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Review Paper

A multigrid discretization scheme for the Steklov eigenproblem

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Abstract—*In this paper, we discusses the finite element approximation for a Steklov eigenvalue problem. Based on the work of Armentano and Padra, we derive an a posteriori error estimate. By constructing auxiliary bubble functions and lifting operators, we prove the reliability and validity of the posterior error estimator. In addition, we verify the robustness of the posterior error estimator under the adaptive grid through numerical experiments.*

Keywords— *Steklov eigenvalue problem, Finite element, Posteriori error estimate*.

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I. INTRODUCTION

The Steklov eigenvalue problem, as an important type of differential equation boundary value problem, is widely used in various fields of mathematical physics and engineering science, such as vibration analysis of elastic structures, modal study of acoustic waveguides, and interface resonance phenomena in electromagnetic fields. Its core feature is that the spectral characteristics of differential operators are dominated by boundary conditions, which gives the solution of the problem a unique physical meaning at the boundary, such as energy concentration or mode localization. However, with the increasing complexity of engineering problems (such as irregular geometric regions, non-uniform material parameters, etc.), traditional analytical methods face severe challenges, and efficient and stable numerical algorithms have become an urgent need for theoretical and applied research.

Many scholars have studied the finite element method for the Steklov eigenvalue problems. In order to address the issue of posterior error estimation in the Steklov eigenvalue problem discussed in Remark 3.11 of reference [1], this paper proposes a posterior error estimator for this problem using the discontinuous finite element method. The reliability and effectiveness of the estimator are theoretically and numerically demonstrated. The main characteristics of the discontinuous finite element method are that the test function is discontinuous along the surface (or edge) in the mesh, which has the advantages of local mass conservation, easy coupling with other methods, hp adaptability, and can work on polygonal meshes. Therefore, the discontinuous finite element method has been used to solve many problems. In addition, the DG method has been used to solve various eigenvalue problems, such as Laplace eigenvalue problem [2], classical self adjoint Steklov eigenvalue problem [1], biharmonic eigenvalue problem [3], Maxwell eigenvalue problem [4], etc.

For the self conjugate Steklov eigenvalue problem, Zeng et al. [1] first studied the discontinuous finite element method and provided its prior error estimate. For the Steklov eigenvalue problem of inverse scattering, Li et al. [5] studied the posterior error estimation and adaptive methods of discontinuous finite element method. On the basis of the above work, this paper further studies the posterior error estimator of the discontinuous finite element method. We have demonstrated the reliability and effectiveness of the posterior error estimator of the characteristic function by utilizing the properties of the lifting operator, and analyzed the reliability of the posterior error estimator of the finite element eigenvalues. The analysis method in this article can be extended to general second-order elliptic eigenvalue problems.

Based on the above work, the remaining part of our article is arranged as follows: In Section 2, we first introduce the model problem and then describe the error estimates. In Section 3, establish a multigrid discretization scheme, and the error estimates of the proposed scheme is presented. In Section 4, conduct a theoretical analysis. Finally, The numerical results show that our method is efficient in Section 5.

II. PRELIMINARIES

We consider the following Steklov eigenvalue problem:

$$-\Delta u + u = 0 \text{ in } \Omega,\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u \text{ on } \partial\Omega \qquad (2.1)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with Lipschitz continuous boundary $\partial\Omega$, $\frac{\partial u}{\partial \mathbf{n}}$ is the outward normal derivation on $\partial\Omega$.

The variational problem associated with (2.1) is given by: Find $\lambda \in \mathbb{R}$ and $0 \neq u \in H^1(\Omega)$, such that

$$a(u,v) = \lambda b(u,v), \forall v \in H^1(\Omega) \qquad (2.2)$$

where

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx, b(u,v) = \int_{\partial\Omega} uvds$$

$$\| u \|_b = (b(u,u))^{\frac{1}{2}} = \| u \|_{0,\partial\Omega}$$

It is clear that $a(\cdot,\cdot)$ is symmetric, continuous and $H^1(\Omega)-$elliptic bilinear form on $H^1(\Omega)\times H^1(\Omega)$.

