Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 7 (July 2025) pp: 57-62
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Identities related to generalized derivations on ideals in prime rings

V. K. Yadav, S. K. Sharma *
Department of Mathematics
D.S. College, Aligarh Affiliated to R M P S University Aligarh, U.P. 202001.

**Corresponding Author, Email address: iitdvishalny@gmail.com*

Abstract

Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F, G, H : R \to R$ are generalized derivations associated with derivations d, g, h respectively. If anyone of the following hold:

(i) $F(xy) + G(x)H(y) \in Z(R)$; (ii) $F(xy) + G(x)H(y) + [\alpha(x), y] = 0$; (iii) $F(xy) + G(y)H(x) + [x, \alpha(y)] = 0$; (iv) $F(xy) + [G(x), y] + G(x)H(y) = 0$; (v) $F(xy) + [x, G(y)] + G(y)H(x) = 0$; for all $x, y \in I$, where α is any map on R and $Z(R)$ is the Centre of R, then R is commutative.

Keywords: *Prime ring, Ideal, Commuting map, Generalized derivation.*
Mathematics Subject Classification (2000): *16W25, 16N60, 16R50.*

Received 15 July, 2025; Revised 28 July, 2025; Accepted 30 July, 2025 © The author(s) 2025. Published with open access at ***www.questjournas.org***

I. Introduction

Throughout this paper, R denotes an associative ring with center $Z(R)$. A ring R is said to be prime ring if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$. We denote operation $\circ$ as a Jordan product which is defined on R as $x \circ y = xy + yx$, for all $x, y \in R$ and Lie product of x, y is defined as $[x, y] = xy - yx$ $\forall x, y \in R$. A mapping $f : R \to R$ is said to be additive if $f(x + y) = f(x) + f(y)$, for all $x, y \in R$. An additive mapping d from R to R is said to be a derivation, if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Let S be a subset of R, then a mapping $f : R \to R$ is said to be commuting on S if $[f(x), x] = 0$ for all

$x \in S$. A mapping $F : R \to R$ is said to be left multiplier if $F(xy) = F(x)y$ for all $x, y \in R$. An additive mapping $F : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. The concept of generalized derivation introduced by Bresar [8]. Obviously, the class of generalized derivation is bigger than the class of derivation as every derivation is generalized derivation but not conversely. The concept of generalized derivation also covers the concept of left multiplier maps. Firstly, E.C. Posner [17] proved pioneer results on derivation in prime rings. He established relation between derivation on a ring and structure of that ring. Many authors have generalized Posner's theorems, for suitable subsets of R, as ideal, left ideal, Lie ideal and Jordan ideal, further information can be found in ([2], [5],[9],[12],[15],[16], [18],[20]). In [6] Bell and Kappe proved that if a derivation d of a prime ring R which acts as homomorphisms or anti- homomorphisms on a nonzero right ideal of R then $d = 0$ on R. In [14] Nadeem- ur rehman generalized Bell and Kappe result by taking generalized derivation instead of derivation. Precisely, he proved, let R be a 2-torsion free prime ring and I be a non zero ideal of R. Suppose $F : R \to R$ is a nonzero generalized derivation with non zero derivation d. If F acts as a homomorphism or anti- homomorphism on I then R is commutative. In this sequence, in 2001 Ashraf and Rehman [4], proved that if R is a prime ring with a non-zero ideal I of R and d is a derivation of R such that either $d(xy) \pm xy \in Z(R)$ for all $x, y \in I$ or $d(xy) \pm yx \in Z(R)$ for all $x, y \in I,$ then R is commutative. Again, Asraf et al. [3] proved that if R is a prime ring which is 2 torsion free and F is a generalized derivation associated with derivation d on R. If F satisfies any one of the following conditions: (i) $F(xy) - xy \in Z(R)$; (ii) $F(xy) - yx \in Z(R)$; (iii) $F(x)F(y) - xy \in Z(R)$; (iv) $F(x)F(y) - yx \in Z(R),$ for all $x, y \in I,$ where I is an ideal of $R,$ then R is commutative. Recently, Albas [17] studied following identities in prime rings: (i) $F(xy) \pm F(x)F(y) \in Z(R)$; (ii) $F(xy) \pm F(y)F(x) \in Z(R),$ for all $x, y \in I,$ a non zero ideal of $R.$

