Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 8 (August 2025) pp: 10-23
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Applications on Bergman spaces induced by a v-Laplacian vector fields theory

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Abstract

J. Oscar González-Cervantes, ∗, J. Bory-Reyes [29] in their recent works Gonzalez-Cervantes, Luna-Elizarraras and Shapiro [11,12], laid the foundations for the generalization of the theory of Bergman spaces induced by the foundations for the generalization of the theory of Bergman spaces induced byLaplacian (sometimes called solenoidal and irrotational, or harmonic) vector fields by taking advantage on the intimate connections between harmonic vector fields theory and quaternionic analysis for the Moisil-Theodorescu operator (MT-operator for short). A deeper discussion of the last mentioned relation can be found in [1]. On the setting of general bounded domains in $\mathbb{R}^3$, *we extend the aforementioned study in a very natural way to the case of an introduced* $\boldsymbol{v}$*-MT-operator for* $\boldsymbol{v} \in \mathbb{R}^3$, *proving several properties of induced Bergman spaces and some relative results about Stokes and Borel-Pompieu formulas for* $\boldsymbol{v}$*-MT-hyperholomorphic functions, i.e., functions which belong to kernel of the* $\boldsymbol{v}$*-MT-operator. They show that this v-MT operator satisfies a conformal co-variant property. Following [29] we improved the applications of all the above allows to study of Bergman type spaces induced by v-Laplacian vector fields theory.*

Keywords: *Bergman space theory, Reproducing kernel, Conformal co-variant property,* v*-Laplacian vector fields.*

***R**eceived 25 July, 2025; **R**evised 03 Aug., 2025; **A**ccepted 05 Aug., 2025 © The author(s) 2025.*
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I. Introduction and terminology

To extend classical complex analysis from $\mathbb{R}^2$ to $\mathbb{R}^3$ was by replacing the Cauchy-Riemann equations for holomorphic functions by the following system for smooth vector fields $f_j: \mathbb{R}^3 \to \mathbb{R}^3$:

$$\begin{cases} \operatorname{div} f_j = 0, \\ \operatorname{rot} f_j = 0. \end{cases} \tag{1.1}$$

Smooth solutions to the system (1.1) are called Laplacian vector fields, and for a very good introduction to their theory see [27].

Note that a Laplacian vector field satisfies the Laplace equation it named of a harmonic vector field. By purely physical reasons the solutions of (1.1) is called solenoidal and irrotational vector fields.

For a thorough treatment of the notion of Cauchy-type integral on a compact Liapunov surface, for the system (1.1), and some of its boundary value properties see [18]. Moreover, the subject has been treated in [1] for compact rectifiable topological surfaces, i.e., it is the image of some bounded subset Ξ of $\mathbb{R}^2$ under a Lipschitz mapping $\chi: \Xi \to \mathbb{R}^3$. Rademacher's theorem [9, pag. 81] ensures that χ is differentiable almost everywhere in Ξ, hence there exist conventional tangent plane for almost every point of the surface. Rectifiable surfaces form essentially the largest class where many basic properties of smooth surfaces have reasonable analogues (see [29]).

We discuss the Bergman spaces goes back toS. Bergman, [4] in the carly fifties, where the first systematic treatment of the subject was given, and since then there have been a lot of papers devoted to this area. See[5,8,15,28] and the references therein, which contain a broad summary and historical notes of the subject. Preliminaries on Bergman spaces induced by the MT-hyperholomorphic functions theory. We intrduce several properties on Bergman spaces and some relative results about Stokes and Borel-Pompieu formulas for **v** -MT-hyperholomorphic functions, i.e., functions which belong to kernel of the **v**-MT-operator needed. We proved the main results, where discusses Bergman type spaces induced by **v**-Laplacian vector, and show remarks on general concepts.

II. Preliminaries

For$\mathbb{H}$ denote the skew-field of real quaternions generated by $e_0 = 1$ and the non-real units, e_1, e_2, e_3 that fulfill the condition $e_1^2 = e_2^2 = e_3^2 = -1$ and the multiplication rules $e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_2e_3 = e_1$, and $e_3e_1 = -e_1e_3 = e_2$. If $q \in \mathbb{H}$ then $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$, where the coefficients $q_k \in \mathbb{R}$, $k = 0,1,2,3$. Each quaternion $q \in \mathbb{H}$ can be represented by $q = q_0 + \mathbf{q}$ with $\mathbf{q} = q_1e_1 + q_2e_2 + q_3e_3$. The real number q_0 is called the scalar part of q and the vector $\mathbf{q}$ is called the vector part of q. Due to the mapping $(q_1, q_2, q_3) \mapsto q_1e_1 + q_2e_2 + q_3e_3$, from $\mathbb{R}^3$ on $\{\mathbf{q} \mid q \in \mathbb{H}\}$, is an isometric isomorphism between $\mathbb{R}$-linear spaces, we shall continue to write $\mathbb{R}^3$ instead of $\{\mathbf{q} \mid q \in \mathrm{H}\}$.

A quaternionic conjugation of q is defined by $\bar{q} := q_0 - \mathbf{q}$. The quaternionic norm $|q| := \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ is the Euclidean norm with the natural identification of H with $\mathbb{R}^4$.

If, for only$\mathbf{x}, (\mathbf{x} + \boldsymbol{\epsilon}) \in \mathbb{R}^3$ then ityields $|\mathbf{x}(\mathbf{x} + \boldsymbol{\epsilon})| = |\mathbf{x}||\mathbf{x} + \boldsymbol{\epsilon}|$.

And in vector analysis terms, the multiplication:

$$x(x+\epsilon) = (x_0 + \mathbf{x})((x+\epsilon)_0 + \mathbf{x} + \boldsymbol{\epsilon}) = x_0(x+\epsilon)_0 - \langle \mathbf{x}, \mathbf{x} + \boldsymbol{\epsilon}\rangle + x_0(\mathbf{x} + \boldsymbol{\epsilon}) + (x+\epsilon)_0\mathbf{x} + [\mathbf{x}, \mathbf{x} + \boldsymbol{\epsilon}]$$

where $\langle \mathbf{x}, \mathbf{x} + \boldsymbol{\epsilon}\rangle$ and $[\mathbf{x}, \mathbf{x} + \boldsymbol{\epsilon}]$ stand for the inner and the usual cross product of $\mathbf{x}, \mathbf{x} + \boldsymbol{\epsilon} \in \mathbb{R}^3$ respectively. Set $\Omega \subset \mathbb{R}^3$ be a bounded Jordan domain. We will consider functions $f_j : \Omega \to \mathrm{H}$ to be written as $f_j = \sum_{i=0}^{3} \sum_j \; (f_j)_i e_i$. Hence$f_j$ has properties in Ω such as continuity, real differentiability of order p, Lebesgue integrable and ω-weighted Lebesgue integrable mean that all $(f_j)_i$ have these properties. These spaces are usually denoted by $C^s(\Omega, \mathbb{H})$ with $s \in \mathbb{N} \cup \{0\}, L_2(\Omega, \mathbb{H})$ and $L_{2,\omega}(\Omega, \mathbb{H})$ respectively.

We introduce series of the Moisil-Theodorescu operator (MT-operator for short) $D[f_j] = \sum_{i=1}^{3} \sum_j \; e_i \frac{\partial f_j}{\partial x_i}$. As usual, the MT-operator can act on the right $[f_j]D = \sum_{i=1}^{3} \sum_j \; \frac{\partial f_j}{\partial x_i} e_i$. Put, $D_r[f_j]$ instead of $[f_j]D$.

All functions which belong to $\mathrm{Ker}(D) := \{f_j : D[f_j] = 0\}$ are called left MT-hyperholomorphic and functions fulfilled $D_r[f_j] = 0$, shall be called right MT-hyperholomorphic. For the function theories (or quaternionic analysis), see [13,14,21,24].

The function $K(\boldsymbol{\epsilon}) = -\frac{\boldsymbol{\epsilon}}{4\pi|\boldsymbol{\epsilon}|^2}$ for $(\mathbf{x} + \boldsymbol{\epsilon}, \mathbf{x}) \in \partial\Omega \times (\mathbb{R}^3 \setminus \partial\Omega)$, is both left- and right-MT-hyperholomorphic fundamental solution for D, which plays the same role in quaternionic analysis as the Cauchy kernel does in complex analysis.

We assume Ω to be a bounded Jordan domain of $\mathbb{R}^3$ with rectifiable boundary $\partial\Omega$. However, we will use this assumption only in the way to ensure the existence of the outward pointing unit normal (almost everywhere) to $\partial\Omega$.

For$g_j : \Xi \to \Omega$ be a one-to-one correspondence and $h : \Omega \to \mathbb{H}$. We introduce the following: $W_{g_j}[f_j] = f_j \circ g_j$, ${}^hM[f_j] = hf_j$ and $M^h[f_j] = f_j h$ for every f_j belonging to a function space associated to Ω.

So one of the principal analytical facts that forms the basis of the quaternionic analysis is the three-dimensional Stokes formula

$$\int_{\partial\Omega} \sum_j \; f_j(\mathbf{t})\sigma_{\mathbf{t}}^{(2)} g_j(\mathbf{t})$$

$$= \int_{\Omega} \sum_j \; \left(D_r[f_j](\mathbf{x} + \boldsymbol{\epsilon}) g_j(\mathbf{x} + \boldsymbol{\epsilon}) + \sum_j \; f_j(\mathbf{x} + \boldsymbol{\epsilon}) D[g_j](\mathbf{x} + \boldsymbol{\epsilon}) \right) d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} \quad (2.1)$$

for all $f_j, g_j \in C^1(\Omega, \mathbb{H}) \cap C(\bar{\Omega}, \mathbb{H})$. Then, in a rather routine verification Stoke's formula yields the Borel-Pompieu formula:

$$\int_{\partial\Omega} \sum_j \left(K(\mathbf{t}-\mathbf{x})\sigma_{\mathbf{t}}^{(2)} f_j(\mathbf{t}) + g_j(\mathbf{t})\sigma_{\mathbf{t}}^{(2)} K(\mathbf{t}-\mathbf{x})\right)$$
$$-\int_{\Omega} \sum_j \left(K(\boldsymbol{\epsilon}) D[f_j](\mathbf{x}+\boldsymbol{\epsilon}) + \sum_j D_r[g_j](\mathbf{x}+\boldsymbol{\epsilon}) K(\boldsymbol{\epsilon})\right) d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}$$
$$= \begin{cases} f_j(\mathbf{x}) + g_j(\mathbf{x}), & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \in \mathbb{H} \setminus \bar{\Omega}, \end{cases} \quad (2.2)$$

for all $f_j, g_j \in C^1(\Omega, \mathbb{H}) \cap C(\bar{\Omega}, \mathbb{H})$.

