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Review Paper

Locally Convex Inductive Limit Topology of Generalised Algebra

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Abstract
The analysis of inductive limit of generalised algebra G(M) is considered, the bornology of the generalised space is introduced with the knowledge of the locally convex (Fréchet) space topology. The result is demonstrated by the generalised space G_X associated with various smoothing parameters ($\alpha \in A$) in the generalised algebra, G(M) that approximates and smoothens out the singularities of the space while preserving the algebraic and topological properties of the algebra, G(M).

Key words and phrases: *Inductive limit, Generalised space, Bornology, Generalised algebra, Singularities, Smoothing parameters.*

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I. Introduction

An inductive limit is a construction used to generate a larger topological space from a sequence of smaller spaces. This concept is often used when dealing with spaces that are not locally compact like the function spaces. During the past few decades, the inductive limit topology have attracted considerable attention due to its relevance in preserving both the algebraic and topological structures of the spaces (algebra). It ranges from the locally convex inductive limits, to the strict inductive limit, weakly compact and nuclear inductive limit topologies. However, the most popular amongst them is the locally convex inductive limits which arises in great superfluity throughout many areas of functional analysis and its applications, especially, in the theory of generalised functions, partial differential equations, e.t.c.. One can consult the works of Schaefer [11], Bourbaki [4], Edwards [5], Bierstedt [3] for detailedintroduction.

Many researchers in the last few decades have done some considerable researches in the area of inductive limit topology especially in the locally convex perspective. Among the earlier researchers are: Warner [14], formulates the analogous notion of the algebraic inductive limit of locally *m*-convex algebras and studied the locally *m*-convex algebras which are algebraic inductive limits of normed algebras. Also, Alberto [1], studies the case of an algebra, **A** carrying the finest locally *m*- convex topology which makes continuous the inclusion maps of a family of subalgebras $\{\boldsymbol{A}_i|_i \in I\}$, where every **A** is endowed with a topological structure of normed algebra. Hirai *etal* [9], examine the harmonic nature of the limit topology $\tau_{ind}\boldsymbol{X}$with the algebraic structure on $\boldsymbol{X}$in an infinite dimensional harmonic analysis, where they encountered inductive limits of certain topological algebraic objects such as the Lie groups, Banach algebra. Also, that the algebraic structure are not necessarily consistent with the inductive limit topologies. In the same vein, Harai*etal* [10], explore the inductive limits in the category of topological groups in various categories (topological algebras, semigroups), where they introduce the Bamboo-shoot topology and its relationship to the locally convex vector topology. The introduction of algebras which are inductive limit of Banach spaces and have inequalities which are different form for the norm in a Banach algebra is examined by Alpay*etal* [2]. Raad [12], studies the classes of inductive limit C^*- algebra considers as the unital algebras that have finitely connected graphs that arise from unital and injective connecting maps which contain inductive limit Cartansubalgebras. The Valdivia's lifting theorem of (pre) compact sets and convergent sequences from a quasi-(LB) space to a metrizable, strictly barrelled space and extension of it to a strictly larger collection of range spaces, where the space of generalised functions have their wavefront sets in a specified close cone is analysed by Gilsdorf [7]. This paper studies the locally convex (Fréchet) inductive limit topology associated with the space of generalised functions G_X and the generalised algebra $G(M)$.

The distribution of this paper is in four section, section 1, we present the introduction and literature review on inductive limit topologies. In section 2, we discuss the preliminaries and definition of necessary terms. In section 3, we present the linear spaces with partial algebraic structure. The case where G_X preserves the algebraic structure of the generalised algebra $G(M)$ is treated in section 4 and section 5 is the conclusion.

II. Preliminaries

2.1 Inductive Limits Topology

Let X and $G_{X\alpha}(\alpha \in \mathrm{A})$ be theFréchetand generalised spaces respectively over K. Let $g\alpha$ be a linear map;

$$g_\alpha : X \to G_X$$

and$\tau\alpha$ be a Fréchet topology on X. The inductive topology on G_X with respect to the family

$$\{(G_{X\alpha}, \tau_\alpha, g_\alpha) : \alpha \in A\}$$

is the strongest topology Fréchet for which each of the mappings $g\alpha(\alpha \in A)$ is continuous on X into $(G_{X\alpha}, \tau\alpha)$. i.e

$$g\alpha : X \to G_{X\alpha,}\tau\alpha$$

In other words, X is a subspace of $_{X\alpha}$.

