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Review Paper

An inventory model for deteriorating items with stock dependent demand rate and nonlinear holding cost

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Abstract
Traditional Economic Order Quantity (EOQ) models assume that the demand rate and holding cost are constant, regardless of the level of inventory or the passage of time. However, in real-world scenarios, the demand for a particular item can be influenced by various factors such as seasonality, selling price, promotional activity, and stock availability. Additionally, holding costs are often not constant — they may vary depending on the quantity of stock held or the duration of storage.This paper develops an EOQ-based inventory model for deteriorating items under two distinct cases: (a) Stock-dependent demand rate with a time-dependent holding cost, (b) Stock-dependent demand rate with a stock-dependent holding cost. In both cases, shortages are permitted. The aim is to formulate mathematical models to determine the optimal order quantity, optimal cycle time, and minimum total cost. Optimal solutions are presented for both cases using numerical examples.

Keywords: *Inventory model;stock dependent demand rate; eterioration; shortage; non-linear holding cost.*

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I. Introduction

Deterministic inventory models in the literature typically consider the demand rate to be either constant, time-dependent, or stock-dependent (Macías et al., 2021; Urban, 2005; Khanra et al., 2013). Common time-varying demand patterns include linear trends (both increasing and decreasing) and exponential trends. In the case of consumer goods, it has been observed that higher inventory levels often lead to increased sales. A well-stocked display tends to attract more customers, thereby boosting demand. This necessitates modeling demand as a function of the available inventory.

In practical business environments, deterioration is inevitable. As a result, numerous studies have focused on incorporating deterioration in inventory management (Duary et al., 2022; Raafat et al., 1991; Sana, 2010a).Inventory deterioration can be classified into three types: direct spoilage, physical depletion, and gradual deterioration. Direct spoilage arises from breakage or accidents during handling and storage — for example, the effectiveness of certain medicines may decline if refrigeration fails due to a power outage. In contrast, gradual deterioration refers to the slow degradation in quality of an item over time, a phenomenon that no inventory item can fully escape. If a constant deterioration rate is assumed, it may lead to inaccuracies in modelling inventory. In this study, we consider a time-dependent deterioration rate, which becomes active after a certain period following the arrival of new stock.

Shortages are also common in practical inventory systems. When demand exceeds available inventory, a shortage occurs, which can disrupt production due to a lack of raw materials or spare parts. While some companies may allow shortages as part of their inventory policy to reduce costs, prolonged or frequent shortages can negatively affect customer satisfaction and brand reputation. Significant contributions addressing inventory models with shortages include studies by Zhou (2003), Manna et al. (2006), Khanra et al. (2013), and Dye et al. (2006).

The objective of this study is to develop an inventory model for a deteriorating item where the demand rate is stock-dependent, holding cost is nonlinear, and shortages are allowed. The deterioration begins after a specific time interval following replenishment, and the rate of deterioration is time-dependent. The model aims to minimize the total cost per unit time of the inventory system over an extended period. Two cases are considered:

1. Nonlinear time-dependent holding cost
2. Nonlinear stock-dependent holding cost

For both cases, the optimal order quantity and corresponding cycle length are determined.

II. Assumption and Notation

The inventory model is developed on the basis of the following assumption and notation:

2.1 Assumption

1. Item cost does not vary with order size.
2. Lead time (The time between placing an order for replenishment stock and its receipts) is assumed to be Zero.
3. Replenishment is instantaneous i.e. replenishment rate is infinite.
4. Inventory consists of only one item. There is only one stocking point each cycle.
5. The time horizon of the inventory system is infinite. Only a typical planning schedule of length T is considered and all remaining cycles are identical.
6. The demand rate is deterministic and is known function of the instantaneous level of inventory q. The functional relationship between demand rate R (q) and the instantaneous inventory level q (t) is given by the following expression.

   $R(q) = Dq^{\beta}$ $D>0$, $0< \beta <1$, $q \geq 0$

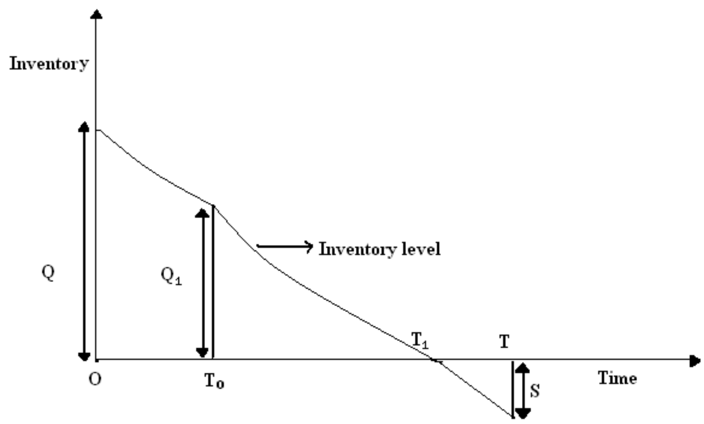
   Where denotes the on-hand inventory level at time t.
7. A time dependent function $\varphi(t) = \theta t$ $(0 <\theta<1)$ of the on hand inventory gets deteriorated per unit time after a time T_0 from the instant arrival of fresh replenishment in stock.
8. Shortages are allowed .