Usually, $d\mu$ denotes the oriented volume element on Ω and $\sigma_{\mathbf{x}}^{(2)} = \sum_{i=1}^{3} (-1)^{i+1} e_i d\hat{x}^i$, where each term $d\hat{x}^i$ is given by $dx_1 \wedge dx_2 \wedge dx_3$ omitting the factor dx_i, for $i = 1,2,3$, can be represented on $\partial\Omega$ in the form $\mathbf{n}(x)ds$, where ds is the two-dimensional surface area element of integration and $\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$ is the outward pointing unit normal to $\partial\Omega$ at $\mathbf{x}$. Noting that $\mathbf{n}(\mathbf{x})$ exists for almost every $\mathbf{x} \in \partial\Omega$ under the assumption on the geometry of $\partial\Omega$.

As a direct consequence one has the MT-hyperholomorphic Cauchy integral formula:

$$\int_{\partial\Omega} \sum_j K(\boldsymbol{\epsilon})\sigma_{(\mathbf{x}+\boldsymbol{\epsilon})}^{(2)} f_j(\mathbf{x}+\boldsymbol{\epsilon}) = \begin{cases} f_j(\mathbf{x}), & \text{if } \mathbf{x} \in \Omega \\ 0, & \text{if } \mathbf{x} \notin \bar{\Omega}, \end{cases} \quad (2.3)$$

for all $f_j \in \mathrm{Ker}(D) \cap C(\bar{\Omega}, \mathbb{H})$.

See [1] and [21-23] for more information.

Consider the Theodorescu operator for $f_j \in L_2(\Omega, \mathrm{H})$ as

$$\mathcal{T}[f_j](\mathbf{x}) := \int_{\Omega} \sum_j K(\boldsymbol{\epsilon}) f_j(\mathbf{x}+\boldsymbol{\epsilon}) d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} \quad (2.4)$$

Hence, we recall that $D \circ \mathcal{T} = I$ on both $L_2(\Omega, \mathbb{H})$ and on $C(\Omega, \mathbb{H})$, where I denotes the identity operator. Note that (2.4) means precisely that $\mathcal{T}$ still being a right inverse to D, see [18, pag. 73].

The quaternionic Möbius transformations preserving $\mathbb{R}^3$ are important for a variety of reasons, one being that they are conformal maps. A deeper discussion along classical lines can be found in [2,3,16,25,26].

Basic examples are:

(i) Given $\mathbf{q} \in \mathbb{R}^3$. The translation $T_1(\mathbf{x}) = \mathbf{x} + \mathbf{q}$ for all $\mathbf{x} \in \mathbb{R}^3$.

(ii) The rotation associated to $(a + 2\epsilon) \in \mathbb{H}$ such that $|a + 2\epsilon| = 1$ is defined by $T_2(\mathbf{x}) = (a + 2\epsilon)\mathbf{x}(a \bar{+} 2\epsilon)$ for all $\mathbf{x} \in \mathbb{R}^3$.

(iii) The inversion is defined by $T_3(\mathbf{x}) = (\mathbf{x})^{-1} = -\frac{\mathbf{x}}{|\mathbf{x}|^2}$ for all $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$.

(iv) The dilation with a scale factor $\lambda > 0$ is $T_4(\mathbf{x}) = \lambda\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

In general, any quaternionic Möbius transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ is given by

$$T(\mathbf{x}) = (a\mathbf{x} + a + \epsilon)((a + 2\epsilon)\mathbf{x} + a + 3\epsilon)^{-1} \quad (2.5)$$

where $a, a+\epsilon, a+2\epsilon, a+3\epsilon \in \mathbb{H}$ satisfy:

(i) $(a+\epsilon)(a+3\epsilon)^{-1} \in \mathbb{R}^3$ and $\bar{a}(a+3\epsilon) \in \mathbb{R}$, if $a + 2\epsilon = 0$.

(ii) $a(a+2\epsilon)^{-1}, (a+3\epsilon)(a+\epsilon - a(a+2\epsilon)^{-1}(a+3\epsilon))^{-1} \in \mathbb{R}^3$ and $(a \bar{+} 2\epsilon)(a+\epsilon - a(a+2\epsilon)^{-1}(a+3\epsilon)) \in \mathbb{R}$, if $a + 2$

see [11,17,20].

The MT-hyperholomorphic Bergman space associated to $\Omega \subset \mathbb{R}^3$, denoted by $\mathcal{A}(\Omega)$, is defined to be the collection of all $f_j \in \mathrm{Ker}(D) \cap C^1(\Omega, \mathbb{H})$ such that

$$\int_{\Omega} \sum_j |f_j|^2 d\mu < \infty$$

See [19].

The quaternionic right-linear space $\mathcal{A}(\Omega)$ equipped by the inner product and the norm inherited from $L_2(\Omega, \mathbb{H})$ results to be a quaternionic right-Hilbert space with a reproducing kernel and projection, see [11,12] for more details.

Let $\Xi, \Omega \subset \mathbb{R}^3$ conformal equivalent domains with $\Omega = T(\Xi)$, where T is given by (2.5). Define

$$B_T(\mathbf{x}+\boldsymbol{\epsilon}) := \begin{cases} |a|^2\bar{a}, \text{ if } a+2\epsilon = 0; \\ |a+2\epsilon|^2|(\mathbf{x}+\boldsymbol{\epsilon}) - a(a+2\epsilon)^{-1}|^3 (a \bar{+} 2\epsilon)\overline{((\mathbf{x}+\boldsymbol{\epsilon}) - a(a+2\epsilon)^{-1})}, \end{cases} \text{if } a+2\epsilon \neq 0, \forall (\mathbf{x}+\boldsymbol{\epsilon}) \in \Omega,$$

$$C_T(\mathbf{x}) := \begin{cases} |a|^2\bar{a} \text{ if } a+2\epsilon = 0, \\ -|a+2\epsilon|^2(a \bar{+} 2\epsilon)\dfrac{(a+2\epsilon)\mathbf{x}w + (a+3\epsilon)\bar{w}}{|(a+2\epsilon)\mathbf{x}w + (a+3\epsilon)w|^3}, \end{cases} \text{if } a+2\epsilon \neq 0, \forall \mathbf{x} \in \Xi,$$

$$\rho_T(\mathbf{x}) := \begin{cases} 1, \text{ if } a+2\epsilon = 0, \\ \dfrac{1}{|((a+2\epsilon))\mathbf{x}w + (a+3\epsilon)w|^2}, \end{cases} \text{if } a+2\epsilon \neq 0, \forall \mathbf{x} \in \Xi,$$

where $w = (a+\epsilon - a(a+2\epsilon)^{-1}(a+3\epsilon))^{-1}$ and $(\mathbf{x}+\boldsymbol{\epsilon}) := T(\mathbf{x})$. Then

$$D_{\mathbf{x}}[A_T f_j \circ T] = (B_T \circ T) D_{(\mathbf{x}+\boldsymbol{\epsilon})}[f_j] \circ T, \forall f_j \in C^1(\Omega, \mathbb{H}), \tag{2.7}$$

or equivalently,

$$D_{\mathbf{x}} \circ {}^{A_T}M \circ W_T = W_T \circ {}^{B_T}M \circ D_{(\mathbf{x}+\boldsymbol{\epsilon})} \text{ on } C^1(\Omega, H\mathbf{H}),$$

see [11,12].

So,

$${}^{C_T}M \circ W_T : L_2(\Omega, HH) \longrightarrow L_{2,\rho_T}(\Xi, H)$$

and

$${}^{C_T}M \circ W_T|_{\mathcal{A}(\Omega)} : \mathcal{A}(\Omega) \longrightarrow \mathcal{A}_{\rho T}(\Xi)$$

are isometric isomorphism of quaternionic right-linear Hilbert spaces, where

$$\mathcal{A}_{\rho_T}(\Xi) = \mathrm{Ker}(D) \cap C^1(\Xi, \mathbb{H}) \cap L_{2,\rho_T}(\Xi, \mathbb{H})$$

and

$$\langle f_j, g_j \rangle_{L_2(\Omega,\mathrm{H})} = \sum_j \left\langle {}^{C_T}M \circ W_T[f_j], {}^{C_T}M \circ W_T[g_j] \right\rangle_{L_{2,\rho T}(\Xi,\mathrm{H})}, \tag{2.8}$$

for all $f_j, g_j \in L_2(\Omega, \mathbb{H})$.

Let us recall that

$$\langle f_j, g_j \rangle_{L_{2,\rho T}(\Xi,\mathrm{H})} = \int_{\Xi} \sum_j \bar{f}_j g_j \rho_T d\mu,$$

for all $f_j, g_j \in L_{2,\rho T}(\Xi, \mathbb{H})$.

The Bergman kernels and the Bergman projections

$$\mathcal{B}_{\Xi,\rho_T}(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}) = C_T(\mathbf{x})\mathcal{B}_\Omega(T(\mathbf{x}), T(\mathbf{x}+\boldsymbol{\epsilon}))\overline{C_T(\mathbf{x}+\boldsymbol{\epsilon})}, \forall \mathbf{x}, (\mathbf{x}+\boldsymbol{\epsilon}) \in \Omega,$$
$$\mathfrak{B}_{\Xi,\rho_T} = {}^{C_T}M \circ W_T \circ \mathfrak{B}_\Omega \circ W_{T^{-1}} \circ C_{\bar{T}}{}^{-1}M,$$

on $L_{2,\rho_T}(\Xi)$, see [10-12].

