Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 4 (April 2025) pp: 36-41
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Collatz Sequence: Negative Parity Invariance, Saturation Point, and Cycles

Danilo Merlini[1, 2, 3], Luca Rusconi[1,3], Massimo Sala[2, 3], Nicoletta Sala[2, 3]
[1](*CERFIM (Research Center for Mathematics and Physics), Locarno, Switzerland)*
[2](*IRFIA (Research Institute in Arithmetic Physics), Locarno, Switzerland)*
[3](*ISSI (Institute for Scientific and Interdisciplinary Studies), Locarno, Switzerland)*
Corresponding Author: nicolettasalavb@gmail.com

ABSTRACT: *In this paper, we consider the Collatz sequence, i.e., the discrete 3n+1 or the (3n+1)/2 sequence, where it is known that the presence of at least one cycle is guaranteed, given by (1→4→2→1) and (1→2→1), respectively.*
The above sequence is noted to lack parity invariance, and the new related sequence has no saturation point.

KEYWORDS: *Collatz Sequence, Discrete Dynamical Model, Parity Symmetry, Cycles, Chalice, Orbits, and Saturation Point.*

R*eceived 09 Apr., 2025;* ***R****evised 17 Apr., 2025;* ***A****ccepted 19 Apr., 2025*

I. INTRODUCTION

In a recent work discussing the differences between left and right [1], specifically regarding parity invariance in quantum mechanics, a parity transformation P - a change in the sign of the coordinates - was revisited in connection with significant results from the past, not only in Statistical Mechanics but also in Weak Interaction (for example, the decay of a meson into three pions). According to Bietenholz [1], the P transformation under consideration here represents a discrete model in the field of dynamical systems, particularly one that complements the well-known dynamical system described by the Collatz Sequence, namely the 3n+1 or (3n+1)/2 sequences. We will apply a P transformation to the sequence defined for positive integers n.

Several researchers recently introduced and studied the newly related model that emerged [2, 3, 4]. We then consider the sequence above, where n is now a negative integer. Furthermore, after a brief computation of the orbits [5], we demonstrate that in the new model, namely that of the sequence [(3n-1)/2, n/2], the saturation point disappears.

The chalice and orbits are also given up to n = 60, and the decay into the three cycles of the new sequence is illustrated.

II. DISAPPEARENCE OF A PARITY SYMMETRY

The (3n+1) sequence (the first of the above forms of the Collatz sequence for positive integers n) is given as:

$$(3n+1) = f(n), \qquad (1)$$
$$(n/2) = g(n) \text{ with } n>0.$$

Notice that f and g are functions of the first degree.

The P transformation is obtained employing the sign = (from the right to the left and vice versa), and we obtain:
$-f(n) = -3n-1 \neq f(-n)$ (2)
$-g(n)=g(-n) = -(n/2)$

Thus, with $-n = m$, now a positive integer m, we obtain:
$h(m) = 3m-1$ (3)
$g(m) = m/2$
Thus, we do not obtain the same sequence by the P transformation above. In the same way, if we define $P(x) = -x$, for all real x, then $P(3n+1))=-3n-1$, and P applied to the Formulas (1) for negative n, i.e., $-n=m>0$ gives:

$P(3n+1) = -3n-1=3m-1=h(m)$
$P(n/2) = P(g(n))=m/2=g(m), m>0$ (4)

As noted above, the model of Eq. (1), for negative values of n, transforms into the model defined by Equations 3 and 4, where m is now positive, and there is a parity symmetry breaking [1, 6, 7, 8]. We note that model (1) has at least one cycle, but for the newly proposed model (3), the existence of at least three cycles [2,10] is given here (in the (3n-1)/2 formulation) by:

C_1: $1\to1$ of length 1
C_2: $5\to7\to10\to5$ of length 3
C_3: $17\to25\to37\to55\to82\to41\to61\to91\to136\to68\to 34\to17$ of length 11.