Let $\mathcal{T}_h = \{T\}$ be a family of regular triangulations of Ω. Let h stand for the mesh-size, namely $h = \max\{h_T : T \in \mathcal{T}_h\}$ is the diameter of $\mathcal{T}_h$, with h_T being the diameter of the triangle T. The diameter of an edge e is dencted by h_e, and the set of edges of elements $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ where $\mathcal{E}_h^i$ denotes the interior edges set and $\mathcal{E}_h^b$ denotes the set of edges lying on the boundary $\partial\Omega$. We denote the average $\{v\}$ and jump $[[v]]$ of v on e by

$$\{v\} = \frac{1}{2}(v^+ + v^-), \quad [[v]] = v^+\mathbf{n}^+ + v^-\mathbf{n}^-$$

where $e \in \partial T^+ \cap \partial T^-, v^+ = v|_{T^+}, v^- = v|_{T^-}, \mathbf{n}$ is the unit outer normal vector from T^+ towards to T^-.

If $e \in \varepsilon_h^b$, define the average and jump of v on e as follows:

$$\{v\} = v, \quad [[v]] = v\mathbf{n}$$

Define the DGFEM space:

$$S^h = \{v \in L^2(\Omega): v|_T \in \mathbb{P}_m(T), \forall T \in \mathcal{T}_h\}$$

where $\mathbb{P}_m(T)$ denotes the space of polynomials defined on T with degree less than or equal to $m \geq 1$. Introduce the piecewise H^s function space of degree s:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega): v|_T \in H^s(T), \forall T \in \mathcal{T}_h\}$$

The DGFEM discretization of (2.2) is to find $\lambda_h \in \mathbb{R}$ and $0 \neq u_h \in S^h$, such that

$$a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h), \forall v_h \in S^h \qquad (2.3)$$

where

$$a_h(u_h,v_h)=\sum_{T\in\mathcal{T}_h}\int_T(\nabla u_h\cdot\nabla v_h+u_hv_h)\mathrm{d}x$$
$$-\sum_{e\in\mathcal{E}_h^i}\int_e\{\nabla u_h\}\cdot[[v_h]]\mathrm{d}s$$
$$-\sum_{e\in\varepsilon_h^i}\int_e\{\nabla v_h\}\cdot[[u_h]]\mathrm{d}s$$
$$+\sum_{e\in\varepsilon_h^i}\frac{\sigma}{h_e}\int_e[[u_h]]\cdot[[v_h]]ds$$

$$b_h(u_h,v_h)=\sum_{e\in\varepsilon_h^b}\int_e u_h\cdot v_h\mathrm{d}s$$

where σ is the interior penalty parameter. We choose σ to be sufficiently large to have coercivity. It is clear that the discretization (2.3) is symmetric which is called symmetric interior penalty Galerkin method (SIPG) in DGFEM.

Introduce the sum space $V(h)=S^h+H^1(\Omega)$ endowed with DG norm

$$\| u_h\|_G^2=\sum_{T\in\mathcal{T}_h}\|\nabla u_h\|_{0,T}^2+\| u_h\|_{0,T}^2)$$
$$+\sum_{e\in\mathcal{E}_h^i}\frac{\sigma}{h_e}\|[[u_h]]\|_{0,e}^2$$

and define the other norm on $H^{1+s}(\mathcal{T}_h)\left(s>\frac{1}{2}\right)$ by

$$\| u_h\|_h^2=\| u_h\|_G^2+\sum_{e\in\mathcal{E}_h^i}h_e\|\{\nabla u_h\}\|_{0,e}^2$$

Note that $\|\cdot\|_G$ is equivalent to $\|\cdot\|_h$ on S^h.

In order to show that the discretization (2.3) is stable, first we will show that $a_h(\cdot,\cdot)$ is coercive on $S^h\times S^h$. It is easy to know that the following continuity and coercivity properties hold:

$$|a_h(u_h,v_h)|\leqslant\| u_h\|_h\| v_h\|_h,\forall u_h,v_h\in S^h+H^{1+s}(\mathcal{T}_h)\quad(2.4)$$

$$\| u_h\|_G^2\lesssim|a_h(u_h,u_h)|,\forall u_h\in S^h\quad(2.5)$$

We consider the following source problem (2.6) associated with (2.2) and the DG approximate source problem (2.7) associated with (2.3), respectively.