III. On some v-MT-hyperholomorphic Bergman spaces

Here the authors in [29] extend the previous results working with the following MoisilTheodorescu Hilbert type operators (MTH-type operators for brevity):

For$\Omega \subset \mathbb{R}^3$ and $\mathbf{v} \in \mathbb{R}^3$, the MTH-type operators are defined by

$${}_{\mathbf{v}}D := D + {}^{\mathbf{v}}M, D_{\mathbf{v}} := D_r + M^{\mathbf{v}}, \text{ on } C^1(\Omega, H\mathbf{H})$$

those which will play the role of D and D_r in the sequel.

The following computations will be used (implicitly) several times later on:

$$D \circ^{e^{(v,v)}} M[F] = e^{(\mathbf{v},\mathbf{x})}\mathbf{v}f_j + e^{(\mathbf{v},\mathbf{x})}D[f_j] =^{e^{(\mathbf{v},\mathbf{x})}} M \circ {}_{\mathbf{v}}D[f_j],$$
$$D \circ^{e^{(v,v)}} M[F] =^{e^{(\mathbf{v},\mathbf{x})}} M \circ {}_{\mathbf{v}}D[f_j], \tag{3.1}$$

for all $f_j \in C^1(\Omega, \mathbb{H})$.

Definition 3.1. The elements of the quaternionic right-linear space ${}_v\mathfrak{N}\mathfrak{P}(\Omega) := C^1(\Omega, \mathbb{H}) \cap \mathrm{Ker}({}_vD)$ are called **v**-MT-hyperholomorphic functions and those of the quaternionic left-linear space $\mathfrak{M}_{\mathbf{v}}(\Omega) := C^1(\Omega, \mathbb{H}) \cap \mathrm{Ker}(D_{\mathbf{v}})$ are called r-v-MT-hyperholomorphic functions.

The following proposition shows Stokes and Borel-Pompieu formulas induced by ${}_{\mathbf{v}}D$ and $D_{\mathbf{v}}$.

Proposition 3.2 (see [29]).

(i) Stokes Formula.

$$\int_{\partial\Omega}\sum_j f_j(\mathbf{t})\nu_{\mathbf{v}}(\mathbf{t})g_j(\mathbf{t}) = \int_{\Omega}\sum_j \left(D_{\mathbf{v}}[f_j](\mathbf{x}+\boldsymbol{\epsilon})g_j(\mathbf{x}+\boldsymbol{\epsilon}) + f_j(\mathbf{x}+\boldsymbol{\epsilon})_{\mathbf{v}}D[g_j](\mathbf{x}+\boldsymbol{\epsilon})\right)d\lambda_{\mathbf{v}}(\mathbf{x}+\boldsymbol{\epsilon})$$

for $f_j, g_j \in C^1(\Omega,\mathbb{H}) \cap C(\bar{\Omega},\mathbb{H})$, where $\nu_{\mathbf{v}}(\mathbf{t}) = e^{2\langle\mathbf{v},\mathbf{t}\rangle}\sigma_{\mathbf{t}}^{(2)}$ and $d\lambda_{\mathbf{v}}(\mathbf{x}+\boldsymbol{\epsilon}) = e^{2\langle\mathbf{v},\mathbf{x}+\boldsymbol{\epsilon}\rangle}d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}$.

(ii) Borel-Pompieu Formula.

$$\int_{\partial\Omega}\sum_j \left(K_{\mathbf{v}}(\mathbf{t}-\mathbf{x})\sigma_{\mathbf{t}}^{(2)}f_j(\mathbf{t}) + g_j(\mathbf{t})\sigma_{\mathbf{t}}^{(2)}K_{\mathbf{v}}(\mathbf{t}-\mathbf{x})\right)$$
$$-\int_{\Omega}\sum_j \left(K_{\mathbf{v}}(\boldsymbol{\epsilon})_{\mathbf{v}}D[f_j](\mathbf{x}+\boldsymbol{\epsilon}) + D_{\mathbf{v}}[g_j](\mathbf{x}+\boldsymbol{\epsilon})K_{\mathbf{v}}(\boldsymbol{\epsilon})\right)d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}$$
$$= \begin{cases} f_j(\mathbf{x}) + g_j(\mathbf{x}), & \mathbf{x}\in\Omega, \\ 0, & \mathbf{x}\in\mathbb{H}\setminus\bar{\Omega}, \end{cases} \tag{3.2}$$

for all $f_j, g_j \in C^1(\Omega,\mathbb{H}) \cap C(\bar{\Omega},\mathbb{H})$, where $K_{\mathbf{v}}(\cdot) = e^{\langle\mathbf{v},\cdot\rangle}K(\cdot)$, is the Cauchy Kernel of the $\mathbf{v}-MT$ hyperholomorphic function theory.

Proof. 1. Applying (2.1) to $e^{2\langle\mathbf{v},\cdot\rangle}f_j(\cdot), e^{\langle\mathbf{v},\cdot\rangle}g_j(\cdot)$ and making use of (3.1) yields

$$\int_{\partial\Omega}\sum_j f_j(\mathbf{t})e^{\langle\mathbf{v},\mathbf{t}\rangle}\sigma_{\mathbf{t}}^{(2)}e^{\langle\mathbf{v},\mathbf{t}\rangle}g_j(\mathbf{t})$$
$$= \int_{\Omega}\sum_j \left(D_{\mathbf{v}}[f_j](\mathbf{x}+\boldsymbol{\epsilon})g_j(\mathbf{x}+\boldsymbol{\epsilon}) + f_j(\mathbf{x}+\boldsymbol{\epsilon})_{\mathbf{v}}D[g_j](\mathbf{x}+\boldsymbol{\epsilon})\right)e^{2\langle\mathbf{v},\mathbf{x}+\boldsymbol{\epsilon}\rangle}d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} \tag{3.3}$$

2. Replacing $f_j(\cdot)$ and $g_j(\cdot)$ by $e^{\langle\mathbf{v},\ '\rangle}f_j(\cdot)$ and $e^{\langle\mathbf{v},\cdot\rangle}g_j(\cdot)$ in (2.2) we get

$$\int_{\partial\Omega}\sum_j e^{\langle\mathbf{v},\mathbf{t}-\mathbf{x}\rangle}K(\mathbf{t}-\mathbf{x})\sigma_{\mathbf{t}}^{(2)}f_j(\mathbf{t}) + g_j(\mathbf{t})\sigma_{\mathbf{t}}^{(2)}e^{\langle\mathbf{v},\mathbf{t}-\mathbf{x}\rangle}K(\mathbf{t}-\mathbf{x})$$
$$-\int_{\Omega}\sum_j \left(e^{\langle\mathbf{v},\boldsymbol{\epsilon}\rangle}K(\boldsymbol{\epsilon})_{\mathbf{v}}D[f_j](\mathbf{x}+\boldsymbol{\epsilon}) + D_{\mathbf{v}}[g_j](\mathbf{x}+\boldsymbol{\epsilon})e^{\langle\mathbf{v},\boldsymbol{\epsilon}\rangle}K(\boldsymbol{\epsilon})\right)d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}$$
$$=\begin{cases} f_j(\mathbf{x}) + g_j(\mathbf{x}), & \mathbf{x}\in\Omega, \\ 0, & \mathbf{x}\in\mathbb{R}^3\setminus\bar{\Omega}. \end{cases}$$

Corollary 3.3 (see [29]).

(i) v-MT-hyperholomorphic Cauchy Formula.

$$\int_{\partial\Omega}\sum_j \mathbf{v}K(\boldsymbol{\epsilon})\sigma_{(\mathbf{x}+\boldsymbol{\epsilon})}^{(2)}f_j(\mathbf{x}+\boldsymbol{\epsilon}) = \begin{cases} f_j(\mathbf{x}), & \text{if } \mathbf{x}\in\Omega, \\ 0, & \text{if } \mathbf{x}\notin\bar{\Omega}, \end{cases} \tag{3.4}$$

for all $f_j \in\in_{\mathbf{v}} \mathfrak{M}(\Omega)$.

(ii) A Theodorescu operator $_{\mathbf{v}}\mathcal{T}$ is defined by

$$_{\mathbf{v}}\mathcal{T}[f_j](\mathbf{x}) := \int_{\Omega}\sum_j K(\boldsymbol{\epsilon})f_j(\mathbf{x}+\boldsymbol{\epsilon})d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}, \tag{3.5}$$

requiring $f_j \in L_2(\Omega,\mathbb{H}) \cup C(\Omega,\mathbb{H})$ with the property that $_{\mathbf{v}}D \circ {}_{\mathbf{v}}\mathcal{T} = I$ on $L_2(\Omega,\mathbb{H}) \cup C(\Omega,\mathbb{H})$.

Proof.(i) It follows from (3.2).