As for model (1), it may be conjectured that there are only the 3 cycles above, the inverse orbits of which are given below with the corresponding chalices [9]. See also Appendix 1 for the length l(n) of the orbits up to n=60 for the (3n-1)/2 sequence, Appendix 2 for the chalice C_1 of the (3n-1)/2, and Appendix 3 concerning the calculation of the growing constant c~4/3 of the same chalice C_1 in Figure 1.

III. ABSENCE OF A SATURATION POINT IN THE NEW SEQUENCE

In the Appendices, we give the orbits with their length l(n) up to n=60, and we note that, as in others models -analogous to the original (3n+1)/2 [10], the saturation point introduced recently [5, 9], is absent, i.e. $l(n) \le n$ "always" (in the interval we have considered: the length l(n) of a trajectory starting at n, i.e., l(n) is defined to be the number of steps to reach one of the numbers of the three cycles C_1, C_2, C_3).

As an example l(87)=28 (reaching 1, in C_1), l(28)=2, reaching 7 in C_2 and l(55)=0, on C_3. Of course, this takes into account the known stronger conjecture concerning model (1). For instance, $l(n)<41.67\cdot\log(n)$, for model (3), taking into account the numerical results given in [10], we suspect that $l(n)\le c\cdot\log(n)$ with c< 41.67.... of the model (1) [9,10, 11]. As for the model (3n+1)/2, our conjecture for the new model (3n-1)/2 is that on the three chalices of height k there are totally present the first natural integers <= to k (notice that l(n)=n only for n=3 in the orbits given in the Appendix1), and for c=41.67... → k> 226.

IV. A SIMILARITY IN THE TWO DYNAMICAL SYSTEMS

The set of odd numbers having l(n)=1, i.e., $d_1(p)$ in the (3n+1)/2 model is given by the solution of $(3d_1(p)+1)/2^k = 1$ where, for a cascade of even numbers, the solution is given by:

$k=2 \to d_1(k) = 1$
$k=4 \to d_1(k) = 5$
$k=6 \to d_1(k) = 21$
$k=8 \to d_1(k) = 85$

We note the recursive relation: $d_1(k)= 4\cdot d_1(k-2)+1$, k=4, 6, ...
Similarly, for the (3n-1)/2 model, where now we have three cycles C_1, C_2, C_3, containing resp. one odd (1), two odd (5, 7) and seven odd (17, 25, 37, 55, 41, 61, 91), we have a similar recursive relation for the three cycles.
For example:

C_1: $d_1(k) = 4\cdot d_1(k-2)-1$
(-1 instead of +1), k=3, 5,...1, $1\cdot4-1=3$, $3\cdot4-1=11$, $11\cdot4-1=43$ (See the chalice for C_1 on the figure 1).

C_2: $d_1(k) = 4\cdot d_1(k-2)-1$, k=3,5,7,.... 5, 5·4-1=19, 19·4-1=75, 75·4-1=299 which all fall in the next odd, i.e.,7 of $C_2 \cdot d_1(k) = 4\cdot d_1(k-2)-1$, k=4,6,8,... 7, 7·4-1=27, 27·4-1=107, 107·4-1=427,...which all fall in the first odd, for example, 5 of C_2.
C_3: $d_1(k) =4\cdot d_1(k-2) -1$, 17,17·4-1=67, 67·4-1=267, .. k==3, 5, 7.. which all fall in the next odd of C_3, i.e., 25 and so on for all the other six odd numbers of C_3, i.e. set d_1 of 25 falls in 37, that of 37 falls in 55, etc.

