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Review Paper

Intertwining the Upwind Differencing and the Newton-Lieberstein's Methods to Solve Mildly Non-linear Boundary Value Problems (MNBVP)

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Abstract
While different numerical and classical methods have been used to solve Boundary Value Problems but not many of these methods have been used to solve Mildly Non-Linear Boundary Value Problems (MNBVP) hence there is a need to think outside the box of ways of achieving this feat. This method requires that two stages that different techniques are involved. The two techniques have different algorithms.

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I. Introduction

In recent years, there were some problems arises from the fields of science and engineering represented by mathematical models. These mathematical models can be written in the form of differential equations, either as a first order or higher order ordinary differential equations (ODEs). This study considers for solving second order

Non-stiff initial value problems (IVPs) of ODEs of the form

$$y'' = f(x, y, y')\ y(a) = y_0, y'(a) = y_0'\ \ x \in [a, b]. \qquad \text{Eqtn. (1)}$$

The approach here is to solve Eq. (1) directly without reducing to first order ODEs using four-point one-step block method. The proposed method has been used to calculate the approximation solution of four points simultaneously in a block. The basic idea of the one-step block method has been studied by Rosser (1967) who introduced a block of new approximation values simultaneously. The approach also been discussed in Worland (1976) and Majid *et al.* (2003). In Majid *et al.* (2003), the authors described a two-point implicit one-step block method for solving first order ODEs based on integration formula using the closest point in the block.

Equation (1) has already been solved directly by several researchers such as Chakravarti and Worland (1971), Suleiman (1989), Fatunla (1991), and Omar and Suleiman, (2005). The system of higher order ODEs can be reduced to a system of first order equation and then solved using first order ODEs. This approach will enlarge the system of first order ODEs and needs more computational work. According to [12], the main idea of this research is to extend the work done by Majid *et al.* (2003) for solving Eq. (1) directly and using variable step size. Numerical results are given to show the efficiency of the proposed method.

Meaning of Boundary Value Problem

1. Mildly Non-Linear Boundary Value Problem

The Mildly Nonlinear Boundary Value Problems (MNBVP) comes in various form. However, the most widely studied variant of (MNBVP) is of the form:

$y'' = f(x, y)$ (2.1)

$y(a) = \alpha,\ y(b) = \beta.$ (2.2)

To ensure that (2.1) and (2.2) have a unique solution, one assumes that

$\frac{\partial f(x,y)}{\partial y} \geq 0,\ a \leq x \leq b$, and $-\infty < y < \infty.$ (2.3)

When (2.3) is valid, (2.1) and (2.2) is called a Mildly Nonlinear problem.

While there are many theories for the numerical solution of linear and nonlinear boundary value problems though is more complex than that for initial value problems, attention will be focused on theorem to support the linear boundary value problems of a special type as in (2.4) and (2.2).

Theorem 1: Let I be an interval $a < x < b$. Let α and β be constants and let $P(x)$, $Q(x)$, and $R(x)$ be continuous on $a \leq x \leq b$. Considering the linear boundary value problem

$y'' + P(x)y' + Q(x)y = R(x),\ x \in I$ (2.4)

$y(a) = \alpha$, and $y(b) = \beta$ (2.4*)

It is important to note that for numerical reasons, *a priori*, that the solution of the problem (2.4) and (2.4*) exists and it has a unique solution. Based on this, it is assumed, in addition to the above that

$Q(x) \leq 0,\ a \leq x \leq b,$ (2.5*)

that is sufficient to ensure the existence of the solution and the uniqueness of the solution.

If one assumes y' to be an independent variable, then

$\frac{\partial}{\partial y}[-P(x)y' - Q(x)y + R(x)] = -Q(x)$ (2.5)

So, the condition (2.3) implies either $-Q(x) \geq 0$ or $Q(x) \leq 0$.

2. Newton-Lieberstein's Method Algorithm

The numerical solution of (2.1) and (2.2) follows in the manner prescribed with the three-point central approximation

$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f(x_i, y_i)$ (2.6)

Replacing (2.6) and (1.3) in (2.4) gives rise to

$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + P(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + Q(x_i)y_i = R(x_i)$ for $i = 1, 2, 3, \dots, n-1.$ (2.7)

The resulting nonlinear system (2.7) can now be solved using the Newton-Lieberstein's method. To illustrate the processes of changing the differential equation into an argument matrix that will later be solved using the Newton-Lieberstein's method the following examples will be employed to demonstrate that.

For $n \geq 2$, the general linear algebraic system of n equations in the n unknowns $x_1, x_2, \dots, x_n$ is given as

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_1 = f_1(x_1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_2 = f_2(x_2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_3 = f_3(x_3) \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = f_n(x_n) \end{matrix}\right\} \quad (3.1)$$

From (3.1) the coefficient matrix A, variable vector x, and solution vector b are defined by

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } \boldsymbol{b} = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \\ \vdots \\ f_n(x_n) \end{pmatrix} \quad (3.2)$$

Then, the system (3.1) can be compactly written as

$\boldsymbol{Ax} = \boldsymbol{b}$ (3.3)

The system (3.1) or its equivalent (3.3) is said to be tridiagonal if all the entries are zero except a_{ii}, $a_{j,j+1}$, $a_{j+1,j}$ for $i = 1, 2, 3, \dots, n$, $j = 1, 2, 3, \dots, n-1$, and none of these is zero. A system of simultaneous algebraic equations with nonzero coefficients only on the main diagonal, the lower diagonal, and the upper diagonal is called a tridiagonal system of equations.