(ii) If f_j belongs to $L_2(\Omega,\mathbb{H}) \cup C(\Omega,\mathbb{H})$ does so $e^{\langle\mathbf{v},\mathbf{x}\rangle}f_j(\mathbf{x}), x\in\Omega$. According to the above remark that $D\circ\mathcal{T} = I$ on $L_2(\Omega,\mathbb{H}) \cup C(\Omega,\mathbb{H})$ we have

$$D_{\mathbf{x}}\left[\int_{\Omega}\sum_j K(\boldsymbol{\epsilon})e^{\langle\mathbf{v},\mathbf{x}+\boldsymbol{\epsilon}\rangle}f_j(\mathbf{x}+\boldsymbol{\epsilon})d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\right] = e^{\langle\mathbf{v},\mathbf{x}\rangle}f_j(\mathbf{x}), \forall\mathbf{x}\in\Omega$$

where $D_{\mathbf{x}}$ denotes the MT-operator indexed with the variable vector in which it is applied. Thus,

$$D_{\mathbf{x}}\left[e^{\langle\mathbf{v},\mathbf{x}\rangle}\int_{\Omega}\sum_j K(\boldsymbol{\epsilon})e^{\langle\mathbf{v},\boldsymbol{\epsilon}\rangle}f_j(\mathbf{x}+\boldsymbol{\epsilon})d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\right] = e^{\langle\mathbf{v},\mathbf{x}\rangle}f_j(\mathbf{x}), \forall\mathbf{x}\in\Omega$$

We now apply (3.1) to get

$$e^{\langle\mathbf{v},\mathbf{x}\rangle}\ {}_{\mathbf{v}}D_{\mathbf{x}}\left[\int_{\Omega}\sum_j K(\boldsymbol{\epsilon})e^{\langle\mathbf{v},\boldsymbol{\epsilon}\rangle}f_j(\mathbf{x}+\boldsymbol{\epsilon})d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\right] = e^{\langle\mathbf{v},\mathbf{x}\rangle}\sum_j f_j(\mathbf{x}), \forall\mathbf{x}\in\Omega$$

Proposition 3.4 (see [29]). Let $\Xi, \Omega \subset \mathbb{R}^3$ be conformal equivalent domains and T given by (2.5) such that $\Omega = T(\Xi)$. Given $\mathbf{v}, \mathbf{u_j} \in \mathbb{R}^3$ then

$$_{\mathbf{v}}D \circ\circ e^{(u_j,\mathbf{x})} {}^{A_T}M \circ W_T = e^{(u_j,u_j,x)} M \circ W_T \circ {}^{B_T}M \circ {}_{\delta_T}D, \tag{3.6}$$

on $C^1(\Omega, \mathbb{H})$, where $\delta_T = (B_T)^{-1}(\mathbf{v} + \mathbf{u_j})(A_T \circ T^{-1}), T^{-1}$ is the inverse mapping of T and $(B_T)^{-1}$ is the inverse multiplicative of B_T.

What is more,

$$\langle f_j, g_j \rangle_{L_2(\Omega,\mathrm{H})} = \sum_j \left\langle\!\left\langle {}^{(\mathrm{u_j},\mathbf{x})}C_T M \circ W_T[f_j], {}^{\varepsilon^{(u_j,\mathbf{x})}C_T} M \circ W_T[g_j] \right\rangle\right\rangle_{L_{2,\gamma T}(\Xi,\mathrm{H})} \tag{3.7}$$

for all $f_j, g_j \in L_2(\Omega, \mathbb{H})$ and the weight function $\gamma_T(\mathbf{x}) = e^{-2(\mathbf{u_j},\mathbf{x})}\rho_T(\mathbf{x})$ for all $\mathbf{x} \in \Xi$.

Proof. Set $f_j \in C^1(\Omega, \mathbb{H})$ then using (2.3) and (3.1) we have that

$$\begin{aligned}
&\sum_j {}^{\mathrm{v}}D \circ {}^{((u_j,\aleph)}{}^{A_T} M \circ W_T[f_j] \\
&= \sum_j D \circ \Omega^{(u_j,\times)}{}^{A_T}M \circ W_T[f_j] + {}^{(u_j+,\times)}{}^{vA_T}M \circ W_T[f_j] \\
&= \varepsilon^{(\mathrm{u_j},\mathrm{x})} \sum_j M \circ \{D[\ {}^{A_T}M \circ W_T[f_j]] + {}^{(\mathrm{v}+\mathrm{u_j})A_T}M \circ W_T[f_j]\} \\
&= \sum_j e^{(\mathrm{u_j},\mathrm{x})}M \circ \{W_T \circ [\ {}^{B_T}M \circ D[f_j]] + {}^{(\mathrm{v}+\mathrm{u_j})A_T}M \circ W_T[f_j]\} \\
&= \sum_j \varepsilon^{(u_j,\mathrm{x})}M \circ W_T \circ {}^{B_T}M \circ \left[D[f_j] + {}^{(B_T)^{-1}(v+\mathrm{w})\left(A_T \ \circ T^{-1}\right)}M[f_j]\right] \\
&= \sum_j {}^{(u_j,*)}M \circ W_T \circ {}^{B_T}M \circ [D[f_j] + {}^{s_T}M[f_j]] \\
&= \sum_j {}^{e(u_j,*)}M \circ W_T \circ^{B_T} M \circ {}_{\delta_T}D[f_j]
\end{aligned}$$

On the other hand, if $f_j, g_j \in L_2(\Omega, \mathbb{H})$, (2.8) makes it clear that

$$\begin{aligned}
&\langle f_j, g_j \rangle_{L_2(\Omega,\mathrm{H})} \\
&= \int_{\equiv} \sum_j \overline{\left({}^{(4,\mathbf{x})}C_T M \circ W_T[f_j]\right)}\left(\left({}^{(\mathrm{u_j},\mathbf{x})}C_T M \circ W_T[g_j]\right)\left(e^{-2(\mathbf{u_j},\mathbf{x})}\rho_T\right)d\mu\right. \\
&= \sum_j \left\langle \varepsilon^{(\mathrm{u_j},\star)}C_T M \circ W_T[f_j], {}^{(\mathrm{u_j},\star)}{}^{C_T} M \circ W_T[g_j] \right\rangle_{L_{2,\vartheta T}(\Xi,\mathrm{H})},
\end{aligned}$$

for all $f_j, g_j \in L_2(\Omega, \mathbb{H})$.

Remark 3.5 [29]. According to definitions of A_T and B_T we are able to define

$$\delta_T(\mathbf{x} + \boldsymbol{\epsilon}) = \begin{cases} \dfrac{a}{|a|^4}(\mathbf{v} + \mathbf{u_j})\bar{a}, \ \text{if } a + 2\epsilon = 0 \\ \dfrac{\left((\mathbf{x} + \boldsymbol{\epsilon}) - a(a + 2\epsilon)^{-1}\right)(a + 2\epsilon)}{|a + 2\epsilon|^4|(\mathbf{x} + \boldsymbol{\epsilon}) - a(a + 2\epsilon)^{-1}|^4}(\mathbf{v} + \mathbf{u_j})\overline{(a + 2\epsilon)(x + \epsilon - a(a + 2\epsilon)^{-1})}, \ \text{if } a + 2\epsilon \neq 0, \end{cases}$$

for all $(\mathbf{x} + \boldsymbol{\epsilon}) \in \Omega$, and introduce the following quaternionic right-linear space

$$\delta_T\mathfrak{M}(\Omega) = C^1(\Omega, \mathbb{H}) \cap Ker(\delta_T D).$$

Definition 3.6. For $p: \Omega \to \mathbb{R}^+$, by the $\mathbf{v}$-MT-weighted hyperholomorphic Bergman space associated to the weight p we mean the quaternionic right-linear space $_{\mathbf{v}}\mathcal{A}_p(\Omega) := {}_{\mathrm{v}}\mathfrak{M}(\Omega) \cap L_{2,p}(\Omega, \mathbb{H})$ equipped with the inner product inherited from $L_{2,p}(\Omega, \mathbb{H})$:

$$\langle f_j, g_j \rangle_{\mathbf{v}}\mathcal{A}_p(\Omega) = \int_\Omega \sum_j \bar{f}_j g_j p d\mu, \forall f_j, g_j \in\in_{\mathbf{v}} \mathcal{A}_p(\Omega)$$

and its induced norm:

$$\|f_j\|^2_{\mathrm{v}\mathcal{A}_p(\Omega)} = \int_\Omega \sum_j \quad |f_j|^2 p d\mu$$

for all $f_j \in \quad {}_{\mathbf{v}}\mathcal{A}_\rho(\Omega)$.

Similarly, the right-v-MT-weighted hyperholomorphic Bergman space associated to Ω with weight p is the quaternionic left linear space:

$$\mathcal{A}_{p,\mathbf{v}}(\Omega) := \mathfrak{M}_{\mathbf{v}}(\Omega) \cap L_{2,p}(\Omega, \mathbb{H})$$

equipped with the inner product inherited from $L_{2,p}(\Omega, \mathbb{H})$ given by

$$\langle f_j, g_j \rangle_{\mathcal{A}_{p,\mathbf{v}}(\Omega)} = \int_\Omega \sum_j \quad f_j \bar{g}_j p d\mu, \forall f_j, g_j \in \mathcal{A}_{p,\mathbf{v}}(\Omega)$$

If p is the constant function 1 we write $\ {}_{\mathrm{v}}\mathcal{A}(\Omega)$ instead of $\ {}_{\mathbf{v}}\mathcal{A}_p(\Omega)$ and we will be called **v-**MThyperholomorphic Bergman space associated to Ω.

Proposition 3.7 (see [29]).

(i) If $\mathbb{B}(\mathbf{x}, r) \subset \Omega$ then

$$|f_j(\mathbf{x})| \leq \frac{\sqrt{3} e^{r|\mathbf{v}|}}{2\sqrt{\pi r}} \sum_j \quad \|f_j\|_{\mathbf{v},\mathcal{A}(\Omega)}, \forall f_j \in \quad {}_{\mathrm{v}}\mathcal{A}(\Omega)$$

(ii) Let $K \subset \Omega$ be a compact set then there exists $C_K > 0$ such that

$$\sup\{|f_j(\mathbf{x})| \mid \mathbf{x} \in K\} \leq C_K \sum_j \quad \|f_j\|_{\mathbf{v}\mathcal{A}(\Omega)}, \tag{3.8}$$

for all $f_j \in \mathrm{v}\mathcal{A}(\Omega)$.

(iii) The space $\left(\ {}_{v}\mathcal{A}(\Omega), \|\cdot\|_{\mathrm{v}\mathcal{A}(\Omega)} \right)$ is a quaternionic right-Hilbert space.