Find $w\in H^1(\Omega)$ such that

$$a(w,v)=b(f,v),\forall v\in H^1(\Omega)\quad(2.6)$$

Find $w_h\in S^h$ such that

$$a_h(w_h,v_h)=b_h(f,v_h),\forall v_h\in S^h\quad(2.7)$$

Since $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ are continuous and coercive on $H^1(\Omega)$ and S^h, respectively. $b(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are bounded, from Lax-Milgram Theorem we know that (2.6) and (2.7) admit the unique solution w and w_h, respectively.

Lemma 2.1. If $f\in L^2(\partial\Omega)$, the solution w of the source problem (2.6) satisfies $w\in H^{1+r_1}(\Omega)$ with $r_1\in(0,\frac{1}{2})$ and

$$\| w \|_{1+r_1} \lesssim \| f \|_{0,\partial\Omega} \qquad (2.8)$$

For the case that $f \in H^{\frac{1}{2}}(\partial\Omega)$, we have $w \in H^{1+r_2}(\Omega)$ and

$$\| w \|_{1+r_2} \leqslant \| f \|_{\frac{1}{2},\partial\Omega} \qquad (2.9)$$

Let w and w_h be the solution of (2.7) and (2.8), respectively, then the SIPG approximation (2.8) is consistent:

$$a_h(w - w_h, v_h) = 0, \forall v_h \in S^h \qquad (2.10)$$

Then, thanks to Lemma 2.1, for the source problem (2.6), let $f \in L^2(\partial\Omega)$, we can define the solution operator $A: L^2(\partial\Omega) \to H^{1+\frac{r}{2}}(\Omega) \subset H^1(\Omega)$ as

$$a(Af, v) = b(f, v), \forall v \in H^1(\Omega) \qquad (2.11)$$

Define the operator $T: L^2(\partial\Omega) \to H^{\frac{1}{2}+\frac{r}{2}}(\partial\Omega)$, such that

$$Tf = Af|_{\partial\Omega} \leftarrow$$

Similarly, from (2.7) we define a discrete solution operator $A_h: L^2(\partial\Omega) \to S^h$ as

$$a_h(A_h f, v) = b_h(f, v), \forall v \in S^h \qquad (2.12)$$

and the discrete operator $T_h: L^2(\partial\Omega) \to \delta S^h \subset L^2(\partial\Omega)$, such that

$$T_h f = A_h f|_{\partial\Omega} \leftarrow$$

where δS^h is the restriction of S^h on $\partial\Omega$.

Hence, (2.2) and (2.3) has the following equivalent operator form, respectively:

$$Au = \mu u, Tu = \mu u \qquad (2.13)$$

$$A_h u_h = \mu_h u_h, T_h u_h = \mu_h u_h \qquad (2.14)$$

where $\mu = \frac{1}{\lambda}, \mu_h = \frac{1}{\lambda_h}$. In this paper, λ, λ_h and μ, μ_h are all called eigenvalues.

From the definition of A_h and (2.5), noticing that $\| \cdot \|_G$ is equivalent to $\| \cdot \|_h$ on S^h, we can derive that

$$\| A_h f \|_h^2 \leqslant a_h(A_h f, A_h f) = b_h(f, A_h f)$$

$$\leqslant \| f \|_{0,\partial\Omega} \| A_h f \|_{0,\partial\Omega} \leqslant \| f \|_{0,\partial\Omega} \| A_h f \|_h$$

which yields

$$\| A_h f \|_h \lesssim \| f \|_{0,\partial\Omega} \lesssim \| f \|_h \qquad (2.15)$$

Lemma 2.2. Suppose that $\varphi \in H^{1+\xi}(T) (0 < \xi < \frac{1}{2})$ and $\Delta\varphi \in L^2(T)$, then there holds

$$\| \nabla\varphi \cdot \mathbf{n} \|_{\xi-\frac{1}{2},e}$$

$$\leqslant \| \nabla\varphi \|_{\xi,T} + h_T^{1-\xi} \| \Delta\varphi \|_{0,T}, \forall T \in \mathcal{T}_h, e \in \partial T$$

(2.16)

Introduce the auxiliary problem: find $\psi \in H^1(\Omega)$ such that

$$a_h(v, \psi) = (v, g), \forall v \in H^1(\Omega) \qquad (2.17)$$

From the elliptic regularity estimates for homogeneous Neumann boundary problem, we know that the following regularity estimate holds: $\forall g\in L^2(\Omega)$, the solution ψ of (2.16) belongs to $H^{1+\beta}(\Omega)(\beta>\frac{1}{2})$ and satisfies

$$\|\psi\|_{1+\beta}\lesssim\|g\|_{0,\Omega} \qquad (2.18)$$

Let $\psi^I\in S^h$ denote the linear interpolation of ψ on $\mathcal{T}_h$.

