilon}} \sum_j \left|\nabla (u_j)_e^\sharp\right|^{2+\epsilon} dx. \tag{2.14}$$

Theorem 2.2 was proved in [29] in the case $\epsilon = 1$ as a key to establish several improved Moser-Trudinger type inequalities on the hyperbolic space.

3. Proof of Theorems 1.1 and 1.2

We provide the proof of Theorem 1.1 and Theorem 1.2. Our proof uses Theorem 2.2 above and the known inequalities on the Euclidean spaces such as the sharp Sobolev, Gagliardo-Nirenberg and Morrey-Sobolev inequalities. We recall them here. The sharp Sobolev inequality on the Euclidean space was independently proved by [1, 30] and has the form

$$S(1+2\epsilon, 1+\epsilon) \parallel \sum_j u_j \parallel_{L^{(1+\epsilon)^*}(\mathbb{R}^{1+2\epsilon})} \leq \parallel \sum_j \nabla u_j \parallel_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}, \; u_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon}), \tag{3.1}$$

for $\epsilon > 0, (1+\epsilon)^* = \frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}$ and the sharp constant $S(1+2\epsilon, 1+\epsilon)$ is given by

$$S(1+2\epsilon, 1+\epsilon) = \left[(1+2\epsilon)^{\epsilon-1}\left(\frac{\Gamma(1+2\epsilon)}{\Gamma\left(\frac{1+2\epsilon}{1+\epsilon}\right)\Gamma\left(2(1+\epsilon) - \frac{1+2\epsilon}{1+\epsilon}\right)\sigma_{(1+2\epsilon)}}\right)^{\frac{1}{1+2\epsilon}}\right]^{-1},$$

where $\Gamma(x) = \int_0^\infty (1+\epsilon)^{x-1} e^{-(1+\epsilon)} d(1+\epsilon), x > 0$ denotes the usual Gamma function. The family of extremal functions is determined uniquely by the function $u_j(x) = \left(1 + |x|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{(\epsilon)}{1+\epsilon}}$ up to a translation, dilatation and multiplying by constant.

Let $\epsilon > 0$ and $(1-\epsilon) \in \left(0, \frac{1+2\epsilon}{\epsilon}\right], \epsilon \neq 0$. The sharp Gagliardo-Nirenberg inequalities in $\mathbb{R}^{1+2\epsilon}$ was established by [14, 15] and has the forms:

(i) for $\epsilon > 0$,

$$\| \sum_j u_j \|_{L^{(1+\epsilon)^2}(\mathbb{R}^{1+2\epsilon})} \leq GN(1+2\epsilon, 1+\epsilon, 1+\epsilon)$$

$$\| \sum_j \nabla u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{1+\epsilon} \| u_j \|_{L^{(1+\epsilon)(\epsilon)+1}(\mathbb{R}^{1+2\epsilon})}^{-\epsilon}, \; u_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon}), \tag{3.2}$$

with $(1+\epsilon) = \frac{(1+2\epsilon)(\epsilon)}{(1+\epsilon)((1+2\epsilon)-((1+\epsilon)^2-\epsilon))(\epsilon))}$, the sharp constant $GN(1+2\epsilon, 1+\epsilon, 1+\epsilon)$ is given by

$$GN(1+2\epsilon,1+\epsilon,1+\epsilon) = \left(\frac{\epsilon+1}{(1+\epsilon)\sqrt{\pi}}\right)^{1+\epsilon} \left(\frac{(1+\epsilon)^2\epsilon+1}{(1+2\epsilon)((1+\epsilon)\epsilon-\epsilon))}\right)^{\frac{1+\epsilon}{1+\epsilon}} \left(\frac{\delta}{(1+\epsilon)^2\epsilon+1}\right)^{\frac{1}{(1+\epsilon)^2}} \left(\frac{\Gamma\left(((1+\epsilon)\epsilon+1)\frac{\epsilon}{((1+\epsilon)\epsilon+1)-(1+\epsilon)}\right)\Gamma\left(\frac{1+2\epsilon}{2}+1\right)}{\Gamma\left(\frac{\epsilon}{1+\epsilon}\frac{\delta}{((1+\epsilon)\epsilon+1)-(1+\epsilon)}\right)\Gamma\left((1+2\epsilon)\frac{\epsilon}{1+\epsilon}+1\right)}\right)^{\frac{1+\epsilon}{1+2\epsilon}},$$

with $q = (1+\epsilon)\epsilon+1, \delta = (1+2\epsilon)(1+\epsilon) - (1+\epsilon)\epsilon^2 + 1$, and an extremal functions is given the function $u_j(x) = \left(1 + |x|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{1}{\epsilon}}$,