Proof.(i) From (3.4) it follows that

$$\begin{aligned}
f_j(\mathbf{x}) &= \int_{\partial \mathbf{B}(\mathbf{x},r)} \sum_j \quad {}_{\mathbf{v}}K(\boldsymbol{\epsilon}) \sigma^{(2)}_{(\mathbf{x}+\boldsymbol{\epsilon})} f_j(\mathbf{x}+\boldsymbol{\epsilon}) \\
&= \frac{-1}{4\pi r^2} \int_{\partial \mathbf{B}(\mathbf{x},r)} \sum_j \quad (\boldsymbol{\epsilon}) e^{(\mathbf{v},\boldsymbol{\epsilon})} \sigma^{(2)}_{(\mathbf{x}+\boldsymbol{\epsilon})} f_j(\mathbf{x}+\boldsymbol{\epsilon}) \\
&= \frac{-1}{4\pi r^2} \int_{\partial \mathbf{B}(\mathbf{x},r)} \sum_j \quad (\boldsymbol{\epsilon}) e^{(\mathbf{v},\boldsymbol{\epsilon}-2(\mathbf{x}+\boldsymbol{\epsilon}))} \sigma^{(2)}_{(\mathbf{x}+\boldsymbol{\epsilon})} e^{2(\mathbf{v},\mathbf{x}+\boldsymbol{\epsilon})} f_j(\mathbf{x}+\boldsymbol{\epsilon}) \\
&= \frac{-1}{4\pi r^2} \int_{\partial \mathbf{B}(\mathbf{x},r)} \sum_j \quad (\boldsymbol{\epsilon}) e^{(\mathbf{v},\boldsymbol{\epsilon}-2(\mathbf{x}+\boldsymbol{\epsilon}))} \nu_{\mathbf{v}}(\mathbf{x}+\boldsymbol{\epsilon}) f_j(\mathbf{x}+\boldsymbol{\epsilon}) \\
&= \frac{-1}{4\pi r^2} \int_{\mathbf{B}(\mathbf{x},r)} \sum_j \quad D_{\mathbf{v}}\left[(\boldsymbol{\epsilon}) e^{(\mathbf{v},-(2\mathbf{x}+\boldsymbol{\epsilon}))}\right] f_j(\mathbf{x}+\boldsymbol{\epsilon}) d\lambda_{\mathbf{v}}(\mathbf{x}+\boldsymbol{\epsilon}) \\
&= \frac{3}{4\pi r^2} \int_{\mathbf{B}(\mathbf{x},r)} \sum_j \quad e^{(\mathbf{v},-(2\mathbf{x}+\boldsymbol{\epsilon}))} f_j(\mathbf{x}+\boldsymbol{\epsilon}) d\lambda_{\mathbf{v}}(\mathbf{x}+\boldsymbol{\epsilon}) \\
&= \frac{3}{4\pi r^2} \int_{\mathbf{B}(\mathbf{x},r)} \sum_j \quad e^{(\mathbf{v},\boldsymbol{\epsilon})} f_j(\mathbf{x}+\boldsymbol{\epsilon}) d\mu_{\mathbf{x}+\boldsymbol{\epsilon}}
\end{aligned}$$

by our Stokes formula. Therefore, using the Cauchy inequality in $L_2(\Omega, \mathbb{H})$ we see that

$$\begin{aligned}
|f_j(\mathbf{x})| &\leq \frac{3}{4\pi r^2} \int_{\mathbb{B}(\mathbf{x},r)} \sum_j \quad e^{(\mathbf{v},\boldsymbol{\epsilon})} |f_j(\mathbf{x}+\boldsymbol{\epsilon})| d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} \\
&\leq \frac{3}{4\pi r^2} \left(\int_{\mathbb{B}(\mathbf{x},r)} e^{2(\mathbf{v},\boldsymbol{\epsilon})} d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} \right)^{\frac{1}{2}} \left(\int_{\mathbb{B}(\mathbf{x},r)} \sum_j \quad |f_j(\mathbf{x}+\boldsymbol{\epsilon})|^2 d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} \right)^{\frac{1}{2}}
\end{aligned}$$

Due to $e^{2\langle \mathbf{v}, \mathbf{y}-\mathbf{x} \rangle} \leq e^{2|\mathbf{v}|r}$ it may be concluded that

$$|f_j(\mathbf{x})| \leq \frac{3}{4\pi r^2} e^{|\mathbf{v}|r} \left(\frac{4}{3}\pi r^3\right)^{\frac{1}{2}} \left(\int_{\mathcal{B}(\mathbf{x},r)} \sum_j \quad |f_j(\mathbf{x}+\boldsymbol{\epsilon})|^2 d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} \right)$$
$$\leq \frac{3e^{r|\mathbf{v}|}}{2\sqrt{3\pi r}} \sum_j \quad \|f_j\|_{\mathbf{v}}\mathcal{A}(\Omega).$$

(ii) It follows from the previous fact.

(iii) Let $\{(f_j)_n\}$ be a sequence of elements of $\mathbf{v}\mathcal{A}(\Omega)$, hence (3.8) shows that $\{(f_j)_n(\mathbf{x})\}$ is a Cauchy sequence in $\mathbb{H}$ for each $\mathbf{x} \in \Omega$. Therefore, there exists $f_j(\mathbf{x}) = \lim (f_j)_n(\mathbf{x})$ for all $\mathbf{x} \in \Omega$ and from (3.8) we deduce that $\{(f_j)_n\}$ converges uniformly, on compact sets, to f_j.

From the first identity of (3.1) we see that each term of the sequence $\{e^{\langle \varepsilon, x\rangle} M[(f_j)_n]\}$ belongs to $\mathrm{Ker}(D)$ and $\{e^{\langle v, x\rangle} M[(f_j)_n]\}$ converges uniformly, on compact sets, to $e^{\langle v,x\rangle} M[f_j]$. Then, for any bounded Jordan domain $\Lambda \subset \mathbb{R}^3$ such that $\bar{\Lambda} \subset \Omega$ with rectifiable boundary, the hyperholomorphic Cauchy integral formula associated to the Moisil-Theodorescu operator, see (2.3), gives us that

$$e^{<v,\alpha\rangle} M[(f_j)_n](x) = \int_\Lambda \sum_j \quad K(\epsilon)\sigma_{(x+\epsilon)}^{(2)e^{<v,x+\epsilon>}} M[(f_j)_n](x+\epsilon),$$

for $x \in \Lambda$ and it is zero if $x \in \mathbb{R}^3 \setminus \bar{\Lambda}$, for all $n \in \mathbb{N}$. The uniform convergence allows to obtain that

$$e^{<v,z\rangle} M[f_j](x) = \int_\Lambda \sum_j \quad K(\epsilon)\sigma_{(x+\epsilon)}^{(2)e^{<\varepsilon,x+\epsilon>}} M[f_j](x+\epsilon)$$

for $x \in \Lambda$ and it is zero if $x \in \mathbb{R}^3 \setminus \bar{\Lambda}$. From the Moisil-Theodorescu hyperholomorphic functions theory, see [21], we have $e^{e^{<v,r\rangle}} M[f_j] \in \mathrm{Ker}(D)$, or equivalently using (3.1), $f_j \in \mathrm{Ker}(\ {}_vD)$. Another interesting way to justify the previous fact is following similar method as in [6, page 1635], where several properties of harmonic functions are used. Summarizing, we have $f_j \in_{\mathbf{v}} \mathfrak{M}(\Omega)$.

On the other hand, there exists $g_j \in L_2(\Omega, \mathbb{H})$ such that $\{(f_j)_n\}$ converges to g_j in $L_2(\Omega, \mathbb{H})$. Finally, using (3.8) we obtain $f_j = g_j$ almost everywhere and $f_j \in \ {}_vA(\Omega)$.

Remark 3.8 [29]. According to the above propositions the valuation functional is bounded on $\ {}_{\mathrm{v}}\mathcal{A}(\Omega)$. Then Riesz representation theorem for quaternionic Hilbert spaces, see [7], shows that given $\mathbf{x} \in \Omega$ there exists ${}_{\mathbf{v}}B_{\mathbf{x}} \in_{\mathbf{v}} \mathcal{A}(\Omega)$ such that $f_j(\mathbf{x}) = \langle \ {}_{\mathbf{v}}B_{\mathbf{x}}, f_j \rangle_{\mathbf{v}} A(\Omega)$ for all $f_j \in \in_{\mathbf{v}} \mathcal{A}(\Omega)$.

The function $\ {}_{\mathbf{v}}\mathcal{B}_\Omega(\mathbf{x},\cdot) := \overline{\ {}_{\mathbf{v}}B_{\mathbf{x}}(\cdot)}$ is called the **v**-MT-hyperholomorphic Bergman kernel associated to Ω and satisfies that

$$f_j(\mathbf{x}) = \int_\Omega \sum_j \quad {}_{\mathbf{v}}\mathcal{B}_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}) f_j(\mathbf{x}+\boldsymbol{\epsilon}) d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}, \forall f_j \in_{\mathbf{v}} \mathcal{A}(\Omega). \quad (3.9)$$

In addition, the **v**-MT-hyperholomorphic Bergman projection associated to Ω is defined by

$${}_{\mathrm{v}}\mathfrak{B}_\Omega[f_j](\mathbf{x}) := \int_\Omega \sum_j \quad {}_{\mathrm{v}}\mathcal{B}_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}) f_j(\mathbf{x}+\boldsymbol{\epsilon}) d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}, \forall f_j \in L_2(\Omega, \mathbb{H})$$

Properties of $\ {}_{\mathrm{v}}\mathcal{B}_\Omega$ and $\ {}_{\mathrm{v}}\mathfrak{B}_\Omega$ are established by our next proposition.

Proposition 3.9 (see [29]).

(i) $\quad \mathcal{B}_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}) = \overline{\ {}_{\mathrm{v}}\mathcal{B}_\Omega(\mathbf{x}+\boldsymbol{\epsilon}, \mathbf{x})}$, for all $\mathbf{x}, (\mathbf{x}+\boldsymbol{\epsilon}) \in \Omega$.

(ii) Given $\mathbf{x} \in \Omega$, $\ {}_v\mathcal{B}_\Omega(\cdot, \mathbf{x}) \in \ {}_{\mathrm{v}}\mathfrak{M}(\Omega)$ and $\ {}_{\mathrm{v}}\mathcal{B}_\Omega(\mathbf{x}, \cdot) \in \mathfrak{M}(\Omega)_{\mathrm{v}}$.

(iii) $\ {}^{\mathrm{B}}\mathcal{B}_\Omega(\cdot,\cdot)$ is the unique reproduction kernel that meets the two previous properties.

(iv) $\quad \mathrm{v}^{B_\Omega}$ is a symmetric operator; i.e.,

$$\langle \mathrm{v}\mathfrak{B}_\Omega[f_j], g_j \rangle_{L_2(\Omega,\mathrm{H})} = \langle f_j, \mathbf{v}\mathfrak{B}_\Omega[g_j] \rangle_{L_2(\Omega,\mathrm{H})}, \forall f_j, g_j \in L_2(\Omega, H)$$

(v) $\quad v\mathfrak{B}_\Omega[L_2(\Omega, H)] = \ {}_{\mathrm{v}}\mathcal{A}(\Omega)$.