Lemma 2.3. Suppose that w and w_h be the solution of (2.6) and (2.7), respectively, $w\in H^{1+s}(\Omega)(0<s<\frac{1}{2})$, then there hold

$$\|w-w_h\|_{0,\Omega}\lesssim h^{\beta}\|w-w_h\|_G \qquad (2.19)$$

Theorem 2.1. Suppose that w and w_h be the solution of (2.6) and (2.7), respectively, $w\in H^{1+s}(\Omega)(0<s<\frac{1}{2})$, then there hold

$$\|w-w_h\|_G\lesssim h^{s}\|w\|_{1+s,\Omega} \qquad (2.20)$$

Theorem 2.2. Suppose that w and w_h be the solution of (2.6) and (2.7), respectively, $w\in H^{1+s}(\Omega)(0<s<\frac{1}{2})$, then there hold

$$\|w-w_h\|_{0,\partial\Omega}\lesssim h^{s}\|w-w_h\|_G \qquad (2.21)$$

Assume that λ is the k th eigenvalue of (2.2), and the algebraic multiplicity is equal to q, $\lambda=\lambda_k=\lambda_{k+1}=\lambda_{k+2}=\cdots=\lambda_{k+q-1}$. When $\|T-T_h\|_{0,\partial\Omega}\to 0(h\to 0)$ [6], q eigenvalues $\lambda_{k,h}$, $\lambda_{k+1,h}$, $\dots$, $\lambda_{k+q-1,h}$ of (2.3) will converge to λ. Let $M(\lambda)$ be the space spanned by all eigenfunctions corresponding to λ and $M_h(\lambda)$ be the direct sum of the eigenspaces corresponding to all eigenvalues of (2.3) that converge to λ. We have the following error estimates[7].

Theorem 2.3. We assume that $M(\lambda)\subset H^{s+1}(\Omega)\left(s>\frac{1}{2}\right)$, t =min(m,s), then there holds

$$|\lambda-\lambda_h|\lesssim h^{2t} \qquad (2.22)$$

Let $u_h\in M_h(\lambda)$ be an eigenfunction of (2.3), then there exists $u\in M(\lambda)$ such that

$$\|u-u_h\|_{0,\partial\Omega}\lesssim h^{t+r} \qquad (2.23)$$

$$\|u-u_h\|_h\lesssim h^{t} \qquad (2.24)$$

III. MULTIGRID DISCRETIZATION

Let $\{\mathcal{T}_{h_i}\}_0^l$ be an family of regular meshes of Ω, $h_{i-1}\gg h_i$, and let S^{h_i} be the DG space defined on $\mathcal{T}_{h_i}$. Denote $T_{h_0}=T_H, S^{h_0}=S^H$. Now, for the eigenvalue problem (2.3) we give the following multigrid discretization scheme of DGFEM based on the shifted inverse iteration.

Scheme 3.1. Given the iterative times l.

Step 1: Solve (2.3) on S^H: Find $(\lambda_H,u_H)\in\mathbb{R}\times S^H$ such that $\|u_H\|_{0,\partial\Omega}=1$ and

$$a_H(u_H,v)=\lambda_H b_H(u_H,v),\ \forall v\in S^H \qquad (3.1)$$

Step 2: $u^{h_0}\Leftarrow u_H,\lambda^{h_0}\Leftarrow\lambda_H, i\Leftarrow 1.$

Step 3: Solve a linear system on S^{h_i}: Find $u'\in S^{h_i}$ such that

$$a_{h_i}(u',v) - \lambda^{h_{i-1}} b_{h_i}(u',v)$$
$$= b_{h_i}(u^{h_{i-1}},v), \forall v \in S^{h_i} \quad (3.2)$$

Let $u^{h_i} = \dfrac{u'}{\| u' \|_{0,\partial\Omega}}$.