2.2 Notation

i)	T	: Length of the cycle.
ii)	TCU	: Total relevant inventory cost per unit time.
iii)	Q	: Initial stock level at the beginning of every inventory.
iv)	S	: Maximum storage level.
v)	$q_1(t)$	: On hand inventory level at time t. where $0\leq t \leq T_0$
vi)	$q_2(t)$	: On hand inventory level at time t , where $T_0 \leq t \leq T_1$
vii)	K	: Ordering cost per order.
viii)	C	:Cost per unit item.
ix)	HC	: Holding cost per cycle.
x)	DC	: Deterioration cost per cycle.
xi)	SC	: Shortage cost per cycle.
xii)	OC	:Ordering cost per cycle
xiii)	C_s	: Shortage cost per unit item

III. Mathematical Model Development

The inventory system developed is depicted by following figure.



The inventory level initial at time t = 0 is Q. At the beginning of each cycle the inventory level decreases rapidly because the quantity demanded is greater at a higher level of inventory. The on hand inventory gradually falls to the level Q_1, at time $t = T_0$, due to demand. After time

$t = T_0$, the stock level further decreases due to demand and deterioration ultimately, the inventory reached to the zero level; at time $t = T_1$ and shortages occur. Accumulated to the level S at time t = T

In the time interval [0, T_0], the inventory level will be depleted at a rate of $D[q(t)]^{\beta}$ where to will be determined by $q(T_0) = Q_1$, the corresponding value of Q_1 will be determined.

The differential equation describing the state of inventory level in [0, T_0] is given by.

$$\frac{d}{dt}q_1(t) = -D[q_1(t)]^{\beta} \text{for} 0 \leq t \leq T_0 \tag{1}$$

with boundary condition $q_1(0) = Q \;\; and \;\; q_1(T_0) = Q_1$

During the interval $[T_0, T_1]$ the inventory level is effected by the demand and deterioration. The inventory falls to zero level at time $t = T_1$

The differential equation describing the state of inventory level on [T_0, T_1] is given by

$$\frac{dq_2(t)}{dt} + \theta t\, q_2(t) = -D\,[q_2(t)]^{\beta} \text{for} T_0 \leq t \leq T_1 \tag{2}$$

with boundary condition $q_2(T_0) = Q_1$ and $q_2(T_1) = 0$

At the time $t = T_1$ shortages are allowed for replenishment up to time t=T, accumulated to the level S.During this interval [T_1,T] the system is affected by demand only

$$\frac{d}{dt}q_3(t) = -D \text{for} \quad T_1 \leq t \leq T \tag{3}$$

with boundary condition $q_3(T_1) = 0$ and $q_3(T) = -S$

the solution of equation (1) with the help of boundary conditionis

$$q_1(t) = [Q^a - taD]^{\frac{1}{a}} \text{for } 0 \leq t \leq T_0 \tag{4}$$

where $a = 1 - \beta$.

$$\text{Therefore,} \quad t = \frac{Q^a - q_1^a}{aD} \tag{5}$$

$$\text{and} q_1(T_0) = Q_1 \text{ gives } Q_1 = [Q^a - T_0 aD]^{\frac{1}{a}} \tag{6}$$

The solution of equation (2) with the help of boundary condition is

$$q_2(t) = [Da(T_1 - t)]^{\frac{1}{a}} [1 + \frac{\theta}{6}(T_1 - t)(T_1 + 2t)] \text{for} T_0 \leq t \leq T_1 \tag{7}$$

[neglecting θ^2 and higher order term]

From the condition $q_2(T_0) = Q_1$ gives

$$q_2(t) = [Da(T_1 - t)]^{\frac{1}{a}} [1 + \frac{\theta}{6}(T_1 - Q_1)(T_1 + 2Q_1)] \tag{8}$$

The solution of equation (3) with the help of boundary condition is

$$q_3(t) = -D(t - T_1) \text{for} T_1 \leq t \leq T \tag{9}$$

At time T shortage accumulate to the level S ie, $q_3(T) = -S$

$$\text{Therefore, } S = D(T - T_1) \tag{10}$$

The total variable cost compromises of the sum of ordering cost, holding cost, deterioration cost, minus backorder cost.