(ii) for $\epsilon < 1$,

$$\| \sum_j u_j \|_{L^{(1+\epsilon)(\epsilon)+1}(\mathbb{R}^{1+2\epsilon})} \leq GN(1+2\epsilon, 1+\epsilon, 1+\epsilon)$$

$$\sum_j \| \nabla u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{1+\epsilon} \| u_j \|_{L^{(1+\epsilon)^2}(\mathbb{R}^{1+2\epsilon})}^{-\epsilon}, \; u_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon}), \tag{3.3}$$

with $(1+\epsilon) = \frac{(1+2\epsilon)(-\epsilon)}{((1+\epsilon)^2-\epsilon)((1+2\epsilon)-(1+\epsilon)\epsilon)}$, the sharp constant $GN(1+2\epsilon, 1+\epsilon, 1+\epsilon)$ is given by

$$GN(1+2\epsilon,1+\epsilon,1+\epsilon) = \left(\frac{1+\epsilon-((1+\epsilon)\epsilon+1)}{(1+\epsilon)\sqrt{\pi}}\right)^{1+\epsilon} \left(\frac{(1+\epsilon)((1+\epsilon)\epsilon+1)}{(1+2\epsilon)(1+\epsilon-((1+\epsilon)\epsilon+1))}\right)^{\frac{1+\epsilon}{1+\epsilon}} \left(\frac{((1+\epsilon)^2\epsilon+1)}{\delta}\right)^{\frac{-\epsilon}{(1+\epsilon)^2}} \left(\frac{\Gamma\left(\frac{\epsilon}{1+\epsilon}\frac{\delta}{1+\epsilon-((1+\epsilon)\epsilon+1)}+1\right)\Gamma\left(\frac{1+2\epsilon}{2}+1\right)}{\Gamma\left(((1+\epsilon)\epsilon+1)\frac{\epsilon}{1+\epsilon-((1+\epsilon)\epsilon+1)}+1\right)\Gamma\left((1+2\epsilon)\frac{\epsilon}{1+\epsilon}+1\right)}\right)^{\frac{1+\epsilon}{1+2\epsilon}},$$

with $((1+\epsilon)\epsilon+1) = (1+\epsilon)(\epsilon)+1, \delta = (1+2\epsilon)(1+\epsilon) - ((1+\epsilon)\epsilon+1)(\epsilon) > 0$, and an extremal functions is given the function $u_j(x) = \left(1 - |x|^{\frac{1+\epsilon}{\epsilon}}\right)_+^{\frac{1}{-\epsilon}}$, where $a_+ = \max\{a, 0\}$ denotes the positive part of a number a.

See [13] for a completely different proof of the sharp Sobolev inequality (3.1) and the sharp Gagliardo-Nirenberg inequality (3.2) and (3.3) by using the mass transportation method.

Finally, we recall the sharp Morrey-Sobolev inequality on $\mathbb{R}^{1+2\epsilon}$. Given $(1+\epsilon) > (1+2\epsilon)$, then for any function $u_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$, the following inequality holds

$$\| \sum_j u_j \|_{L^\infty(\mathbb{R}^{1+2\epsilon})} \leq b_{1+2\epsilon,1+\epsilon} \sum_j \mathrm{Vol}(\mathrm{supp}\, u_j)^{\frac{-\epsilon}{(1+2\epsilon)(1+\epsilon)}} \| \nabla u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}, \tag{3.4}$$

here Vol denotes the Lebesgue measure of any measurable subset of $\mathbb{R}^{1+2\epsilon}$, the sharp constant $b_{1+2\epsilon,1+\epsilon}$ is given by

$$b_{1+2\epsilon,1+\epsilon} = (1+2\epsilon)^{-\frac{1}{1+\epsilon}} \sigma_{(1+2\epsilon)}^{\frac{1}{\frac{1+2\epsilon}{}}} (-1)^{\frac{\epsilon}{1+\epsilon}},$$

and an extremal function is given by $u_j(x) = \left(1 - |x|^{\frac{-\epsilon}{\epsilon}}\right)_+$. For more about this inequality, see [31].