Step 4: Compute the Rayleigh quotient

$$\lambda^{h_i} = \frac{a_{h_i}(u^{h_i},u^{h_i})}{b_{h_i}(u^{h_i},u^{h_i})}$$

Step 5: If $i = l$, then output (λ^{h_l}, u^{h_l}), stop; else, $i \Leftarrow i+1$ and return to Step 3.

From (2.10) we define the Ritz-Galerkin projection operator $P_h : H^1(\Omega) \to S^h$ by

$$a_h(u - P_h u, v_h) = 0, \forall v_h \in S^h \quad (3.3)$$

Hence, for any $f \in H^1(\Omega)$

$$a_h(A_h f - P_h(Af), v_h)$$
$$= a_h(A_h f - Af + Af - P_h(Af), v_h)$$
$$= 0, \forall v_h \in S^h$$

Then, $A_h f = P_h A f, \forall v \in H^1(\Omega)$, thus $A_h = P_h A$.

Lemma 3.1. Let (λ, u) be an eigenpair of (2.2), then for any $v \in S^h$ and $\| v \|_b \neq 0$, the Rayleigh quotient $R(v) = \dfrac{a_h(v,v)}{\| v \|_b^2}$ such that

$$\frac{a_h(v,v)}{\| v \|_b^2} - \lambda$$
$$= \frac{a_h(v-u,v-u)}{\| v \|_b^2} - \lambda \frac{\| v-u \|_b^2}{\| v \|_b^2} \quad (3.4)$$

Lemma 3.2. For any nonzero $u, v \in S^h$,

$$\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \| \leq 2 \frac{\| u-v \|}{\| u \|},$$
$$\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \| \leq 2 \frac{\| u-v \|}{\| v \|} \quad (3.5)$$

Let (λ_H, u_H) be the kth eigenpair of (3.1), then (λ^{h_l}, u^{h_l}) derived from Scheme 3.1 is the k kth eigenpair approximation of (2.2). In what follows we also denote $(\lambda_H, u_H) = (\lambda_{k,H}, u_{k,H}), (\lambda^{h_l}, u^{h_l}) = (\lambda_k^{h_l}, u_k^{h_l})$.

IV. A POSTERIORI ERROR ESTIMATE

Let (λ_h, u_h) be the eigen pair of (2.3), On each unit $T \in \mathcal{T}_h$ and each edge $e \in \mathcal{E}_h$, there are the following element residuals and surface residuals.

$$R_T = -\Delta u_h + u_h$$
$$J_{F,1} = [[\nabla u_h]], \forall e \in \mathcal{E}_h^i$$
$$J_{F,1} = \lambda_h u_h - \frac{\partial u_h}{\partial \mathbf{n}}, \forall e \in \mathcal{E}_h^b$$
$$J_{F,2} = [[u_h]], \forall e \in \mathcal{E}_h^i$$

The local error estimation on an element is defined as:

$$\eta_T^2 = h_T^2 \|-\Delta u_h + u_h\|_{0,T}^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_h^i\cap\partial T} he\|J_{F,1}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h^b\cap\partial T} he\,\|J_{F,1}\|_{0,e}^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_h^i\cap\partial T}\sigma he^{-1}\|J_{F,2}\|_{0,e}^2$$

(4.1)

The global error estimator is as follows:

$$\eta(u_h) = \left(\sum_{T\in\mathcal{T}_h}\eta_T^2\right)^{1/2} \quad (4.2)$$

Introduce a stable lifting operator $\Lambda: V_h \to [[S^h]]^2$

$$\int_\Omega \Lambda(v)\cdot w\,\mathrm{d}x = \sum_{e\in\mathcal{E}_h^i}\int_e [[v]]\cdot\{w\}\mathrm{d}s, \forall w\in[[S^h]]^2 \quad (4.3)$$

$$\|\Lambda(v)\|_{0,\Omega}^2 \le \sum_{e\in\mathcal{E}_h^i}\|he^{-1/2}[[v]]\|_{0,e}^2 \quad (4.4)$$