These computations demonstrate the similarity of the recursive equations, with every odd number in one of the three cycles behaving like the number 1 in C_1. A broader relationship between two consecutive sets of odd numbers (d_1 and d_2), which represent related cascades of even numbers, is expressed by the solution of the equations:

$$\frac{3\cdot d_1(k)-1}{2^k} = d_2(k) \quad k=1, 2, 3\ldots \tag{5}$$

We obtain:

k=1	$d_1 = 1 + 2^2\cdot p$	$d_2 = 1+ 6\cdot p$	p=0,1,2,3....
k=2	$d_1 = 7+ 2^3\cdot p$	$d_2= 5+6\cdot p$	p=0,1,2,3...
k=3:	$d_1 = 3+ 2^4\cdot p$	$d_2 = 1+6\cdot p$	p=0,....
k=4:	$d_1 =27+ 2^5\cdot p$	$d_2 = 5+6\cdot p$	p=0,...
k=5:	$d_1 =11+ 2^6\cdot p$	$d_2 = 1+6\cdot p$	p=0,.
k=6	$d_1 =107+2^7\cdot p$	$d_2= 5+6\cdot p$	p=0,.
k=7	$d_1 =43 + 2^8\cdot p$	$d_2= 1+6\cdot p$	p=0,.
k=8	$d_1 =427+2^9\cdot p$	$d_2= 5+6\cdot p$	p=0,...
k=9:	$d_1=171+ 2^{10}\cdot p$	$d_2 = 1+6\cdot p$	p=0,..
k=10:	$d_1=1707+ 2^{11}\cdot p$	$d_2= 5+6\cdot p$	p=0
k=11:	$d_1= 683+ 2^{12}\cdot p$	$d_2= 1+6\cdot p$	p=0
k=12:	$d_1= 6827+2^{13}\cdot p$	$d_2= 5+6\cdot p$	p=0

We note now that the coefficient of p in d_1 is 2^{k+1} and more than this, that two emerging "invariant" sets given by 1+6·p (for k odd, 1 modulo 6) and 5+6·p (for k even, 5 modulo 6) which, after additional applications of the rule ((3n-1))/2 and n/2), converge to the 3 "eigensets", i.e., to the three cycles C_1, C_2, C_3 given above. and illustrated below without the cascades of the even up to 17 (the smallest integer in C_3).

→	1,	5,	7,	11,	13,	17,
	1,	7,	5,	1,	19,	25,......
	1,	5,	7,	1,	7,	37,...
	1.	7,	5,	1,	5,	55,...
	1,	5,	7,	1,	7,	41,...
	1,	7,	5,	1.	5,	61,,
	1,	5,	7,	1,	7,	91,..
	1	7,	5,	1,	5,	17,..
thus:						
	C_1	C_2	C_2	C_1	C_2	C_3

Matrix representation of the three cycles in the (3n-1)/2.

Notice that the above "matrix", (for n up to 87 =7+ 8·p, p=10, we have for k=2: $((3\cdot 87-1) /2^2) = 65= 5+6\cdot p$ for p=10) and since l(87) = 27, falls in C_1, the matrix has the first line ending in 65 (22 numbers) and height 13 i.e. (65, 97, 145, 217, 325, 487, 365, 547, 205, 307, 115, 43, 1).

Finally, it is noticed that the cycle of the smallest length that appears is C_1 of length 1 (1→1) as illustrated below with the chalice, where the growth constant is calculated to be c=(4/3=1.33...) as for the (3n+1)/2 sequence [9].

The computations indicate that regarding (3n+1)/2, the sequence of (3n-1)/2 is described by a Fibonacci sequence given by:

With the characteristic equation:

$$x^2 - x - \frac{4}{9} = 0$$

The solutions are:

$$x_1 = \frac{4}{3} = 1.\overline{3} \text{ and } x_2 = -\frac{1}{3} = -0.\overline{3}$$

To the best of our knowledge, the content of this work is new or has not been given along the above lines.

V. CONCLUSION

We have applied the P transformation to the ((3n+1)/2, n/2) discrete dynamical model and discovered it. Then, we investigated some properties of the new model ((3n-1)/2, n/2) (n>0), specifically its structure and orbits. We have confirmed the disappearance of the saturation point for a numerical experiment up to n=60.