For the moment, the individual costs are now evaluated before they are group together:-

1) Ordering cost per cycle(OC) = K
2) Shortage cost per cycle (SC)= $C_s\int_{T_1}^{T} q_3(t)\, dt = -\frac{DC_S(T-T_1)^2}{2}$
3) The deterioration cost in the interval [0 , T_1] is given by
   $= C[Q - \int_0^{T_1} D[q(t)]^{\beta}\, dt]$

$= C[Q - \int_0^{T_0} D[q_1(t)]^\beta dt - \int_{T_0}^{T_1} D[q_2(t)]^\beta dt]$

$= C[Q - I_1 - I_2]$

$\therefore$ Where, $I_1 = \int_0^{T_0} D[q_1(t)]^\beta dt = - \int_Q^{Q_1} dq_1 = Q - Q_1$

and $I_2 = \int_{T_0}^{T_1} D[q_2(t)]^\beta dt$

$= (Da)^{\frac{1}{a}}(T_1 - T_0)^{\frac{1}{a}}[1 + \frac{\theta\beta}{6}[\frac{3T_1(T_1-T_0)}{a+1} - \frac{2(T_1-T_0)^2}{2a+1}]]$

Hence deterioration cost (DC) per cycle is

$= C\Big[Q_1 - [Da(T_1 - T_0)]^{\frac{1}{a}}\ [1 + \frac{\theta\beta}{6}(T_1 - T_0)\left(\frac{3T_1}{1+a} - \frac{2(T_1-T_0)}{1+2a}\right)]\Big]$

4)Holding cost per cycle (HC): Here we consider two possibilities of variation in the holding cost function namely a numerical function of length of time for which the item is held in stock and a nonlinear function of the amount of on hand inventory.

3.1 Case 1 (when holding costis non-linear time dependent):

In this formulation we treat the holding cost as a power function of the length of time

$\therefore$ Holding Cost per cycle (HC_1)

$=[\int_0^{T_0} ht^n\, q_1(t)dt + \int_{T_0}^{T_1} ht^n\, q_2(t)dt]$ where $n \in Z^+ \setminus \{1\}$

$= I_3 + I_4$

Where $I_3 = \frac{h}{(aD)^n D} \int_{Q_1}^{Q} Q^{na}\left[1 - \left(\frac{q_1}{Q}\right)^a\right]^n q_1^a\, dq_1$

$= \frac{h}{(a)^n D^{n+1}} \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q^{a(n-r)} \left[\frac{Q^{a(r+1)+1} - Q_1^{a(r+1)+1}}{a(r+1)+1}\right]$ [as $\left|\frac{q_1}{Q}\right| < 1$]

$I_4 = \int_{T_0}^{T_1} h(Da)^{\frac{1}{a}} t^n (T_1 - t)^{\frac{1}{a}} \left[1 + \frac{\theta}{6}(T_1 - t)(T_1 + 2t)\right] dt$

$= h(Da)^{\frac{1}{a}}(T_1)^n \sum_{r=o}^{n} \binom{n}{r}(-1)^r \left(\frac{1}{T_1}\right)^r \left[\frac{(T_1-T_0)^{k_1}}{k_1} + \frac{\theta T_1(T_1-T_0)^{k_2}}{2k_2} - \frac{\theta}{3}\frac{(T_1-T_0)^{k_3}}{k_3}\right]$

Where $k_1 = \frac{(r+1)a+1}{a}$ $k_2 = \frac{(r+2)a+1}{a}$ $k_3 = \frac{(r+3)a+1}{a}$

$\therefore$ Holding cost per cycle (HC) = $I_3 + I_4$

$= h\Bigg[(Da)^{\frac{1}{a}}(T_1)^n \sum_{r=o}^{n} \binom{n}{r}(-1)^r \left(\frac{1}{T_1}\right)^r \left[\frac{(T_1-T_0)^{k_1}}{k_1} + \frac{\theta T_1(T_1-T_0)^{k_2}}{2k_2} - \frac{\theta}{3}\frac{(T_1-T_0)^{k_3}}{k_3}\right] +$

$\frac{1}{(a)^n D^{n+1}} \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q^{a(n-r)} \left[\frac{Q^{a(r+1)+1} - Q_1^{a(r+1)+1}}{a(r+1)+1}\right]\Bigg]$

Where $k_1 = \frac{(r+1)a+1}{a}$ $k_2 = \frac{(r+2)a+1}{a}$ $k_3 = \frac{(r+3)a+1}{a}$