Simultaneously, defining an auxiliary bilinear form $\overline{a_h}(\cdot,\cdot): V_h\times V_h\to R$

$$\overline{a_h}(w,v) = \sum_{T\in\mathcal{E}_h}\int_T \nabla w\cdot\nabla v\,\mathrm{d}x + \sum_{T\in\mathcal{E}_h}\int_T w\cdot v\,\mathrm{d}x - \sum_{T\in\mathcal{E}_h}\int_T\nabla w\cdot\Lambda(v)\mathrm{d}x + \theta\sum_{e\in\mathcal{E}_h^i}\int_T\Lambda(w)\cdot\nabla v\,\mathrm{d}x + \sigma\cdot he^{-1}\sum_{e\in\mathcal{E}_h^i}\int_e[[w]][[v]]\mathrm{d}s \quad (4.5)$$

It is easy to know that $\overline{a_h} = a_h$ on $S^h\times S^h$, $\overline{a_h} = a$ on $H^1(\Omega)\times H^1(\Omega)$, and

$$\|\overline{a_h}(w,v\| \le \|w\|_G + \|v\|_G, \forall w,v\in V(h)$$

Lemma 4.1. For any $v\in S^h$, we have $E_h: S^h\to S^h\cap H^1(\Omega)$, such that

$$\sum_{T\in\mathcal{T}_h}\left(h_T^{-2}\|v-E_hv\|_{0,T}^2 + \|\nabla(v-E_hv\|_{0,T}^2\right) \le \sum_{e\in\mathcal{E}_h^i} he^{-1}\|[v]\|_{0,e}^2 \quad (4.6)$$

Lemma 4.2. Using U_h to represent linear finite element space, for any $\varphi\in H^1(\Omega)$, there is a segmented linear interpolation I^h, satisfy

$$\|\varphi - I^h\varphi\|_{0,T} + h_T\|\nabla(\varphi - I^h\varphi\|_{0,T} \le h_T\|\nabla\varphi\|_{0,w_T}, \forall T\in\mathcal{T}_h \quad (4.7)$$

$$\|\varphi - I^h\varphi\|_{0,T} \leq h_e^{1/2}\|\nabla\varphi\|_{0,w_e}, \forall e \in \mathcal{T}_h \quad (4.8)$$

where w_T is the union of all units that share a node with T, w_e is the union of all units that share a node with e.

Theorem 4.1. Suppose that (λ, u) and (λ_h, u_h) be the eigen pair of (2.2) and (2.3), respectively, $u \in H^{1+s}(\Omega)\left(s > \frac{1}{2}\right)$, for any $v \in H^1(\Omega)$, then there hold

$$\begin{aligned} &\| u - u_h \|_G \\ &\leq \sup_{v \in H^1(\Omega)} \frac{|\langle \lambda_h u_h, v\rangle + \overline{a_h}(u_h, v)|}{\| v \|_G} \\ &+ \inf_{v \in H^1(\Omega)} \| u_h - v \|_G \\ &+ \| \lambda_h u_h - \lambda u \|_{0,\partial\Omega} + \| u_h - u \|_{0,\Omega} \end{aligned} \quad (4.9)$$

Theorem 4.2. Under the conditions of Theorem 4.1, the following inequality holds:

$$\begin{aligned} &\| u - u_h \|_{\mathrm{G}} \\ &\leq \eta(u_h) + \| \lambda u - \lambda_h u_h \|_{0,\partial\Omega} \\ &+ \| u - u_h \|_{0,\Omega} \end{aligned} \quad (4.10)$$

Lemma 4.3. For all polynomial functions $v \in \mathbb{P}_m(T)$, $w \in \mathbb{P}_m(e)$, we have

$$\| v \|_{0,T} \leq \| \mathrm{b}_T^{1/2} v \|_{0,T} \quad (4.11)$$

$$\| w \|_{0,e} \leq \| \mathrm{b}_T^{1/2} w \|_{0,e} \quad (4.12)$$

Lemma 4.4. Suppose that (λ, u) and (λ_h, u_h) be the eigen pair of (2.2) and (2.3), respectively, there are the following local upper bounds:

$$\begin{aligned} h_T \| -\Delta u_h + u_h \|_{0,T} &\leq \| -\nabla(u - u_h) \|_{0,T} \\ &+ h_T \| u - u_h \|_{0,T} \end{aligned} \quad (4.13)$$

Let $e \in \mathcal{E}_h^i$ be the inner edge of adjacent units T^+ and T^-

$$\begin{aligned} h_e^{1/2} \| \mathrm{J}_{\mathrm{F},1} \|_{0,e} \leq \sum_{T \in U_e} (&\| -\nabla(u - u_h) \|_{0,T} \\ &+ h_T \| u - u_h \|_{0,T}) \end{aligned} \quad (4.14)$$

where $U_e = \{T^+, T^-\}$.