Proof of Theorem 1.1 (see [34]). Suppose u_j is a function on $W^{1,2+\epsilon}(\mathbb{H}^{1+2\epsilon})$. Let us define two new functions $(u_j)_g^\sharp$ and $(u_j)_e^\sharp$ by (2.1) and (2.2) respectively. Theorem 2.2 implies

$$\sum_j \|\nabla_g (u_j)_g^\sharp\|_{L^{2+\epsilon}(\mathbb{H}^{1+2\epsilon})}^{2+\epsilon} - \left(\frac{2\epsilon}{2+\epsilon}\right)^{2+\epsilon} \sum_j \|(u_j)_g^\sharp\|_{L^{2+\epsilon}(\mathbb{H}^{1+2\epsilon})}^{2+\epsilon} \geq \left\|\sum_j \nabla (u_j)_e^\sharp\right\|_{L^{2+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{2+\epsilon},$$

for any $\epsilon \geq 0$ if $\epsilon = 1$, and for any $2+\epsilon \geq \frac{2(3+\epsilon)}{2+\epsilon}$ if $\epsilon \geq 0$. Note that $\|(u_j)_g^\sharp\|_{L^{2+\epsilon}(\mathbb{H}^{3+\epsilon})} = \| u_j \|_{L^{2+\epsilon}(\mathbb{H}^{3+\epsilon})}$.
Hence, applying Pólya-Szegö principle (2.4) and equality (2.3), we get

$$\sum_j \|\nabla_g u_j\|_{L^{2+\epsilon}(\mathbb{H}^{3+\epsilon})}^{2+\epsilon} - \left(\frac{2+\epsilon}{2+\epsilon}\right)^{2+\epsilon} \sum_j \| u_j \|_{L^{2+\epsilon}(\mathbb{H}^{3+\epsilon})}^{2+\epsilon} \geq \left\|\sum_j \nabla (u_j)_e^\sharp\right\|_{L^{2+\epsilon}(\mathbb{R}^{3+\epsilon})}^{2+\epsilon}. \tag{3.5}$$

Suppose that $\epsilon \geq 0$ and $\frac{2(4+\epsilon)}{3+\epsilon} \leq 2+\epsilon < 4+\epsilon$. Using the sharp Sobolev inequality (3.1) for $(u_j)_e^\sharp$ and using the equality $\|(u_j)_e^\sharp\|_{L^{(2+\epsilon)^*}(\mathbb{R}^{4+\epsilon})} = \| u_j \|_{L^{(2+\epsilon)^*}(\mathbb{H}^{4+\epsilon})}$, we obtain the desired inequality (1.1).

Suppose $u_j \in W^{1,2+\epsilon}(\mathbb{H}^{4+\epsilon})$ such that the equality in (1.1) holds for u_j. Let $v_j = u_j^*$ the decreasing rearrangement function of u_j on $[0,\infty)$, and define $(u_j)_g^\sharp$ and $(u_j)_e^\sharp$ by (2.1) and (2.2) respectively. Since the equality in (1.1) holds for u_j, we must have $\|\nabla_g u_j\|_{L^{2+\epsilon}(\mathbb{H}^{4+\epsilon})} = \|\nabla_g (u_j)_g^\sharp\|_{L^{2+\epsilon}(\mathbb{H}^{4+\epsilon})}$ and

$$\sum_{,} \|\nabla_g (u_j)_g^\sharp\|_{L^{2+\epsilon}(\mathbb{H}^{4+\epsilon})}^{2+\epsilon} - \left(\frac{3+\epsilon}{2+\epsilon}\right)^{2+\epsilon} \sum_{,} \|(u_j)_g^\sharp\|_{L^{2+\epsilon}(\mathbb{H}^{4+\epsilon})}^{2+\epsilon} = \sum_{,} \|\nabla (u_j)_e^\sharp\|_{L^{2+\epsilon}(\mathbb{R}^{4+\epsilon})}^{2+\epsilon}.$$

From the proof of Theorem 2.2, we see that the second condition implies

$$\int_0^\infty \sum_{,} \left|\left(v_j(1+\epsilon)(1+\epsilon)^{\frac{1}{2+\epsilon}}\right)'\right|^{2+\epsilon} (1+\epsilon)^{1+\epsilon} d(1+\epsilon) = 0.$$

Thus, we have $v_j(1+\epsilon) = c(1+\epsilon)^{-\frac{1}{2+\epsilon}}$ for some constant $c \in \mathbb{R}$. However, $\int_0^\infty \sum_{,} v_j(1+\epsilon)^{2+\epsilon} d(1+\epsilon) < \infty$ which forces $c = 0$. This finishes our proof of Theorem 1.1.

Proof of Theorem 1.2 (see [34]). The proof of Theorem 1.2 is similar with the one of Theorem 1.1. Suppose that $\epsilon \geq 0$ and $\frac{2(4+\epsilon)}{3+\epsilon} \leq 2(1+\epsilon) < 4+\epsilon$. By (3.5), we can apply the sharp Gagliardo-Nirenberg inequalities (3.2) and (3.3) for function $(u_j)_e^\sharp$ to derive the desired inequalities (1.6) and (1.7) as done for the inequality (1.1), respectively.