$\therefore$ Total cost per unit time is given by

$$TCU_1(T_1, T) = \frac{1}{T}(OC + HC_1 + DC - SC)$$

3.2 Case 2 (when holding cost is non-linear stock dependent):

In this formulation, we treat the holding cost rate as a power function of on hand inventory,

ie, $\frac{d}{dt}(HC_2) = hq^n$ where $n \in Z^+ \setminus \{1\}$

$\therefore$ Holding cost per cycle (HC_2) is

$= \int_o^{T_0} hq_1{}^n dt + \int_{T_0}^{T_1} hq_2{}^n\, dt$

$= I_5 + I_6$

Where $I_5 = \int_0^{T_0} hq_1^n dt = \frac{h}{(n+a)D}\left(Q^{n+a} - Q_1{}^{n+a}\right)$

$I_6 = \int_{T_0}^{T_1} hq_2{}^n dt$

$$= \mathrm{h}(Da)^{\frac{n}{a}}\left[\frac{a(T_1-T_0)^{\frac{n+a}{a}}}{n+a} + \frac{n\theta}{6}\left[\frac{3T_1a(T_1-T_0)^{\frac{n+2a}{a}}}{n+2a} - \frac{2a(T_1-T_0)^{\frac{n+3a}{a}}}{n+3a}\right]\right]$$

∴ Total cost per unit time is

$$TCU_2(T_1,T) = \frac{1}{T}(OC + HC_2 + DC - SC)$$

The solutions for the optimal values of T_0 and T (say T_0* and T*) can be found by solving the following equations simultaneously:

$$\frac{\partial\, TCU(T_1,T)}{\partial T_0} = 0, \text{and } \frac{\partial\, TCU(T_1,T)}{\partial T} = 0$$

The cost function $TCU\ (T_1, T)$ will be a convex functionif

$\left[\frac{\partial^2 TCU\ (T_0,T)}{\partial T_0^2}\right]$at ($T_0$*,T*) > 0 ,and

$\begin{vmatrix} \frac{\partial^2 TCU\ (T_1,T)}{\partial T_1^2} & \frac{\partial^2 TCU\ (T_1,T)}{\partial T_1 \partial T} \\ \frac{\partial^2 TCU(T_1,T)}{\partial T \partial T_1} & \frac{\partial^2 TCU(T_1,T)}{\partial T^2} \end{vmatrix}$> 0 at ($T_1$*,T*) are satisfied

IV. Numerical Analysis:

In this section, numerical examples are considered to illustrate the proposed model. Let D = 2.0; h= 0.3; C_s= 0.5; θ= .03; C = 10.00; K = 200; T_0= 5 days

For the case 1, optimal solutions are T = 35 days, T_1 = 12 days, Q = 250, TCU= 440 and the optimality conditions $\left[\frac{\partial^2 TCU(T_1,T)}{\partial T_1^2}\right]_{(T_1^*,T^*)} = 150 > 0$ and $\left[\left[\frac{\partial^2 TCU(T_1,T)}{\partial T_1^2}\right]\left[\frac{\partial^2 TCU(T_1,T)}{\partial T^2}\right] - \left[\frac{\partial^2 TCU(T_1,T)}{\partial T_1 \partial T}\right]^2\right]_{(T_1^*,T^*)} = 12.5 \times 10^3 > 0$ are satisfied.

For Case 2, optimal solutions are T = 59 days, T_1 = 12.5 days, Q = 200 units, TCU = 579.0025the optimality conditions $\left[\frac{\partial^2 TCU(T_1,T)}{\partial T_1^2}\right]_{(T_1^*,T^*)} = 167 > 0$ and $\left[\left[\frac{\partial^2 TCU(T_1,T)}{\partial T_1^2}\right]\left[\frac{\partial^2 TCU(T_1,T)}{\partial T^2}\right] - \left[\frac{\partial^2 TCU(T_1,T)}{\partial T_1 \partial T}\right]^2\right]_{(T_1^*,T^*)} = 14.7 \times 10^3 > 0$ are satisfied.

V. Conclusion

In this paper, an EOQ-based inventory model over an infinite time horizon has been developed, incorporating stock-dependent demand and nonlinear holding costs, with shortages allowed. Two variations of holding cost structures were considered: (a) a nonlinear function of the time and (b) A nonlinear function of the stock level. Approximate optimal solutions were obtained for both cases. The results indicate that the total inventory cost is lower when the holding cost is modeled as a function of the on-hand inventory, highlighting the significance of incorporating stock-dependency in holding cost structures. The proposed model contributes to the literature by more accurately reflecting real-world inventory dynamics, especially for deteriorating items.

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