For each boundary edge $e \in \mathcal{E}_h^i$ and $e \in \partial\Omega$, we have

$$\begin{aligned} h_e^{1/2} \| \mathrm{J}_{\mathrm{F},1} \|_{0,e} &\leq \| -\nabla(u - u_h) \|_{0,T} \\ &+ h_e \| u - u_h \|_{0,T} \\ &+ h_e^{1/2} \| \lambda_h u_h - \lambda u \|_{0,e} \end{aligned} \quad (4.15)$$

For any $e \in \mathcal{E}_h^i$

$$h_e^{-1} \| [[u_h]] \|_{0,e}^2 = h_e^{-1} \| [[u - u_h]] \|_{0,e}^2 \quad (4.16)$$

Theorem 4.3. Under the conditions of Theorem 4.1, the following inequality holds:

$$\mathbf{n}_T \leq \sum_{T \in w_T} (\| -\nabla(u - u_h) \|_{0,T} + h_T \| u - u_h \|_{0,T}) \quad + \sum_{e \in \mathcal{E}_h^i \cap \partial\Omega} h e^{1/2} \| [[u - u_h]] \|_{0,e}$$

$$+\sum_{e\in\mathcal{E}_h^i\cap\partial\Omega} h e^{1/2}\|\lambda_h u_h-\lambda u\|_{0,e}\eta(u_h)$$
$$\le\|u-u_h\|_G+h^{1/2}\|\lambda_h u_h-\lambda u\|_{0,\partial\Omega} \tag{4.17}$$

V. NUMERICAL EXPERIMENTS

Consider the Steklov problem (2.1), The test domains are set to be the unit square $\Omega_S:=(0,1)^2$ with vertices are (0,1), (1,0), (0,0), (1,1) and the L-shaped domain $\Omega_L:=[0,1]\times[0,\frac{1}{2}]\cup[0,\frac{1}{2}]\times[\frac{1}{2},1]$, respectively. The four smallest approximate eigenvalues on Ω_S are

$\lambda_1\approx 0.240079085421,\lambda_2\approx 1.492303134531$

$\lambda_3\approx 1.492303134531,\lambda_4\approx 2.082647054031$

The four smallest approximate eigenvalues on Ω_L are

$\lambda_1\approx 0.182964236872,\lambda_2\approx 0.893672918808$

$\lambda_3\approx 1.688600483582,\lambda_4\approx 3.217859788054$

This paper presents a study on the multigrid discretization of Steklov eigenvalue problems. Based on our approach, we solve the eigenvalue problem on the fine grid $\mathcal{T}_h$ using linear elements and also provide solutions using Scheme 3.1 Numerical experiments are conducted on Ω_S and Ω_L. From TABLE I and TABLE III, it can be seen that when the mesh size increases, the advantages of the multigrid discretization method with shifted inverse iteration become more apparent, indicating the efficiency of our approach.

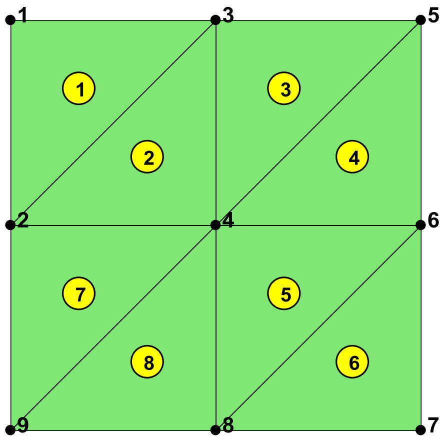

Fig. 1. The unit square domain Ω_S.