Suppose that $\epsilon \geq 0$. We note that (3.5) still holds under this condition. We now can apply the sharp Morrey-Sobolev inequality (3.4) for $(u_j)_e^\sharp$ to yield the inequality (1.8) with remark that $\|(u_j)_e^\sharp\|_{L^\infty(\mathbb{R}^{2+\epsilon})} = \| u_j \|_{L^\infty(\mathbb{H}^{2+\epsilon})}$ and $\mathrm{Vol}(\mathrm{supp}(u_j)_e^\sharp) = V(\mathrm{supp}\, u_j)$.

We conclude by a remark in the case $\epsilon \to 1^+$ of the inequality (1.6). Taking the limit as done in [15], we obtain the following Poincaré-Sobolev logarithmic inequality in $\mathbb{H}^{2+\epsilon}$ which is an extension of the optimal Euclidean $L^{2(1+\epsilon)}$-Sobolev logarithmic inequality [14, 15] to the hyperbolic spaces. Suppose $u_j \in W^{1,2(1+\epsilon)}(\mathbb{H}^{2+\epsilon})$ with $\| u_j \|_{L^{2(1+\epsilon)}(\mathbb{H}^{2+\epsilon})} = 1$, it holds

$$\int_{\mathbb{B}^{2+\epsilon}} \sum_{,} |u_j|^{2(1+\epsilon)} \ln\left(|u_j|^{2(1+\epsilon)}\right) dV$$

$$\leq \frac{2+\epsilon}{2(1+\epsilon)} \ln\left(\mathcal{L}_{2+\epsilon,2(1+\epsilon)} \int_{\mathbb{B}^{2+\epsilon}} \sum_{,} \left(|\nabla_g u_j|_g^{2(1+\epsilon)} - \left(\frac{1+\epsilon}{2+\epsilon}\right)^{2(1+\epsilon)} |u_j|^{2(1+\epsilon)}\right) dV\right) \tag{3.6}$$

for any $\epsilon \geq 0$ and $\frac{2(4+\epsilon)}{3+\epsilon} \leq 2(1+\epsilon) < 4+\epsilon$ with the constant $\mathcal{L}_{4+\epsilon,2(1+\epsilon)}$ is given by

$$\mathcal{L}_{4+\epsilon,2(1+\epsilon)} = \frac{2(1+\epsilon)}{4+\epsilon}\left(\frac{1+2\epsilon}{e}\right)^{1+2\epsilon} \pi^{(1+\epsilon)} \left[\frac{\Gamma\left(\frac{4+\epsilon}{2}+1\right)}{\Gamma\left((4+\epsilon)\frac{1+2\epsilon}{2(1+\epsilon)}+1\right)}\right]^{\frac{2(1+\epsilon)}{4+\epsilon}}.$$

4. Other Sobolev Inequalities on the Hyperbolic Spaces

We establish several Sobolev inequalities on the hyperbolic spaces $\mathbb{H}^{4+\epsilon}$. These inequalities generalize the results of [27] on $\mathbb{H}^2$ to higher dimensional spaces. The main results read as follows.

Theorem 4.1 (see [34]). Let $0 \leq \epsilon < \infty$. Then for any function $u_j \in W^{1,1+\epsilon}(\mathbb{H}^{2+\epsilon})$, the following inequalities holds.

(i) If $\epsilon = 0$ then

$$(1+\epsilon)^{2+\epsilon}\left(\int_{\mathbb{B}^{2+\epsilon}} \sum_{,} |u_j| dV\right)^{2+\epsilon} + S(2+\epsilon,1)^{2+\epsilon}\left(\int_{\mathbb{B}^{2+\epsilon}} \sum_{,} |u_j|^{\frac{2+\epsilon}{1+\epsilon}} dV\right)^{1+\epsilon}$$
$$\leq \left(\int_{\mathbb{B}^{2+\epsilon}} \sum_{,} |\nabla_g u_j|_g dV\right)^{2+\epsilon}. \tag{4.1}$$

(ii) If $\epsilon > 0$ then

$$\left(\frac{2\epsilon}{1+\epsilon}\right)^{1+2\epsilon}\left(\int_{\mathbb{B}^{1+2\epsilon}} \sum_{,} |u_j|^{1+\epsilon} dV\right)^{\frac{1+2\epsilon}{1+\epsilon}} + S(1+2\epsilon,1+\epsilon)^{1+2\epsilon}\left(\int_{\mathbb{B}^{1+2\epsilon}} \sum_{,} |u_j|^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}} dV\right)^{\frac{\epsilon}{1+\epsilon}}$$

$$\leq \left(\int_{\mathbb{B}^{1+2\epsilon}} \sum_j \left|\nabla_g u_j\right|_g^{1+\epsilon} dV\right)^{\frac{1+2\epsilon}{1+\epsilon}}. \quad (4.2)$$