Let $X\alpha$ be the family of subspaces of the generalised space $G_X \ni X\alpha = X_\beta$for$\alpha = \beta$, a direct set under inclusion satisfying $X = \cup_\alpha G_{X\alpha}$, where $\alpha,\beta \in A$ is directed set under $\alpha \leq \beta$ if $X\alpha \subseteq X_\beta$. Furthermore, on each $G_{X\alpha}(\alpha \in A)$, there is a seminorm $\| \cdot \|_{X\alpha}$given, such that whenever $\alpha \leq \beta$, the topology induced by $\| \cdot \|_{X\beta}$ is coarser than the topology induced by $\| \cdot \|_{X\alpha}$. Then, X can be topologise with the inductive limit topology of generalised space, G_X

Definition 2.1.A locally convex Fréchetspace X isbornological, if every convex, balanced subset $V \subseteq X$ that absorbs every bounded set in X is a neighbourhood of the origin. In other words, a bornological space is a locally convex (Fréchet) space on which each seminorm that is bounded on bounded set is continuous.

Bourbaki [4] summarises that every Fréchet space is bornological.

2.2 Moderate and Negligible Functions

Definition 2.2. Moderate and Negligible Functions ([8])

We set

$$\mathrm{E}(M) := \left(C^\infty(X)\right)^I \varepsilon_u(M) := \{(\alpha_\epsilon)_\epsilon \in \varepsilon(M) | \forall K \subset\subset X \forall \beta \in \mathbb{N}_0^n \exists N \in \mathbb{N}$$
$$with \sup_{x\in K}\left|\partial^\beta \alpha_\epsilon(x)\right| = \gamma(\epsilon^{-N})\ as\ \epsilon \to 0\}.$$
$$N(M) := \{(\alpha_\epsilon)_\epsilon \in \varepsilon(M) | \forall K \subset\subset M \forall \beta \in \mathbb{N}_0^n \forall u \in \mathbb{N}: \sup_{x\in K} \partial^\beta \alpha_\epsilon(x)|$$
$$= \gamma(\epsilon^u)\ as\ \epsilon \to 0\}$$

Elements of $\mathrm{E}(M)$ and $N(M)$ are called moderate and negligible functions respectively. Where, elements of moderate functions constitute the differential algebras and the negligible function is a differential ideal. To this end, the generalised algebra

$G(M)$ is defined as

$$\mathrm{G}(M) = \mathrm{E}(M)/N(M),$$

where the $\mathrm{G}(M)$ is an associative, commutative differential algebra.

Definition 2.3. Let X be a Fréchet space. Then, we can associate a generalised space $\mathrm{G_X}$ as follows:

Let $I \subset \mathrm{R}$ be the interval (0,1]. Then, we define the moderate nets in X to be

$$\mathrm{E(X)} := (\alpha_\epsilon)_\epsilon \in_I :$$

for continuous seminormsρ on X there exist Nsuch that $\rho(\alpha\epsilon) \sim 0(\epsilon^{\mathrm{N}})$.

The negligible nets as

$$N(\mathrm{X}) = (\alpha\epsilon)_\epsilon \in I :$$

for continuous seminormsρ on X and for all m

$$|\rho(\alpha\epsilon)\epsilon| \sim 0(\epsilon^m).$$

Then, the generalised space of X is defined as the quotient $\mathrm{G_X} = \mathrm{E(X)}/N(\mathrm{X})$ ([13]).

III. Linear Spaces with Partial Algebraic Structure

Ekhaguere [6] acknowledges that partial algebras arise due to its applications especially in quantum theory where observables are presented by unbounded linear maps that does not matched for multiplication. Among such linear maps is the generalised space (algebra) with point wise multiplication. The generalised space (algebras) are characterised as partial algebra.

Definition 3.1.Partial Algebra (See, [6]).
A partial algebra is a triplet (U,Γ,·) which comprises a linear space U, a partial multiplication · on U and a relation$\gamma \subseteq$ U × U given by

$$\gamma = \{(x,y) \in U \times U : x \cdot y \in U\} \quad (3.1)$$

such that
$(x,v),(x,z), (y,z) \in U \Rightarrow (x,\alpha v + \beta z),(\alpha x + \beta y,z) \in \Gamma$, and then

$$(\alpha x+ \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z) \quad (3.2)$$

and

$$x\cdot (\alpha v + \beta z) = \alpha(x \cdot v) + \beta(x \cdot z) \forall \alpha,\beta \in \mathcal{C} \quad (3.3)$$

where$\mathcal{C}$is the complex numbers.