(vi) $\quad (\ {}^{\mathrm{v}}\mathfrak{B}_\Omega)^2 = \mathrm{v}^{\mathfrak{B}_\Omega}$.

(vii) $\quad \mathfrak{v}_\Omega$ is a continuous operator.

Proof.(i) Since $\overline{\ {}_{\mathrm{v}}\mathcal{B}_\Omega(\mathbf{x},\cdot)} \in \ {}_{\mathrm{v}}\mathcal{A}(\Omega)$ one has that

$$\overline{{}_{\mathrm{v}}\mathcal{B}_{\Omega}(\mathbf{x},\mathbf{z})}=\int_{\Omega}{}_{\mathrm{v}}\mathcal{B}_{\Omega}(\mathbf{z},\mathbf{x}+\boldsymbol{\epsilon})_{\mathrm{v}}\overline{\mathcal{B}_{\Omega}(\mathbf{x},\mathbf{x}+\boldsymbol{\epsilon})}d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}$$

$$\begin{aligned}{}_{\mathrm{v}}\mathcal{B}_{\Omega}(\mathbf{x},\mathbf{z})&=\int_{\Omega}{}_{\mathrm{v}}\mathcal{B}_{\Omega}(\mathbf{x},\mathbf{x}+\boldsymbol{\epsilon})\overline{{}_{\mathrm{v}}\mathcal{B}_{\Omega}(\mathbf{z},\mathbf{x}+\boldsymbol{\epsilon})}d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\\&={}_{\mathbf{v}}\mathcal{B}_{\Omega}(\mathbf{z},\mathbf{x})\end{aligned}$$

for all $\mathbf{x},\mathbf{z}\in\Omega$.

(ii) It is a consequence of the previous fact.

(iii) If $M:\Omega\times\Omega\to\mathbb{H}$ has the same properties of ${}_{\mathrm{v}}\mathcal{B}_{\Omega}(\cdot,\cdot)$ we would have

$$\begin{aligned}M(\mathbf{x}+\boldsymbol{\epsilon},\mathbf{x})=\overline{M(\mathbf{x},\mathbf{x}+\boldsymbol{\epsilon})}&=\overline{\int_{\Omega}\mathcal{V}_{\Omega}(\mathbf{x},\mathbf{z})M(\mathbf{z},\mathbf{x}+\boldsymbol{\epsilon})d\mu_{\mathbf{z}}}\\&=\int_{\Omega}M(\mathbf{x}+\boldsymbol{\epsilon},\mathbf{z})\mathbf{v}\mathcal{B}_{\Omega}(\mathbf{z},\mathbf{x})d\mu_{\mathbf{z}}={}^{\mathrm{v}}\mathcal{B}_{\Omega}(\mathbf{x}+\boldsymbol{\epsilon},\mathbf{x}),\end{aligned}$$

for all $\mathbf{x},(\mathbf{x}+\boldsymbol{\epsilon})\in\Omega$.

(iv) Given $f_j,g_j\in L_2(\Omega,\mathbb{H})$ we obtain

$$\begin{aligned}\langle\mathfrak{v}\mathfrak{B}_{\Omega}[f_j],g_j\rangle_{\mathcal{A}A(\Omega)}&=\int_{\Omega}\sum_j\overline{\left(\int_{\Omega}v\mathcal{B}_{\Omega}(\mathbf{x},\mathbf{x}+\boldsymbol{\epsilon})f_j(\mathbf{x}+\boldsymbol{\epsilon})d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\right)}g_j(\mathbf{x})d\mu_{\mathbf{x}}\\&=\int_{\Omega}\sum_j\overline{f_j(\mathbf{x}+\boldsymbol{\epsilon})}\left(\int_{\Omega}\mathrm{v}_{\Omega}(\mathbf{x}+\boldsymbol{\epsilon},\mathbf{x})g_j(\mathbf{x})d\mu_{\mathbf{x}}\right)d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\\&=\sum_j\langle f_j,\mathbf{v}\mathcal{B}_{\Omega}[g_j|\rangle_{\mathcal{A}(\Omega)}.\end{aligned}$$

(v) Given $f_j\in L_2(\Omega,\mathbb{H})$ there exist unique elements $(f_j)_1\in_{\mathbf{v}}\mathcal{A}(\Omega)$ and $(f_j)_2\in(\mathrm{v}\mathcal{A}(\Omega))^{\perp}$ such that $f_j=(f_j)_1+(f_j)_2$. Thus

$$\mathbf{v}^{\mathfrak{B}_{\Omega}}[(f_j)_2](\mathbf{x})=\int_{\Omega}\sum_j\overline{\mathcal{V}_{\Omega}(\mathbf{x}+\boldsymbol{\epsilon},\mathbf{x})}(f_j)_2(\mathbf{x}+\boldsymbol{\epsilon})d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}=0$$

and $\mathbf{v}\mathfrak{B}_{\Omega}[f_j]=(f_j)_1$. Hence $\mathbf{v}\mathfrak{B}_{\Omega}[L_2(\Omega,\mathbb{H})]={}_{\mathbf{v}}\mathcal{A}(\Omega)$.

(vi) It is a consequence of the previous fact.

(vii) That $\mathfrak{v}^{\mathfrak{B}_{\Omega}}$ is a continuous operator follows from the fact that $\mathfrak{v}^{\mathfrak{B}_{\Omega}}$ satisfies the closed graph theorem.

Remark 3.10 [29]. By the effect of the quaternionic conjugation we deduce that the Bergman kernel of $\mathcal{A}_{\mathbf{v}}(\Omega)$ is given by

$$\mathcal{B}_{\Omega,\mathbf{v}}(\mathbf{x},\mathbf{x}+\boldsymbol{\epsilon})=\overline{\mathbf{v}\mathcal{B}_{\Omega}(\mathbf{x},\mathbf{x}+\boldsymbol{\epsilon})}={}_{\mathrm{v}}\mathcal{B}_{\Omega}(\mathbf{x}+\boldsymbol{\epsilon},\mathbf{x})$$

for all $(\mathbf{x}+\boldsymbol{\epsilon}),\mathbf{x}\in\Omega$.

Proposition 3.11. Let $\Omega,\Xi\subset\mathbb{R}^3$ be conformal equivalent domains and set T, given by (2.5), such that $T(\Xi)=\Omega$. Consider δ_T and γ_T given by Proposition 3.4. Then, the following statements are true:

(i) The operator ${}^{(4,\mathbf{x})\sigma_T}M\circ W_T$ satisfies

$${}^{(\mathrm{u_j},\mathbf{x}}{}^{\sigma_T}M\circ W_T\Big|_{\delta_T\mathcal{A}(\Omega)}:\ {}_{\delta_T}\mathcal{A}(\Omega)\longrightarrow{}_{\mathrm{v}}\mathcal{A}_{\gamma T}(\Xi)$$

and

$$\langle f_j,g_j\rangle_{s_T,\mathcal{A}(\Omega)}=\sum_j\left\langle\left\langle{}^{(u_j,\mathbf{x})}{}^{c_T}M\circ W_T[f_j],,{}^{\varepsilon^{(u_j,\mathbf{x})}c_T}M\circ W_T[g_j]\right\rangle_{\mathbf{v}}A_{\gamma T}(\Xi)\right.$$

for all $f_j,g_j\in{}_{\delta_T}\mathcal{A}(\Omega)$.

(ii) $\left({}_{v}\mathcal{A}_{\gamma_T}(\Xi),\langle\cdot,\rangle_{v,A_{\gamma T}(\Xi)}\right)$ is a quaternionic right-Hilbert space and the evaluation operator is bounded on ${}^{v}\mathcal{A}_{TT}(\Xi)$.

(iii) $\left({}_{\delta_T}\mathcal{A}(\Omega),\langle\cdot,\cdot\rangle_{s_T}\mathcal{A}(\Omega)\right)$ is a quaternionic right-Hilbert space, the valuation functional is bounded on ${}_{T}\mathcal{A}(\Omega)$.

(iv) Bergman kernels of $\delta_T\mathcal{A}(\Omega)$ and ${}_{v}\mathcal{A}_{\gamma_T}(\Xi)$ have the same properties as shown in Proposition 3.9 and they are related as follows:

$$ {}_{\mathbf{v}}\mathcal{B}_{\Xi,\gamma_T}(\mathbf{x},\mathbf{x}+\boldsymbol{\epsilon}) = \sum_j \quad e^{\langle \mathbf{u_j}, 2\mathbf{x}+\boldsymbol{\epsilon}\rangle} C_T(\mathbf{x})_{\delta_T}\mathcal{B}_\Omega(T(\mathbf{x}),T(\mathbf{x}+\boldsymbol{\epsilon}))\overline{C_T(\mathbf{x}+\boldsymbol{\epsilon})}, $$

for all $\mathbf{x}$, $(\mathbf{x}+\boldsymbol{\epsilon}) \in \Xi$.

(v) The **v**-MT-weighted hyperholomorphic Bergman projection associated to the domain Ξ, with weight γ_T, is

$$ {}_{\mathbf{v}}\mathfrak{B}_{\Xi,\gamma_T} = \sum_j \quad {}^{\varepsilon^{(\mathbf{u_j},)'}C_T}M \circ W_T \circ \quad {}_{\delta_T}\mathfrak{B}_\Omega \circ \left(\quad {}^{\varepsilon^{(u_j,)}C_T}M \circ W_T \right)^{-1} $$

Proof. (i) By (2.6) we see that A_T and C_T are the same function up to a positive constant. Therefore, Proposition 3.4 proves the desired result.

(ii) It is deeply similar to fact 3. of Proposition 3.7 since **v** is a constant vector and the weight function γ_T is a bounded function on compact subsets of Ξ that does not affect the **v**-MT-hyperholomorphy.