The growing constant of the chalice is estimated (as in a similar model) to be c≅4/3. Finally, the sequence is well approximated (as other sequences) by a Fibonacci one, given by:

$$f_n = f_{n-1} + \frac{4}{9} f_{n-2} \,.$$

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Appendix 1

The next Table 1 collects the orbits from n=1 to n=60 in the ((3n-1)/2).

n	l(n)	Cycle	n	l(n)	Cycle
1	0	C_1	31	5	C_3
2	1	C_1	32	5	C_1
3	3	C_1	33	8	C_3
4	2	C_1	34	0	C_3
5	0	C_2	35	7	C_2
6	4	C_1	36	7	C_2
7	0	C_2	37	0	C_3
8	3	C_1	38	4	C_2
9	5	C_2	39	11	C_1
10	1	C_2	40	3	C_2
11	5	C_1	41	1	C_3
12	5	C_1	42	6	C_3
13	4	C_2	43	7	C_1
14	1	C_2	44	7	C_1
15	8	C_1	45	5	C_3
16	4	C_1	46	3	C_3
17	0	C_3	47	6	C_3
18	6	C_2	48	6	C_1
19	3	C_2	49	6	C_3
20	2	C_2	50	1	C_3
21	5	C_3	51	6	C_2
22	6	C_1	52	6	C_2
23	2	C_3	53	12	C_1
24	6	C_1	54	5	C_2
25	0	C_3	55	3	C_3
26	6	C_2	56	3	C_2
27	4	C_2	57	20	C_1
28	2	C_2	58	10	C_1
29	9	C_1	59	9	C_1
30	9	C_1	60	10	C_1

Table 1.

Appendix 2

Figure 1 reproduces the chalice C_1 of the sequence. In this case, the sequence up to k= 11 highlights that the constant:

$$c = \frac{n_{k+1}}{n_k}$$

approaches the value:

$$c = \frac{4}{3}$$

As sequence [10]:

$$\left[\frac{3n+1}{2}, \frac{n}{2}\right]$$

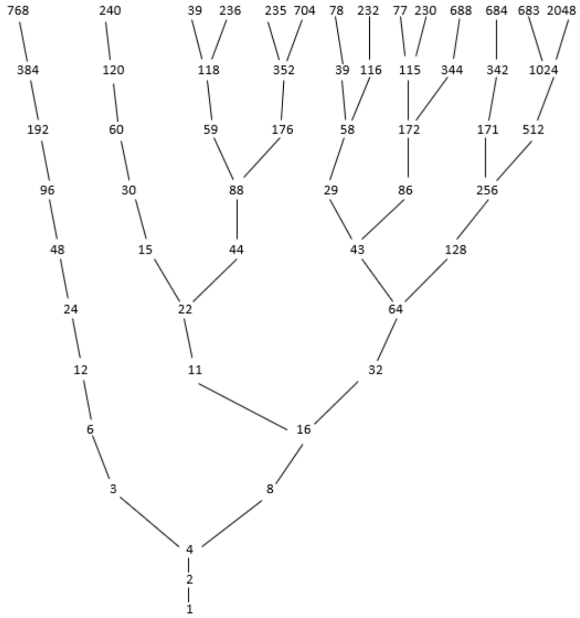

Figure 1. Chalice of C_1.

Appendix 3

Observing Figure 1, we determine the following Table 2, with the calculation of the growing constant c=4/3 =1.3333....

k	$\mathbf{n_k}$	$\mathbf{\sum n_k}$
11	14	56
10	10	42
9	8	32
8	6	24
7	5	18
6	3	13
5	3	10
4	2	7
3	2	5
2	1	3
1	1	2
0	1	1

Table 2

For example, observing Table 2, if we consider the level k=11 and k=10, we have:
For k=11: $\sum n_k$=56. For k= 10: $\sum n_k$=42.

Their ratio is $56{:}\,42 = 1.\overline{3} = \frac{4}{3}$ which is the constant c.