(iii) If $0 < \epsilon < \infty$ then

$$\sup_{x\in\mathbb{B}^{1+2\epsilon}} \left|u_j(x)\right| \leq C(1+2\epsilon, 1+\epsilon)\left(\int_{\mathbb{B}^{1+2\epsilon}} \sum_j \left|\nabla_g u_j\right|_g^{1+\epsilon} dV\right)^{\frac{1}{1+\epsilon}}, \quad (4.3)$$

with

$$C(1+2\epsilon, 1+\epsilon) = \left(2^{2\epsilon}(1+2\epsilon)\sigma_{(1+2\epsilon)}\right)^{-\frac{1}{1+\epsilon}}\left(\frac{\Gamma\left(\frac{-1}{2}\right)\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}\right)^{\frac{\epsilon}{1+\epsilon}}.$$

Furthermore, the equality holds in (4.3) if $u_j(x) = v_j\left(V\left(B_g(0,\rho(x))\right)\right)$ with

$$v_j(1+\epsilon) = c\int_{1+\epsilon}^{\infty}\left(\sinh\Phi^{-1}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)\right)^{-2(1+\epsilon)} d(1+\epsilon). \quad (4.4)$$

Obviously, $(a+b)^{1+\epsilon} \leq a^{(1+\epsilon)} + b^{(1+\epsilon)}$ for any $0 \leq \epsilon \leq 1$ and $a, b \geq 0$. As a consequence, the inequality (4.2) is weaker than the inequality (1.1). However,the inequality (4.2) is valid for any $\epsilon \geq 0$ and $\epsilon > 0$.

Proof. The part (i) follows from part (ii) by letting $\epsilon \downarrow 0$. We next prove part (ii). Let $u_j \in W^{1,1+\epsilon}(\mathbb{H}^{1+2\epsilon})$, we define two new functions $(u_j)_g^{\sharp}$ and $(u_j)_e^{\sharp}$ by (2.1) and (2.2) respectively. Denote $v_j = u_j^*$ and recall the function Φ from (2.6). By (2.7), we have

$$\begin{aligned}
\int_{\mathbb{B}^{1+2\epsilon}} \sum_j \left|\nabla_g (u_j)_g^{\sharp}\right|_g^{1+\epsilon} dV &= \left((1+2\epsilon)\sigma_{(1+2\epsilon)}\right)^{1+\epsilon}\int_0^{\infty}\sum_j \left|v_j'(1+\epsilon)\right|^{1+\epsilon}\left(\sinh\Phi^{-1}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)\right)^{(1+\epsilon)(2\epsilon)} d(1+\epsilon) \\
&= \left((1+2\epsilon)\sigma_{(1+2\epsilon)}\right)^{1+\epsilon}\int_0^{\infty}\sum_j \left|v_j'(1+\epsilon)\right|^{1+\epsilon}\left(\left(\sinh\Phi^{-1}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)\right)^{(1+2\epsilon)(2\epsilon)}\right)^{\frac{1+\epsilon}{1+2\epsilon}} d(1+\epsilon) \\
&= \left((1+2\epsilon)\sigma_{(1+2\epsilon)}\right)^{1+\epsilon}\int_0^{\infty}\sum_j \left|v_j'(1+\epsilon)\right|^{1+\epsilon}\left(k_{1+2\epsilon,1+2\epsilon}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)+\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)^{2\epsilon}\right)^{\frac{1+\epsilon}{1+2\epsilon}} d(1+\epsilon) \\
&\geq \left((1+2\epsilon)\sigma_{(1+2\epsilon)}\right)^{1+\epsilon}\int_0^{\infty}\sum_j \left|v_j'(1+\epsilon)\right|^{1+\epsilon}\left(\left(\frac{2\epsilon}{1+2\epsilon}\right)^{1+2\epsilon}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)^{1+2\epsilon}+\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)^{2\epsilon}\right)^{\frac{1+\epsilon}{1+2\epsilon}} d(1+\epsilon),
\end{aligned}$$

here we use Lemma 2.1 to bound $k_{1+2\epsilon,1+2\epsilon}$ from below. It is easy to see that for $0 < \epsilon < 1$, it holds

$$(a+b)^{1+\epsilon} = \sup_{0<\epsilon<1}\left((1-\epsilon)^{-\epsilon}a^{1+\epsilon} + (\epsilon)^{-\epsilon}b^{1+\epsilon}\right). \quad (4.5)$$