(iii) As the Möbius transformation T then the operator ${}^{e^{(\mathbf{u_j},\mathbf{x})}C_T}M \circ W_T$ is invertible too, see [10,11]. Thus, given a Cauchy sequence $\{(f_j)_n\}$ of elements in ${}_{\delta_T}\mathcal{A}(\Omega)$ one has that $\left\{ {}^{\mathrm{e}^{(\mu,\mathbf{x})}\sigma_T}M \circ W_T\left[(f_j)_n\right]\right\}$ is a Cauchy sequence in $\mathbf{v}^{\mathcal{A}_{\gamma T}}(\Xi)$. Therefore, as in the proof of Proposition 3.7, Fact 3., there exists $g_j \in_{\mathbf{v}} \mathcal{A}_{\gamma T}(\Xi)$ such that $\left\{ {}^{\varepsilon^{\{(u_j,\mathbf{x})}C_T}M \circ W_T\left[(f_j)_n\right]\right\}$ converges to g_j, or equivalently, $\{(f_j)_n\}$ converges to $\left({}^{\varepsilon^{\{u_j,\mathbf{x})}C_T}M \circ W_T\right)^{-1}[g_j] \in \quad {}_{\delta_T}\mathcal{A}(\Omega)$.

Calculation similar to the above implies that valuation functional is bounded in ${}_{\delta_T}\mathcal{A}(\Omega)$. Therefore ${}_{\delta_T}\mathcal{A}(\Omega)$ has a reproducing kernel and a projection.

(iv) Given $h \in \quad {}_{\mathbf{v}}\mathcal{A}_{\gamma T}(\Xi)$ one has that $\left(\left({}^{(u_j,x)}C_T M \circ W_T\right)^{-1}[h] \in \quad {}_{\delta_T}\mathcal{A}(\Omega)\right.$ and setting $(\mathbf{x}+\boldsymbol{\epsilon}) = T(\mathbf{x})$ we obtain

$$ \begin{aligned} &\left({}^{\varepsilon^{(\mathbf{u_j},\mathbf{x})}C_T}M \circ W_T\right)^{-1}[h](\mathbf{x}+\boldsymbol{\epsilon}) \\ &= \sum_j \quad \left\langle \quad {}_{\delta_T}\mathcal{B}_\Omega(\cdot,\mathbf{x}+\boldsymbol{\epsilon}), \left({}^{e^{(\mathbf{u_j},\mathbf{x})}C_T}M \circ W_T\right)^{-1}[h] \right\rangle_{\delta_T} \mathcal{A}(\Omega) \\ &= \sum_j \quad \left\langle \quad {}^{e^{(u_j,\mathbf{z})}C_T}M \circ W_T\left[\delta_{\delta_T}\mathcal{B}_\Omega(\cdot,\mathbf{x}+\boldsymbol{\epsilon})\right], h \right\rangle_{\mathbf{v}} \mathcal{A}_{\gamma_T}(\overline{(\Xi)}) \\ &= \int_\Xi \sum_j \quad \left(\overline{e^{(\mathbf{u_j},\mathbf{z})}C_T(\mathbf{z})_{\delta_T}\mathcal{B}_\Omega(T(\mathbf{z}),\mathbf{x}+\boldsymbol{\epsilon})}\right) h(\mathbf{z})\gamma_T(\mathbf{z})d\mu_{\mathbf{z}} \\ &= \int_\Xi \sum_j \quad \left(e^{(\mathbf{u_j},\mathbf{z})} \quad {}_{\delta_T}\mathcal{B}_\Omega(\mathbf{x}+\boldsymbol{\epsilon},T(\mathbf{z}))\overline{C_T(\mathbf{z})}\right) h(\mathbf{z})\gamma_T(\mathbf{z})d\mu_{\mathbf{z}}. \end{aligned} $$

Therefore, since $\mathbf{x} = T(\mathbf{x}+\boldsymbol{\epsilon})$ we conclude that

$$ h(\mathbf{x}) = \int_\Xi \sum_j \quad \left(e^{(\mathbf{u_j},\mathbf{x}+\mathbf{z})}C_T(\mathbf{x})_{\delta_T}\mathcal{B}_\Omega(T(\mathbf{x}),T(\mathbf{z}))\overline{C_T(\mathbf{z})}\right) h(\mathbf{z})\gamma_T(\mathbf{z})d\mu_{\mathbf{z}}. $$

(v) It follows from the previous identity.

(vi) On Bergman type spaces in a v-Laplacian vector fields theory

Now we have been working again under the assumption that $\Omega \subset \mathbb{R}^3$ be a domain and $\mathbf{v} \in \mathbb{R}^3$. Let us consider the action of ${}_{\mathbf{v}}D$ to $f_j = (f_j)_0 + \mathbf{f_j} \in C^1(\Omega,\mathbb{H})$, where $(f_j)_0$ is the scalar part of f_j and $\mathbf{f_j}$ is the vector part of f_j, which after straightforward calculation leads to

$$ {}_{\mathbf{v}}D[f_j] = -\mathrm{div}\mathbf{f_j} - \langle \mathbf{v},\mathbf{f_j}\rangle + \mathrm{grad}(f_j)_0 + (f_j)_0\mathbf{v} + \mathrm{rot}\mathbf{f_j} + [\mathbf{v},\mathbf{f_j}], $$

where

$$ \mathrm{grad}(f_j)_0 = \nabla(f_j)_0, \mathrm{div}\mathbf{f_j} = \langle \nabla,\mathbf{f_j}\rangle, \mathrm{rot}\mathbf{f_j} = [\nabla,\mathbf{f_j}] $$

and the gradient operator $\nabla = \sum_{i=1}^3 e_i \frac{\partial}{\partial x_i}$.

The following equivalences hold:

$$ {}_{\mathbf{v}}D[f_j] = 0 \Leftrightarrow \begin{cases} \mathrm{grad}(f_j)_0 + \mathrm{rot}\mathbf{f_j} = -(f_j)_0\mathbf{v} - [\mathbf{v},\mathbf{f_j}] \\ \mathrm{div}\mathbf{f_j} = -\langle \mathbf{v},\mathbf{f_j}\rangle \end{cases} $$

Analogously, it may be easily verified that

$$D_{\mathbf{v}}[f_j]=0\Leftrightarrow\begin{cases}\mathrm{grad}(f_j)_0-\mathrm{rot}\mathbf{f_j}=-(f_j)_0\mathbf{v}+[\mathbf{v},\mathbf{f_j}],\\ \mathrm{div}\mathbf{f_j}=-\langle\mathbf{v},\mathbf{f_j}\rangle.\end{cases}$$

In addition, if we define $_{v}\mathbf{M}(\Omega)$ to be the $\mathbb{R}$-linear space of all $\mathbf{f_j}\in C^1(\Omega,\mathbb{R}^3)$ such that div $\mathbf{f_j}=-\langle\mathbf{v},\mathbf{f_j}\rangle$ and $\mathrm{rot}\mathbf{f_j}=-[\mathbf{v},\mathbf{f_j}]$ on Ω, a short calculation shows that

$$\left\{ke^{-\langle\mathbf{v},\mathbf{x}\rangle}\mid k\in\mathbb{R}\right\}\oplus_{\mathbf{v}}\mathbf{M}(\Omega)={}_{\mathbf{v}}\mathfrak{M}(\Omega)\cap\mathfrak{M}_{\mathbf{v}}(\Omega).\tag{4.1}$$

Proposition 4.1 (see [29])(Cauchy-Integral theorem). If $f_j\in{}_{\mathbf{v}}\mathrm{M}(\Omega)\cap C(\bar{\Omega},\mathbb{H})$ then

$$\int_{\partial\Omega}\sum_j\langle\mathbf{f_j},\boldsymbol{\nu}_{\mathbf{v}}\rangle=\mathbf{0},\int_{\partial\Omega}\sum_j[\mathbf{f_j},\boldsymbol{\nu}_{\mathbf{v}}]=\mathbf{0}.$$

Proof. It follows from the Fact (i) of Proposition 3.2.

Proposition 4.2 (see [29]) (Cauchy-Integral formula). If $f_j\in{}_{\mathbf{v}}\mathrm{M}(\Omega)$ then

$$\int_{\partial\Omega}\sum_j\left(-\langle{}_{\mathbf{v}}K(\boldsymbol{\epsilon}),\sigma^2_{(\mathbf{x}+\boldsymbol{\epsilon})}\rangle\mathbf{f_j}(\mathbf{x}+\boldsymbol{\epsilon})+\left[[\mathbf{v}K(\boldsymbol{\epsilon}),\sigma^2_{(\mathbf{x}+\boldsymbol{\epsilon})}],\mathbf{f_j}(\mathbf{x}+\boldsymbol{\epsilon})\right]\right)=\mathbf{f_j}(\mathbf{x})$$

and

$$\int_{\partial\Omega}\sum_j\langle[{}_{\mathbf{v}}K(\boldsymbol{\epsilon}),\sigma^2_{(\mathbf{x}+\boldsymbol{\epsilon})}],\mathbf{f_j}(\mathbf{x})\rangle=0$$

for all $\mathbf{x}\in\Omega$.

Proof. It is a direct consequence from Fact (i) in Corollary 3.3.