In 2015, S.K Tiwari et al. [19] considered the following situations: (i) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$; (ii) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$; (iii) $G(xy) \pm F(y)F(x) \pm xy \in Z(R)$;(iv) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$; (v) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$; (vi) $G(xy) \pm F(x)F(y) \pm [\alpha(x), y] \in Z(R),$ for all $x, y \in I,$ where I is a non-zero ideal in prime ring R, $\alpha : R \to R$ is any mapping and F, G are two generalized derivations associated with derivations d, g respectively.

Motivated by above results, in this paper we are considering following situations: (i) $F(xy) + G(x)H(y) \in Z(R)$; (ii) $F(xy) + G(x)H(y) + [\alpha(x), y] = 0$; (iii) $F(xy) + G(y)H(x) + [x, \alpha(y)] = 0$; (iv) $F(xy) + [G(x), y] + G(x)H(y) = 0$; (v)$F(xy)+[x, G(y)]+G(y)H(x) = 0$; for all $x, y \in I,$ where $I,$ a non zero ideal of $R,$ where α is any map on R and F, G, H are generalized derivations associated with derivations d, g, h respectively.

II. Preliminaries Results

The following Lemmas will be used in our results:

Lemma 2.1 ([11], Lemma 1.1). *Let R be a prime ring of characteristic different from 2, L a noncentral Lie ideal of R, d a nonzero derivation of R, $n \geq 1$. If d satisfies $[d(u), u]^n = 0,$ for any $u \in L,$ then R is commutative.*

Lemma 2.2 ([12], Lemma 3). *If a prime ring R contains a commutative nonzero right ideal, then R is commutative.*

III.Main Results

Theorem 3.1. *Let R be a prime ring of characteristic different from 2 and I be a nonzero ideal of R. Suppose that $F, G, H : R \to R$ are non zero generalized derivations, associated with derivations d, g, h : $R \to R$ respectively such that $F(xy) + G(x)H(y) \in Z(R),$ for all $x, y \in I.$ If g, h are non zero derivation then R is commutative.*

Proof. We have

$$F(xy) + G(x)H(y) \in Z(R) \tag{1}$$

for all $x, y \in I.$ Replacing y by yz in (1), we get

$$(F(xy) - G(x)H(y))z + xyd(z) + G(x)yh(z) \in Z(R) \tag{2}$$

for all $x, y, z \in I.$ Commuting (2) with $z,$ we get

$$[xyd(z), z] + [G(x)yh(z), z] = 0 \tag{3}$$

for all $x, y, z \in I.$ Replacing x by xw in (3), we get

$$[xwyd(z), z] + [G(x)wyh(z), z] + [xg(w)yh(z)] = 0 \tag{4}$$

for all $x, y, z, w \in I.$ Again replacing y by wy in (3), we get

$$[xwyd(z), z] + [G(x)wyh(z), z] = 0 \quad (5)$$

for all $x, y, z, w \in I$. Subtracting equation (5) from equation (4), we get

$$[xg(w)yh(z), z] = 0 \quad (6)$$

for all $x, y, z, w \in I$. Replacing y by $yh(z)u$ in equation (6), we get

$$xg(w)yh(z)[uh(z), z] + [xg(w)yh(z), z]uh(z) = 0 \quad (7)$$

for all $x, y, z, u, w \in I$. Using equation (6), we obtain $xg(w)yh(z)[uh(z), z] = 0$, for all $x, y, z, u, w \in I$. Since R is prime and I a non zero ideal, therefore either $xg(w) = 0$ or $h(z)[uh(z), z] = 0$, for all $x, y, z, u, w \in I$. In case $xg(w) = 0$, for all $x, w \in I$. we obtain $g = 0$, a contradiction. Therefore $h(z)[uh(z), z] = 0$, for all $z, u \in I$. It can we written as,