Consequently, for any $0 < \epsilon < 1$, we get

$$\begin{aligned}
\int_{\mathbb{B}^{1+2\epsilon}} \sum_j \left|\nabla_g (u_j)_g^{\sharp}\right|_g^{1+\epsilon} dV &\geq (1-\epsilon)^{1-\frac{1+\epsilon}{1+2\epsilon}}(2\epsilon)^{1+\epsilon}\int_0^{\infty}\sum_j \left|v_j'(1+\epsilon)\right|^{1+\epsilon}(1+\epsilon)^{1+\epsilon}d(1+\epsilon) + (\epsilon)^{1-\frac{1+\epsilon}{1+2\epsilon}}\left((1+2\epsilon)\sigma_{(1+2\epsilon)}\right)^{1+\epsilon}\int_0^{\infty}\sum_j \left|v_j'(1+\epsilon)\right|^{1+\epsilon}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)^{\frac{(1+\epsilon)(2\epsilon)}{1+2\epsilon}} d(1+\epsilon) \\
&\geq (1-\epsilon)^{1-\frac{1+\epsilon}{1+2\epsilon}}\frac{(2\epsilon)^{1+\epsilon}}{p^{(1+\epsilon)}}\int_0^{\infty}\sum_j \left|v_j(1+\epsilon)\right|^{1+\epsilon}d(1+\epsilon) + (\epsilon)^{1-\frac{1+\epsilon}{1+2\epsilon}}\sum_j \left\|\nabla(u_j)_e^{\sharp}\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{1+\epsilon} \\
&\geq (1-\epsilon)^{1-\frac{1+\epsilon}{1+2\epsilon}}\frac{(2\epsilon)^{1+\epsilon}}{p^{(1+\epsilon)}}\sum_j \left\|(u_j)_g^{\sharp}\right\|_{L^{1+\epsilon}(\mathbb{H}^{1+2\epsilon})}^{1+\epsilon} + (\epsilon)^{1-\frac{1+\epsilon}{1+2\epsilon}}S(1+2\epsilon,1+\epsilon)^{1+\epsilon}\sum_j \left\|(u_j)_e^{\sharp}\right\|_{L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}(\mathbb{R}^{1+2\epsilon})}^{1+\epsilon},
\end{aligned}$$

here we use the Hardy inequality in $[0,\infty)$ for the first inequality, and the sharp Sobolev inequality for the second inequality. Taking the supremum over $0 < \epsilon < 1$ and using again (4.5) and the fact $\| u_j \|_{L^{1+\epsilon}(\mathbb{H}^{1+2\epsilon})} = \left\|(u_j)_g^{\sharp}\right\|_{L^{1+\epsilon}(\mathbb{H}^{1+2\epsilon})}$ and $\| u_j \|_{L^{(1+\epsilon)^*}(\mathbb{H}^{1+2\epsilon})} = \left\|(u_j)_e^{\sharp}\right\|_{L^{(1+\epsilon)^*}(\mathbb{R}^{1+2\epsilon})}$ with $(1+\epsilon)^* = \frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}$, we obtain

$$\int_{\mathbb{B}^{1+2\epsilon}} \sum_j \left|\nabla_g (u_j)_g^{\sharp}\right|_g^{1+\epsilon} dV \geq \sum_j\left[\left(\frac{2\epsilon}{1+\epsilon}\right)^{1+2\epsilon}\| u_j \|_{L^{1+\epsilon}(\mathbb{H}^{1+2\epsilon})}^{1+2\epsilon} + S(1+2\epsilon,1+\epsilon)^{1+2\epsilon}\| u_j \|_{L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}(\mathbb{H}^{1+2\epsilon})}^{1+2\epsilon}\right]^{\frac{1+\epsilon}{1+2\epsilon}},$$

which implies (4.2) by the Pólya-Szegö principle on $\mathbb{H}^{1+2\epsilon}$.

We next prove part (iii). Denote

$$l(1+\epsilon) = \left[\sinh\Phi^{-1}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)\right]^{2\epsilon}, \; \epsilon \geq -1,$$

where Φ is defined by (2.6). It is easy to check that $l(1+\epsilon) \sim (1+\epsilon)^{\frac{2\epsilon}{1+2\epsilon}}$ as $\epsilon \to 0$ and $l(1+\epsilon) \sim (1+\epsilon)$ as $\epsilon \to \infty$.