The term tridiagonal is most appropriate for the coefficient matrix in (3.3) if it has the form that the main diagonal a_{ii}, the super-diagonal $a_{j,j+1}$, (diagonal above the diagonal), and the sub-diagonal $a_{j+1,j}$, (diagonal below the diagonal) are the only entries that are non-zeros.

For practical importance, in ensuring that the tridiagonal system has a unique solution, it is expedient the following conditions are satisfied.

Theorem 2: Let (3.3) be a tridiagonal system. Let the main diagonal entries be negative while the sub-diagonal and super-diagonal entries are positive such that

$$a_{ii} < 0, i = 1, 2, 3, \dots, n \tag{3.4}$$

$$a_{i,i+1} > 0, i = 1, 2, 3, \dots, n-1 \tag{3.5}$$

$$a_{i+1,i} > 0, i = 1, 2, 3, \dots, n-1 \tag{3.6}$$

Furthermore, let the main diagonal entries dominate the matrix in that, let the absolute value of the main diagonal entries be greater than or equal to the sum of all other row entries, with strict inequality for at least one row of the coefficient matrix. Precisely, let

$$-a_{11} \geq a_{12} \tag{3.7}$$

$$-a_{n,n} \geq a_{n,n-1} \tag{3.8}$$

$$-a_{ii} \geq a_{i,i-1},\ i = 2, 3, \dots, n-1, \tag{3.9}$$

With strict inequality holding for at least one of (3.7) through (3.9). then, the resulting linear algebraic system has one and only one solution.

The availability of Theorem 2 before beginning computations cannot be underestimated. If a system were to have no solution, computations can yield nonsense numbers. If a system were to have more than one solution, certain computational techniques could drift from one solution to another.

Considering the system in (3.1) in which the coefficient is

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & & & & & & 0 \\ a_{21} & a_{22} & a_{23} & & & & & \\ & a_{32} & a_{33} & a_{34} & & & & \\ & & \dots & \dots & \dots & & & \\ & & & \dots & \dots & \dots & & \\ & & & & \dots & \dots & \dots & \\ & & & & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & & & & & & a_{n,n-1} & a_{nn} \end{pmatrix} \tag{3.9*}$$

satisfying the conditions in Theorem 2. If $f_i(x_i)$ for all $i = 1, 2, \dots, n$, are not all constants and the derivative, $f_i'(x_i) \geq 0$ then the system is classified as a mildly nonlinear. According to [Ortega and Rheinboldt (1980)], all such mildly nonlinear system has solutions and their solutions are unique.

The Newton-Lieberstein method for solving a mildly nonlinear system is given in the following algorithm.

Step 1: Guess an initial value for $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots, x_n^{(0)}$ and a value for ω in the range $[0, 2]$.

Step 2: For $k = 0, 1, 2, 3, \dots$, iterate with the relations

$$x_1^{(k+1)} = x_1^{(k)} - \omega\left(\frac{a_{11}x_1^{(k)} + a_{12}x_2^{(k)} - f_1(x_1^{(k)})}{a_{11} - f_1'(x_1^{(k)})}\right)$$

$$x_2^{(k+1)} = x_2^{(k)} - \omega\left(\frac{a_{21}x_1^{(k)} + a_{22}x_2^{(k)} + a_{23}x_3^{(k)} - f_2(x_2^{(k)})}{a_{22} - f_2'(x_2^{(k)})}\right)$$

$$x_3^{(k+1)} = x_3^{(k)} - \omega\left(\frac{a_{32}x_2^{(k)} + a_{33}x_3^{(k)} + a_{34}x_4^{(k)} - f_3(x_3^{(k)})}{a_{33} - f_3'(x_3^{(k)})}\right)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$x_{n-1}^{(k+1)} = x_{n-1}^{(k)} - \omega\left(\frac{a_{n-1,n-2}x_{n-2}^{(k)} + a_{n-1,n-1}x_{n-1}^{(k)} + a_{n-1,n}x_n^{(k)} - f_{n-1}(x_{n-1}^{(k)})}{a_{n-1,n-1} - f_{n-1}'(x_{n-1}^{(k)})}\right)$$

$$x_n^{(k+1)} = x_n^{(k)} - \omega\left(\frac{a_{n,n-1}x_{n-1}^{(k)} + a_{nn}x_n^{(k)} - f_n(x_n^{(k)})}{a_{nn} - f_n'(x_n^{(k)})}\right)$$

Step 3: Set a prescribed convergence tolerance, ϵ. Determine if $\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon$, for $i = 1, 2, 3, \dots, n$. Else, repeat step 2.

Step 4: If step 3 is satisfied then, substitute the values of $x_1^{(k+1)},\ x_2^{(k+1)}, x_3^{(k+1)}, \dots, x_n^{(k+1)}$ into the original system of equations to establish that they are an approximate solution to the given problem.

3. Upwind Differencing

4. Supporting Theorem

5. Results and Discussions

6. Conclusion

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