$$h(z)u[h(z), z] + h(z)[u, z]h(z) = 0 \quad (8)$$

for all $z, u \in I$. Replacing u by $uh(z)$ in equation (8), we get

$$h(z)uh(z)[h(z), z] = 0 \quad (9)$$

for all $z, u \in I$. Again replacing u by z in equation (8), we get $h(z)uh(z)z[h(z), z] = 0$, for all $z, u \in I$. Replacing u by uz in equation (9), then subtracting from last equation, we obtain $h(z)u[h(z), z]^2 = 0$, for all $z, u \in I$. Therefore we get $[h(z), z]^2u[h(z), z]^2 = 0$, for all $z, u \in I$. Using primness of R, we get $[h(z), z]^2 = 0$, for all $z, u \in I$. As h is a non zero derivation and if I in noncentral ideal of R sing then by Lemma 2.1, R is commutative. If I is central then by Lemma 2.2, again R is commutative. This proves the theorem. □

Theorem 3.2. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F, G, H : R \to R$ are generalized derivations, associated with derivations $d, g, h : R \to R$ respectively and $\alpha : R \to R$ is any map such that $F(xy) + G(x)H(y) + [\alpha(x), y] = 0$, for all $x, y \in I$. If g, h are non zero derivation then R is commutative.*

Proof. We have

$$F(xy) + G(x)H(y) + [\alpha(x), y] = 0 \quad (10)$$

for all $x, y \in I$. Replacing y by yz in (10), we get

$$(F(xy) + G(x)H(y) + [\alpha(x), y])z + xyd(z) + G(x)yh(z) + y[\alpha(x), z] = 0 \quad (11)$$

for all $x, y, z \in I$. Using equation (10),we get

$$xyd(z) + G(x)yh(z) + y[\alpha(x), z] = 0 \quad (12)$$

for all $x, y, z, w \in I$. Replacing y by wy in (12), we get

$$xwyd(z) + G(x)wyh(z) + wy[\alpha(x), z] = 0 \quad (13)$$

for all $x, y, z, w \in I$. Multiplying equation (12), by w from the right and sub- tracting from the equation (13), we get

$$[x, w]yd(z) + [G(x), w]yh(z) = 0 \quad (14)$$

for all $x, y, z, w \in I$. Replacing $x = w$ in (14), we get $[G(x), x]yh(z) = 0$, for all $x, y, z \in I$. Since R is prime then either $h = 0$ or $[G(x), x] = 0$, for all $x \in I$. As h is anon zero derivation on R, therefore

$$[G(x), x] = 0 \quad (15)$$

for all $x \in I$. Linearising above equation we get $[G(x), y] + [G(y), x] = 0$, for all $x, y \in I$. Replacing x by xy, we obtain, $[xg(y), y] = 0$, for all $x, y \in I$. Again replacing x by tx, we obtain $[t, y]xg(y) = 0$, for all $x, t, y \in I$. Using primness of R and let $A = \{y \in I | g(y) = 0\}$ and $B = \{y \in I | [t, y] = 0\}$. Then A and B are proper additive subgroup of I and $I = A \cup B$. Since a group can not be union of two proper subgroups therefore either $I = A$ or $I = B$. If $I = A$, we get $g = 0$, contradiction therefore $I = B$, hence R is commutative. This proves the theorem. □

Theorem 3.3. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F, G, H : R \to R$ are generalized derivations, associated with derivations $d, g, h : R \to R$ respectively and $\alpha : R \to R$ is any map such that $F(xy) + G(y)H(x) + [x, \alpha(y)] = 0$, for all $x, y \in I$. If g, h are non zero derivations then R is commutative.*