Consequently, we get

$$\int_0^\infty l(1+\epsilon)^{-\frac{1+\epsilon}{\epsilon}} d(1+\epsilon) < \infty.$$

Moreover, making the change $(1+\epsilon) = \sigma_{(1+2\epsilon)}\Phi(1-\epsilon)$ with $d(1+\epsilon) = (1+2\epsilon)\sigma_{(1+2\epsilon)}(\sinh(1-\epsilon))^{2\epsilon}$, we have

$$\begin{aligned}\int_0^\infty l(1+\epsilon)^{-\frac{1+\epsilon}{\epsilon}} d(1+\epsilon) &= \int_0^\infty \left[\sinh \Phi^{-1}\left(\frac{1+\epsilon}{\sigma_{(1+2\epsilon)}}\right)\right]^{-\frac{(1+\epsilon)(2\epsilon)}{\epsilon}} d(1+\epsilon) = (1+2\epsilon)\sigma_{(1+2\epsilon)} \int_0^\infty (\sinh(1-\epsilon))^{-\frac{2\epsilon}{\epsilon}} d(1-\epsilon) \\ &= (1+2\epsilon)\sigma_{(1+2\epsilon)} 2^{\frac{\epsilon}{2\epsilon}} \frac{\Gamma(-1)\Gamma(1)}{\Gamma(0)}.\end{aligned}$$

Using the identity $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\Gamma(2x)$, we get

$$\int_0^\infty l(1+\epsilon)^{-\frac{1+\epsilon}{\epsilon}} d(1+\epsilon) = (1+2\epsilon)\sigma_{(1+2\epsilon)} 2^{\frac{2\epsilon}{\epsilon}} \frac{\Gamma\left(\frac{-1}{2}\right)\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}. \tag{4.6}$$

Denote $v_j = u_j^*$, we have $\lim_{\epsilon\to\infty} \sum_j v_j(1+\epsilon) = 0$, and hence

$$v_j(1+\epsilon) = \int_{1+\epsilon}^\infty \sum_j v_j'(1+\epsilon) d(1+\epsilon) = \int_{1+\epsilon}^\infty \sum_j v_j'(1+\epsilon) l(1+\epsilon) l(1+\epsilon)^{-1} d(1+\epsilon).$$

Thank to Hölder inequality, we get

$$\begin{aligned}\sup_{x\in\mathbb{H}^{1+2\epsilon}} |u_j(x)| = v_j(0) &\le \left(\int_0^\infty \sum_j |v_j'(1+\epsilon)|^{1+\epsilon} l(1+\epsilon)^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{1+\epsilon}} \left(\int_0^\infty l(1+\epsilon)^{-\frac{1+\epsilon}{\epsilon}} d(1+\epsilon)\right)^{\frac{\epsilon}{1+\epsilon}} \\ &= \frac{1}{(1+2\epsilon)\sigma_{(1+2\epsilon)}} \left(\int_0^\infty l(1+\epsilon)^{-\frac{1+\epsilon}{\epsilon}} d(1+\epsilon)\right)^{\frac{\epsilon}{1+\epsilon}} \left(((1+2\epsilon)\sigma_{(1+2\epsilon)})^{1+\epsilon} \int_0^\infty \sum_j |v_j'(1+\epsilon)|^{1+\epsilon} l(1+\epsilon)^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{1+\epsilon}} \\ &= C(1+2\epsilon, 1+\epsilon) \sum_j \|\nabla(u_j)_g^\sharp\|_{L^{1+\epsilon}(\mathbb{H}^{1+2\epsilon})},\end{aligned}$$

here the second equality comes from (2.7) and (4.6). This proves (4.3).

To check the sharpness of $C(1+2\epsilon, 1+\epsilon)$, we see that if $u_j(x) = v_j\left(V\left(B_g(0,\rho(x))\right)\right)$ with v_j defined by (4.4), then $u_j^* = v_j$. Hence $\sup_{x\in\mathbb{H}^{1+2\epsilon}} |u_j(x)| = v_j(0)$. Moreover, for such a choice of function v_j, we have equality in the Hölder inequality above. This proves the sharpness of $C(1+2\epsilon, 1+\epsilon)$ and the equality holds for this function u_j.