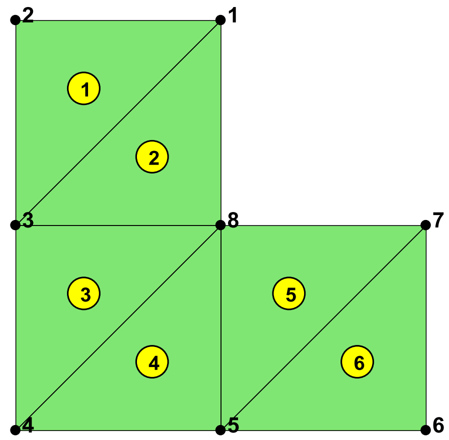

Fig. 2. The unit square domain Ω_L.

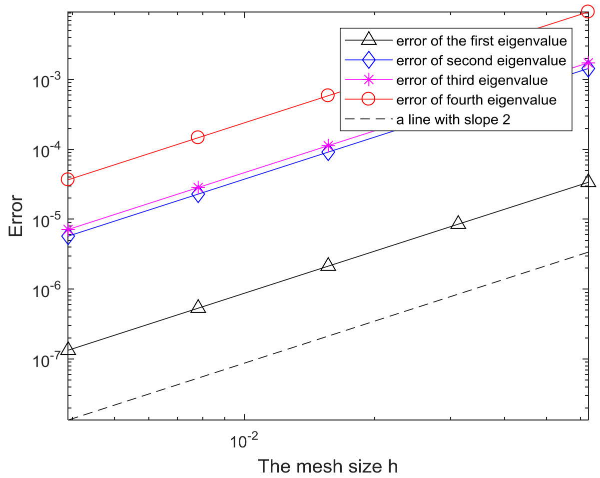

Fig. 3. The error curves in Ω_S.

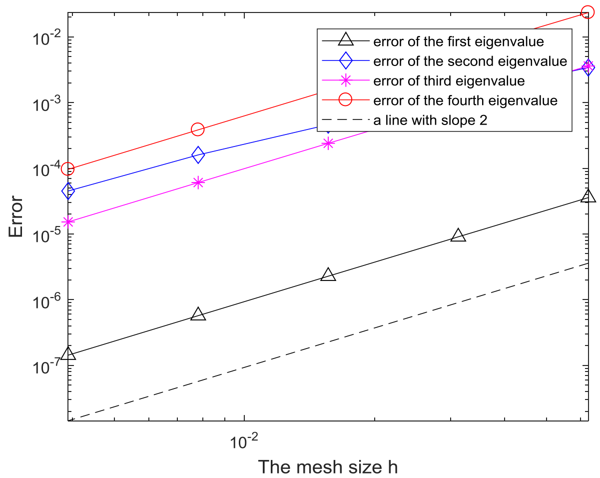

Fig. 4. The error curves in Ω_L.

TABLE III. The first fourth eigenvalues of (2.1) solved using linear elements on domain Ω_S, based on scheme 3.1.

h	$\lambda_{1,H}$	$\lambda_{2,H}$	$\lambda_{3,H}$	$\lambda_{4,H}$
1/16	0.240112826863608	1.493737311069134	1.494032112369319	2.091898207310659
1/32	0.240087583630703	1.492666919245926	1.492748911957080	2.084980101243282
1/64	0.240081216887900	1.492394549965722	1.492416140866160	2.083232720433993
1/128	0.240079619065589	1.492326033499974	1.492331570017943	2.082793751735668
1/256	0.240079218879208	1.492308863965719	1.492310265635612	2.082683761802871

TABLE IVI. The first fourth eigenvalues of (2.1) solved using linear elements on domain Ω_L, based on scheme 3.1.

h	$\lambda_{1,H}$	$\lambda_{2,H}$	$\lambda_{3,H}$	$\lambda_{4,H}$
1/16	0.182999984078385	0.897087899512774	1.692214637235418	3.241311541144290
1/32	0.182973282649124	0.894937883634405	1.689540378762870	3.223904609240853
1/64	0.182966511922253	0.894134444101236	1.688840311680383	3.219389751050658
1/128	0.182964807413177	0.893832650873182	1.688661107773306	3.218244256048426
1/256	0.182964379708959	0.893717985383559	1.688615730973319	3.217956122839075

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