Proof. We have

$$F(xy) + G(y)H(x) + [x, \alpha(y)] = 0 \quad (16)$$

for all $x, y \in I$. Replacing x by xz in (16), we get

$$(F(xy) + G(y)H(x) + [\alpha(x), y])z + F(x)zy - F(x)yz + xd(zy) - xd(y)z$$
$$+G(y)xh(z) + x[z, \alpha(y)] = 0 \quad (17)$$

for all $x, y, z \in I$. Using equation (10),we get

$$F(x)[z, y] + xd(zy) - xd(y)z + G(y)xh(z) + x[z, \alpha(y)] = 0 \quad (18)$$

for all $x, y, z \in I$. Replacing $z = y$ in equation (18), we get

$$xzd(z) + G(z)xh(z) + x[z, \alpha(z)] = 0 \quad (19)$$

for all $x, y, z \in I$. Replacing x by ux in equation (19), we get

$$uxzd(z) + G(z)uxh(z) + ux[z, \alpha(z)] = 0 \quad (20)$$

for all $x, y, z, u \in I$. Multiplying equation (19) by u from left then subtracting from equation (20), we get

$$[G(z), u]xh(z) = 0 \quad (21)$$

for all $x, z, u \in I$. Thus for each $z \in I$ by primeness of R either $[G(z), u] = 0$ or $h(z) = 0$, for all $x, z, u \in I$. Let $A = \{z \in I | [G(z), u] = 0\}$ and $B = \{z \in I | h(z) = 0\}$. Then A and B are proper additive subgroup of I and $I = A \cup B$. But an additive group can not be union of two proper additive subgroups. Therefore either $A = I$ or $B = I$. In case $B = I$, we get $h(z) = 0$, for all $z \in I$, this implies that $h = 0$, a contradiction. Hence $[G(z), u] = 0$, for all $z, u \in I$. Replacing z by zu we obtain $[G(z)u + zg(u), u] = 0$, for all $z, u \in I$, this implies that $[zg(u), u] = 0$, for all $u, z \in I$. Replacing z by tz, we obtain $[t, u]zg(u) = 0$, for all $z, u, t \in I$. Using primeness of R, we get either $g = 0$ or R is commutative. As g is a non zero derivation therefore R is commutative. This proves the theorem.

Corollary 3.4. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. Suppose $F : R \to R$ is a nonzero generalized derivation with d,*

a. *If F acts as a homomorphism on I and if $d \neq 0$, then R is commutative.* □

b. *If F acts as an anti-homomorphism on I and if $d \neq 0$,, then R is commu- tative.*

Proof. Replacing $\alpha = 0$, and $F = G = H$ in Theorem 3.2 and Theorem 3.3, we get the required result. □

Theorem 3.5. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that $F, G, H : R \to R$*

are generalized derivations, associated with derivations d, g, h : $R \to R$ *respectively and* $\alpha : R \to R$ *is any map. If g, h are non zero derivation on R such that*

(i) $F(xy) + [G(x), y] + G(x)H(y) = 0,$ *for all* $x, y \in I.$

(ii) $F(xy) + [x, G(y)] + G(y)H(x) = 0,$ *for all* $x, y \in I.$

Then R is commutative. Proof. (i) We have

$$F(xy) + [G(x), y] + G(x)H(y) = 0 \quad (22)$$

for all $x, y \in I.$ Replacing y by yz in (22), we get

$(F(xy) + [G(x), y] + G(x)H(y))z + xyd(z) + y[G(x), z] + G(x)yh(z) = 0$ (23) for all $x, y, z \in I.$

Using equation (22),we get

$$xyd(z) + y[G(x), z] + G(x)yh(z) = 0 \quad (24)$$

for all $x, y, z \in I.$ Replacing y by ty in equation (24), we get

$$xtyd(z) + ty[G(x), z] + G(x)tyh(z) = 0 \quad (25)$$

for all $x, y, z, t \in I.$ Multiplying equation (24) from left then subtracting from equation (25), we obtain

$$[x, t]yd(z) + [G(x), t]yh(z) = 0 \quad (26)$$

for all $x, y, z, t \in I.$ Replacing $x = t$ in equation (26) we obtain $[G(x), x]yh(z) = 0,$ for all $x, y, z \in I.$ Using primeness of $R,$ we obtain either $h = 0$ or $[G(x), x] = 0,$ for all $x \in I.$ As h is a non zero derivation on $R,$ therefore $[G(x), x] = 0,$ for all $x \in I,$ this equation is same as the equation (15),therefore either $g = 0$ or R is commutative. If $g = 0,$ a contradiction, therefore R is commutative . This proves our result.