References

[1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.
[2] T. Aubin, Y.Y. Li, On the best Sobolev inequality, J.Math. Pures Appl. 78 (1999) 353–387.
[3] T. Aubin, O. Druet, E. Hebey, Best constants in Sobolev inequalities for compact manifolds of nonpositive curvature, C.R. Acad. Sci. Paris Sér. I Math. 326 (1998) 1117–1121.
[4] A. Baernstein II, A unifiedapproach to symmetrisation, in: Partial Differential Equations of Elliptic Type, Cortona, 1992, in: Sympos. Math., vol.XXXV, Cambridge Univ. Press, Cambridge, 1994, pp.47–91.
[5] R.D. Benguria, R.L. Frank, M. Loss, The sharp constant in the Hardy–Sobolev–Maz'ya inequality in the three dimensional upper half-space, Math. Res. Lett. 15 (2008) 613–622.
[6] E. Berchio, D. Ganguly, Improved higher order Poincaré inequalities on the hyperbolic space via Hardy-type remainder terms, Commun. Pure Appl. Anal. 15 (2016) 1871–1892.
[7] E. Berchio, L. D'Ambrosio, D. Ganguly, G. Grillo, Improved Lp-Poincaré inequalities on the hyperbolic space, Nonlinear Anal. 157 (2017) 146–166.
[8] E. Berchio, D. Ganguly, G. Grillo, Sharp Poincaré–Hardy and Poincaré–Rellich inequalities on the hyperbolic space, J.Funct. Anal. 272 (2017) 1661–1703.
[9] J. Ceccon, M. Montenegro, Optimal Lp-Riemannian Gagliardo–Nirenberg inequalities, Math. Z. 258 (2008) 851–873.
[10] J. Ceccon, M. Montenegro, Optimal Riemannian Lp-Gagliardo–Nirenberg inequalities revisited, J. Differential Equations 254 (2013) 2532–2555.
[11] J. Ceccon, M. Montenegro, Sharp Lp-entropy inequalities on manifolds, J. Funct. Anal. 269 (2015) 1591–1619.
[12] J. Ceccon, M. Montenegro, Sharp constants in Riemannian Lp-Gagliardo–Nirenberg inequalities, J. Math. Anal. Appl. 433 (2016) 260–281.
[13] D. Cordero-Erausquin, B. Nazaret, C. Villani, A mass-transportation approach to sharp Sobolev and Gagliardo–Nirenberg inequalities, Adv. Math. 182 (2004) 307–332.

[14] M. del Pino, J. Dolbeault, Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions, J.Math. Pures Appl. 81 (2002) 847–875.
[15] M. del Pino, J. Dolbeault, The optimal Euclidean Lp-Sobolev logarithmic inequality, J.Funct. Anal. 197 (2003) 151–161.
[16] O. Druet, The best constants problem in Sobolev inequalities, Math. Ann. 314 (1999) 327–346.
[17] O. Druet, E. Hebey, AB program in geometric analysis: sharp Sobolev inequalities and related problems, Mem. Amer. Math. Soc. 160(761) (2002), viii+98pp.
[18] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, vol.5, AMS, Providence, RI, 1999.
[19] E. Hebey, M. Vaugon, The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds, Duke Math. J. 79 (1995) 235–279.
[20] E. Hebey, M. Vaugon, Meilleures constantes dans le théorème d'inclusion de Sobolev, Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996) 57–93.
[21] G. Lu, Q. Yang, Sharp Hardy–Adams inequalities for bi–Laplacian on hyperbolic space of dimension four, Adv. Math. 319 (2017) 567–598.
[22] G. Lu, Q. Yang, Hardy–Sobolev–Maz'ya inequalities for higher order derivatives on half spaces, preprint, arXiv :1703 .08171.
[23] G. Mancini, K. Sandeep, On a semilinear elliptic equation in $\mathbb{H}^n$, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 7 (2008) 635–671.
[24] G. Mancini, K. Sandeep, Moser–Trudinger inequality on conformal discs, Commun. Contemp. Math. 12 (2010) 1055–1068.
[25] G. Mancini, K. Sandeep, C. Tintarev, Trudinger–Moser inequality in the hyperbolic space Hn, Adv. Nonlinear Anal. 2 (2013) 309–324.
[26] V.G. Maz'ya, Sobolev Spaces, Springer Verlag, Berlin, New York, 1985.
[27] F. Mugelli, G. Talenti, Sobolev inequalities in 2-dimensional hyperbolic space, in: General Inequalities, 7, Oberwolfach, 1995, in: Internat. Ser. Numer. Math., vol.123, Birkhäuser, Basel, 1997, pp.201–216.
[28] V.H. Nguyen, The sharp Hardy–Moser–Trudinger inequality in dimension n, preprint.
[29] V.H. Nguyen, Improved Moser–Trudinger type inequalities in the hyperbolic space $\mathbb{H}^n$, Nonlinear Anal. 168 (2018) 67–80.
[30] G. Talenti, Best constants in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[31] G. Talenti, On isoperimetric theorems of mathematical physics, in: P.M. Gruber, J.M. Wills (Eds.), Handbook of Convex Geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp.1131–1147.
[32] A. Tertikas, K. Tintarev, On existence of minimizers for the Hardy–Sobolev–Maz'ya inequality, Ann. Mat. Pura Appl. (4) 186 (2007) 645–662.
[33] G. Wang, D. Ye, A Hardy–Moser–Trudinger inequality, Adv. Math. 230 (2012) 294–320.
[34] Van Hoang Nguyen, The sharp Poincaré–Sobolev type inequalities in the hyperbolic spaces $\mathbb{H}^n$, J. Math. Anal. Appl. 462 (2018) 1570–1584.