Definition 3.2. Partial Subalgebra
Let (U,Γ,·) be a partial algebra. Then, a subspace $\mathfrak{B}$is called a partial subalgebra of U if $x \cdot y \in \mathfrak{B}$whenever $x,y \in \mathfrak{B}$and $(x,y) \in \Gamma$.
In the same vein, Ekhaguere remarked that if (U,Γ,·) is a partial algebra and Γ =
U×U, then, (U,Γ,·) is an algebra. In other words, every algebra is a partial algebra. Based on this remark and definition, we say that generalised functions (algebra) are classified as partial algebra since point wise multiplication is involved. Not only that, from the Preliminary section, the functions also enjoy the property of inductive limit topology,i.e (bornology).
Ekhaguere [6], made a characterisation that provides a linkage between algebras and partial algebra in the following theorem.

Theorem 3.3.*Ekhaguere2007 [6]*
Every algebra (*resp.*−*algebra*) *is an inductive limit of partial algebra* (*resp. partial* −*algebra*).
Consequently, in mathematical analysis, the spaces of test functions and generalised functions are topological vector spaces. Also, given a map that takes a Fréchetspace X into a space of generalised functions G_X, i.e, $g : X \longrightarrow G_X$, that is continuous,$G_X$ is endowed with a topology (inductive limit).

Proposition 3.4.G_X *is continuous for an inductive limit topology, τ on a* Fréchet*space, X, if and only if, the composition map is continuous for a given linear map g on a* Fréchet*space, X into the generalised space G_X.*
*Proof.*Let V be a neighbourhood at the origin in (G_X,τ). Suppose g is continuous, then the composition $g \circ g\alpha$ is continuous for each index set ($\alpha \in A$). Conversely, if

$$g \circ g\alpha : (X\alpha,\tau\alpha) \rightarrow G_X$$

is continuous for each $\alpha \in A$ and we recall V is a neighbourhood at the origin, such that we have

$$g\circ g\alpha^{-1}=g\alpha^{-1}\circ g^{-1}(V)$$
$$=(g\alpha^{-1}\circ g^{-1}(V)$$
$$=g\alpha^{-1}(g^{-1}(V))$$

This shows that $g_\alpha^{-1}(g^{-1}(V))$ is the neighbourhood at the origin, that is, it is an inductive limit topology τ on G_X. Hence, $g$is continuous.

IV. Results

Proposition 4.1.G_X *is a bornological space.*
*Proof.*Suppose, G_X is bornological, by the Propositions 3.4, we show that its topology can be induced by a family of seminorms since it is a directed system. Now, given a generalised functions space, G_X which is a continuous linear functional such that$y \in G_X$ is an open set in the topology, τ and ρ is a seminorm on G_X which is bounded on the bounded set. Next, we show that this family of seminorms induces a topology, τ on G_X. Since, $y \in G_X$ is an open set in the topology τ generated by ρ, we express $y \in G_X$ as a union of the sets defined by finite number of seminorms. This we do by introducing the concept of a unit ball
$Br(0)$ centred at 0 with radius r defined on the set of all points $y \in G_X$ by;

$$Br(0) = \{y \in G_X : \|y\| \leq r\}. \quad (4.1)$$

We take the union of the corresponding unit balls. i.e.

$$\bigcup_{y\in G_X}\{y\} = Br_1(0) \cup Br_2(0) \cup \cdots \cup Br_n(0). \quad (4.2)$$

By Proposition 3.4, this implies that y can be expressed as a union of the set defined by a finite number of seminorms which shows that the topology, τ on G_X is bornological. Hence, G_X isbornological.

Remark 4.2G_X being a bornologicalspace, allows the flow of an inductive limit topology. Also, it isFréchet, though not metrisable but can converge in a weak*-topology.

With respect to Theorem 3.3and the space of generalised functions, G_X, satisfying the bornological properties of the Fréchet space, we prove the following theorem.

Theorem 4.3.G_X *is an inductive limit of the algebra of generalised functions,* $G(M)$.

*Proof.*Let $\{G\alpha(M), \pi_{\alpha,\beta}\}$ be a directed system of the generalised algebra, where $\alpha,\beta \in A$ (an indexed set) and $\pi_{\alpha,\beta}$ is a map. Then, we define $G_X = \cup_\alpha G\alpha(M)/\sim$. This is the space determined by equivalence relations on the union of the generalised algebra and: $[a] \sim [b]$, if and only if;

$$\pi_{\alpha,\beta}([a]) = \pi_{\alpha,\beta}([b]) \quad (4.3)$$

$\forall \alpha,\beta \in$ A, $a,b \in G(M)$. By Theorem 4.1,for $\alpha > 0$, we have an inclusion map that maps an element of $G\alpha(M)$ to the corresponding generalised functions, G_X. That is

$\iota\alpha : G\alpha(M) \rightarrow G_X$, that satisfies for any $\alpha > \beta > 0$, the inclusion map ι_β is

compatible with the inclusion map $\iota\alpha$in the sense that$\iota\alpha(g\alpha) = \iota_\beta(g_\beta)$, where $g\alpha, g_\beta$are the representative of the equivalence class in $G\alpha(M)$. Now, the inclusion maps preserve the algebraic structure of the generalised algebra $G\alpha(M)$.