(*ii*) We have $F(xy) + [x, G(y)] + G(y)H(x) = 0,$ for all $x, y \in I.$ Replacing x by xz, we get

$$F(x)[z, y] + xd(zy) - xd(y)z + (F(xy) + [x, G(y)] + G(y)H(x))z$$
$$+x[z, G(y)] + G(y)xh(z) = 0 \quad (27)$$

for all $x, y, z, t \in I.$ Using hypothesis and replacing $y = z,$ in equation (27), we get

$$xzd(z) + x[z, G(z)] + G(z)xh(z) = 0 \quad (28)$$

for all $x, y, z \in I.$ Replacing x by ux in equation (28),we get

$$uxzd(z) + ux[z, G(z)] + G(z)uxh(z) = 0 \quad (29)$$

for all $u, x, y, z \in I.$ Multiplying equation (28), by u from left then subtracting from equation (29),we get

$$[G(z), u]xh(z) = 0 \quad (30)$$

for all $u, x, z \in I.$ The equation (30) is same as the equation (21), we get R is commutative. This proves the result. □

References

[1]. E, Albas, Generalized derivations on ideals of prime rings. *Miskolc Math. Notes* 14(1), 39 (2013).

[2]. A.Ali, N.Rehman, S. Ali, On lie ideals with derivations as homomorphisms and anti-homomorphisms *Acta Math. Hungar.* 101(12), 7982 (2003).

[3]. M. Asraf et al., Some commutativity theorems for rings with generalized derivations, *South. Asian. Bull. Math.* 31,415-421 (2007)
[4]. M. Ashraf,N. Rehman, On derivations and commutativity in prime rings *East-West J. Math.* 3(1), 8791(2001).
[5]. R. Awtar, Lie and Jordan structure in prime rings with derivations, *Proc. Amer. Math. Soc.* 41 (1973) 67-74.
[6]. H.E. Bell, L.C.Kappe, Ring in which derivation satisfy certain algebraic conditions *Acta Math. Hung.* 53, 339-346 (1989).
[7]. A. Boua et al., Joradan ideals and derivations in prime near rings, *Comm. Math. Univ. Carolin.* 31, 131-139 (2014).
[8]. M.Bresar, On the distance of the composition of two derivations to the generalized derivations, *Glasgow Math.J.* 33, 89-93(1991).
[9]. B. Dhara, Power values of derivations with annihilator conditions on Lie ideals in prime rings, *Comm. Algebra,* 37 (6), 2159-2167 (2009).
[10]. T. K. Lee , Semiprime rings with hypercentral derivation, *Canada. Bull. Math.* 38, 445-449(1995).
[11]. L. Carini and V. De Filippis, Commutators with power central values on a Lie ideal *Paci. Jour. Math.*,193, 269-278(2000).
[12]. J. Mayne, Centralizing automorphisms of Lie ideals in prime rings, *Canada. Bull. Math.* 35, 510-514 (1992).
[13]. Nadeem ur rehman, On commutativity of rings with generalized derivation
[14]. *Glas. Math.* 44, 43-49(2002).
[15]. Nadeem ur rehman, On generalized derivation as homomorphism and anti homomorphisms *Glas. Math.* 39, 27-30(2004).
[16]. L. Oukhtite, A. mamouni, Commutativity theorems for prime rings with generalized derivations on Jordan ideals, *Jour. of Taibah Univ.* 9, 314- 319(2015).
[17]. L. Oukhtite, A. mamouni, Generalized derivations centralizing on Jordan ideals of rings with involution, *Turk. J. Math.,* 38 , 233-239(2014).
[18]. E. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* 8,1093-1100 (1957).
[19]. R. K. Sharma, et al., Left annihilator of commutator identity with general- ized derivations and multilinear polynomials in prime rings, *Comm. Algebra* (2015), doi: 10.1080/00927872.2015.1085996.
[20]. S. K. Tiwari et al., Identities related to generalized derivation on ideal in prime rings, *Beitr. Algebra Geom.* 1-13(2015).
[21]. S. K. Tiwari et al., Multiplicative (generalized)-derivation in semiprime rings, *Beitr. Algebra Geom.* , 1-15,(2015).