Let, $a_{1\alpha}, a_{2\alpha} \in G\alpha(M)$ and$\lambda \in \mathcal{C}$, then;

$$\iota_\alpha(a_{1\alpha} + a_{2\alpha}) = \iota_\alpha(a_{1\alpha}) + (a_{2\alpha}) \quad (4.4)$$

and

$$\iota_\alpha(\lambda a_{1\alpha}) = \lambda \iota_\alpha(a_{1\alpha}). \quad (4.5)$$

For any vector space V and a collection of linear maps $\phi_\alpha : G_\alpha(M) \rightarrow V$, satisfying the linearity of the vector space, there exists a linear map; $\phi : G_X \rightarrow V$ such that $\phi \circ \iota_\alpha = \phi_\alpha \forall \alpha > 0$. By Theorem 4.1, we consider the inductive limit of G_X by the equivalence classes of the elements in the direct sum,$\bigoplus_\alpha G\alpha(M)$, where the two elements are equivalent if their difference belongs to the kernel of the inclusion maps ι_α , that is $[a \sim b]$ if and only if $a - b$ is negligible.

Remark 4.4 The equivalence relation ensures that the two elements in the generalised algebra are considered equivalent if they agree with each other under the bonding maps.

V. Conclusion

The inductive limit, gives room for interpolation between the space of generalised functions, $G_X$associated with different smoothing parameters ($\alpha \in A$) in the generalised algebra $G(M)$. This interpolation shows how singularities in the generalised functions can be approximated and smoothened out while preserving the algebraic and topological properties of the generalised algebra, $G(M)$.

References

[1] Alberto, A. *Locally convex inductive limits of normed algebras*. Rendiconti del seminarioMatematicodellaUniversita di Padova, NUMDAM. 1974.

[2] Alpay, D.& G. Salomon. *On Algebras which are Inductive Limits of Banach Spaces.*Integral Equations and Operator Theory, 2013.arXiv:1302.3372v2 [Math.FA].

[3] Bierstedt, D. K. *Introduction to Locally Convex Inductive Limits.* Functional Analysis and Its Applications Lecture Note. ICPAM, 1956. Nice, p 35-133.

[4] Bourbaki, N. *Topological Vector Spaces Chapter 1 to 5* . Elements of mathematics, 2003. Springer.

[5] Edwards, R. E. *Fourier Series.* A Modern Introduction, Volume 2, 1981. Springer -Verlag, New York.

[6] Ekhaguere, G. O. S.*Some Characterisation of Partial Algebras*. Journal Math.Annal.App. Elsevier. 2007. (337),p 1295-1301.

[7] Gilsdorf, T. E. *Valdivia's Lifting Theorem For Non-Metrisable Spaces. 2022.*Preprint. arXiv:2205:1549v1 [Math.FA].

[8] Grosser, M. , M. Kunzinger, M. Oberguggenberger, & R. Steinbauer, .*Geometric Theory of Generalized Functions with Applications to General Relativity*. 2001. Springer.

[9] Harai, T. , H. Shimomura, N. Tatsuuma, & E. Hirai. *On Inductive Limits of Topological Algebraic Structures in Relation to the Product Topologies.* Infinite Dimensional Harmonic Analysis: Transaction of a Japanese - German Symposium. 2000.

[10] Harai, T., H. Shimomura, N. Tatsuuma, & E. Hirai. *The Inductive Limit of Topologies, their Direct Products and Problem Related to Algebraic Structure.* J. Math Kyota University (JMKYAZ). 2001. 41(3),p 475-505.

[11] Schaefer, H. H. *Topological vector spaces*. Springer, 1986. New York.

[12] Raad, Ali. *Existence and Uniqueness of Inductive Limit CartanSubalgebras on Inductive Limit C^*- algebras. 2021.* A PhD Thesis. Enlighten: these https:// these.gla.ac.uk/research-enlighten @ glasgow.ac.uk.

[13] Shantanu, Dave.*Geometrical Embeddings of Distributions into Algebras of Generalised Functions. 2007*. arXiv:0709.2039v1 [math.AP].

[14] Warner, S. *Inductive Limit of Normed Algebra*. Havard University, Cambridge, 1955. p1-27.