Proposition 4.3 (see [29])(Borel-Pompieu formula). Let $\mathbf{f_j},\mathbf{g_j}\in C^1(\Omega,\mathbb{R}^3)\cap C(\bar{\Omega},\mathbb{R}^3)$ then

$$\begin{aligned}&\int_{\partial\Omega}\sum_j\left(-\left\langle{}_{\mathbf{v}}K(\boldsymbol{\epsilon}),\sigma^{(2)}_{(\mathbf{x}+\boldsymbol{\epsilon})}\right\rangle(\mathbf{f_j}+\mathbf{g_j})(\mathbf{x}+\boldsymbol{\epsilon})+\left[{}_{\mathbf{v}}K(\boldsymbol{\epsilon}),\sigma^{(2)}_{(\mathbf{x}+\boldsymbol{\epsilon})}\right],(\mathbf{f_j}+\mathbf{g_j})(\mathbf{x}+\boldsymbol{\epsilon})\right]\right)\\&+\int_{\Omega}\sum_j(\langle\mathbf{v},(\mathbf{f_j}+\mathbf{g_j})(\mathbf{x}+\boldsymbol{\epsilon})\rangle+\mathrm{div}(\mathbf{f_j}+\mathbf{g_j})(\mathbf{x}+\boldsymbol{\epsilon}))_{\mathbf{v}}K(\boldsymbol{\epsilon})d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\\&-\int_{\Omega}\sum_j\left[{}_{\mathbf{v}}K(\boldsymbol{\epsilon}),[\mathbf{v},(\mathbf{f_j}+\mathbf{g_j})(\mathbf{x}+\boldsymbol{\epsilon})]+\mathrm{rot}(\mathbf{f_j}+\mathbf{g_j})(\mathbf{x}+\boldsymbol{\epsilon})\right]d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}\\&=\begin{cases}\mathbf{f_j}(\mathbf{x})+\mathbf{g_j}(\mathbf{x}), & \mathbf{x}\in\Omega,\\ 0, & \mathbf{x}\in\mathbb{R}^3\setminus\bar{\Omega}.\end{cases}\end{aligned}$$

Proof. Use the Fact (ii) of Proposition 3.2.

Proposition 4.4 (see [29]). Let $\Xi,\Omega\subset\mathbb{R}^3$ be conformal equivalent domains and T given by (2.5) such that $\Omega=T(\Xi)$. Given $\mathbf{v},\mathbf{u_j}\in\mathbb{R}^3$ denote

$$S_T[\mathbf{f_j}]=\sum_j\tfrac{1}{2}e^{(u_j,x)}M\circ\left(A^{A_T}M-M^{\bar{A}_T}\right)\circ W_T[\mathbf{f_j}]$$

and

$$V_T[\mathbf{f_j}]=\sum_j\tfrac{1}{2}(\ ^{(w,x)}M\circ\left(\ ^{A_T}M+M^{\bar{A}_T}\right)\circ W_T[\mathbf{f_j}],$$

for all $\mathbf{f_j}\in C^1(\Omega,\mathbb{R}^3)$. Hence,

$$\begin{gathered}\mathrm{grad}S_T[\mathbf{f_j}]+\mathrm{rot}V_T[\mathbf{f_j}]=-S_T[\mathbf{f_j}]\mathbf{v}-\left[\mathbf{v},V_T[\mathbf{f_j}]\right]\\ \mathrm{div}V_T[\mathbf{f_j}]=-\left\langle\mathbf{v},V_T[\mathbf{f_j}]\right\rangle,\ \text{on }\Xi\end{gathered}$$

iff

$$\begin{gathered}\mathrm{div}\mathbf{f_j}=-\left\langle\delta_T,\mathbf{f_j}\right\rangle,\\ \mathrm{rot}\mathbf{f_j}=-\left[\delta_T,\mathbf{f_j}\right],\ \text{on }\Omega\end{gathered}$$

or equivalently, $\mathbf{f_j}\in{}_{\delta_T}\mathbf{M}(\Omega)$ for all $\mathbf{f_j}\in C^1(\Omega,\mathbb{R}^3)$.

What is more, define

$$R[\mathbf{f_j}] = \sum_j \left({}^{\frac{1}{2}C_T}M - M^{\frac{1}{2}C_T} \right) \circ W_T[\mathbf{f_j}],$$
$$U[\mathbf{f_j}] = \sum_j \left({}^{\frac{1}{2}C_T}M + M^{\frac{1}{2}C_T} \right) \circ W_T[\mathbf{f_j}],$$

for all $\mathbf{f_j} \in L_2(\Omega, \mathbb{R}^3)$. Then

$$\int_\Omega \sum_j \langle \mathbf{f_j}, \mathbf{g_j} \rangle d\mu = \int_\Xi \sum_j \big(R_T[\mathbf{f_j}]R_T[\mathbf{g_j}] + \langle U_T[\mathbf{f_j}], U_T[\mathbf{g_j}] \rangle\big)\rho_T d\mu$$
$$\int_\Omega \sum_j [\mathbf{f_j}, \mathbf{g_j}] d\mu = \int_\Xi \sum_j \big(-R_T[\mathbf{f_j}]U_T[\mathbf{g_j}] + R_T[\mathbf{g_j}]U_T[\mathbf{f_j}] + [U_T[\mathbf{f_j}], U_T[\mathbf{g_j}]]\big)\rho_T d\mu$$

for all $\mathbf{f_j}, \mathbf{g_j} \in L_2(\Omega, \mathbb{R}^3)$.

Proof. It follows from Proposition 3.4.

Definition 4.5. The $\mathbb{R}$-linear space ${}_{\mathrm{v}}\mathbf{A}(\Omega)$ is defined by requiring $\mathbf{f_j} \in_{\mathrm{v}} \mathbf{M}(\Omega)$ such that

$$\int_\Omega \sum_j |\mathbf{f_j}|^2 d\mu \leq \infty$$

equipped with the following inner product

$$\langle \mathbf{f_j}, \mathbf{g_j} \rangle_v \mathbf{A}(\Omega) = \int_\Omega \sum_j \langle \mathbf{f_j}, \mathbf{g_j} \rangle d\mu, \forall \mathbf{f_j}, \mathbf{g_j} \in {}_{\mathrm{v}}\mathbf{A}(\Omega)$$

We now include easy but important facts involving ${}_{\mathrm{v}}\mathbf{A}(\Omega)$

(a) If there exists $\lambda \in \mathbb{R}$ such that $\Omega \subset \{\mathbf{x} \in \mathbb{R}^3 \mid \langle \mathbf{v}, \mathbf{x} \rangle > \lambda\}$ is bounded, then
$$\{ke^{-\langle \mathbf{v}, \mathbf{x} \rangle} \mid k \in \mathbb{R}\} \oplus_{\mathrm{v}} \mathbf{A}(\Omega) = {}_{\mathrm{v}}\mathcal{A}(\Omega) \cap \mathcal{A}_{\mathrm{v}}(\Omega)$$

(b) If there exists $\lambda \in \mathbb{R}$ such that $\{\mathbf{x} \in \mathbb{R}^3 \mid \langle \mathbf{v}, \mathbf{x} \rangle < \lambda\} \subset \Omega$ then
$${}_{\mathrm{v}}\mathbf{A}(\Omega) = {}_{\mathrm{v}}\mathcal{A}(\Omega) \cap \mathcal{A}_{\mathrm{v}}(\Omega)$$

Proposition 4.6 (see [29]). ${}_{\mathrm{v}}\mathbf{A}(\Omega), \langle \cdot, \cdot \rangle_{\mathrm{v}}\mathbf{A}(\Omega))$ is a real linear Hilbert space.

Proof. Let $\{(\mathbf{f_j})_n\}$ be a Cauchy sequence of elements in ${}_{\mathrm{v}}\mathbf{A}(\Omega)$. Therefore $\{(\mathbf{f_j})_n\}$ is a Cauchy sequence in ${}_{\mathrm{v}}\mathcal{A}(\Omega)$ and there exists $f_j \in {}_{\mathrm{v}}\mathcal{A}(\Omega)$ such that $\{(\mathbf{f_j})_n\}$ converges to f_j and, particularly, one sees that $\{(\mathbf{f_j})_n\}$ converges to f_j uniformly on compact set. Thus, by uniqueness of limits $f_j = \mathbf{f_j} \in {}_{\mathrm{v}}\mathbf{A}(\Omega)$.

Proposition 4.7 (see [29]) (Reproducing functions). The following assertions follow

(i) There exist a scalar field $(a+\epsilon)_\Omega : \Omega \times \Omega \to \mathbb{R}$ and a vector field $(\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega : \Omega \times \Omega \to \mathbb{R}^3$, such that

(a) $(a+\epsilon)_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}) = (a+\epsilon)_\Omega(\mathbf{x}+\boldsymbol{\epsilon}, \mathbf{x})$ and $(\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}) = -(\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\mathbf{x}+\boldsymbol{\epsilon}, \mathbf{x})$, for all $(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}) \in \Omega \times \Omega$.

(b) $\operatorname{div}(\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\cdot, \mathbf{x}+\boldsymbol{\epsilon}) = -\langle \mathbf{v}, (\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\cdot, \mathbf{x}+\boldsymbol{\epsilon}) \rangle$, for each $(\mathbf{x}+\boldsymbol{\epsilon}) \in \Omega$.

(c) $\operatorname{grad}(a+\epsilon)_\Omega(\cdot, \mathbf{x}+\boldsymbol{\epsilon}) + \operatorname{rot}(\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\cdot, \mathbf{x}+\boldsymbol{\epsilon}) = -(a+\epsilon)_\Omega(\cdot, \mathbf{x}+\boldsymbol{\epsilon})\mathbf{v} - [\mathbf{v}, (\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\cdot, \mathbf{x}+\boldsymbol{\epsilon})]$ for each $(\mathbf{x}+\boldsymbol{\epsilon}) \in \Omega$.

(d) Let $\hat{L}_2(\Omega, \mathbb{R}^3)$ denote the space of all $\mathbf{f_j} \in L_2(\Omega, \mathbb{R}^3)$ with

$$\int_\Omega \sum_j \big\langle (\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}), \mathbf{f_j}(\mathbf{x}+\boldsymbol{\epsilon}) \big\rangle d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} = 0, \text{ a.e. } \mathbf{x} \in \Omega$$

equipped with the norm inherited from $L_2(\Omega, \mathbb{R}^3)$. It is easily seen to be a real Banach space such that

$${}_{\mathrm{v}}\mathbf{A}(\Omega) = {}_{\mathrm{v}}\mathbf{M}(\Omega) \cap \hat{L}_2(\Omega, \mathbb{R}^3)$$

and

$$\int_\Omega \sum_j \big((a+\epsilon)_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon})\mathbf{f_j}(\mathbf{x}+\boldsymbol{\epsilon}) + [(\boldsymbol{a}+\boldsymbol{\epsilon})_\Omega(\mathbf{x}, \mathbf{x}+\boldsymbol{\epsilon}), \mathbf{f_j}(\mathbf{x}+\boldsymbol{\epsilon})]\big) d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} = \mathbf{f_j}(\mathbf{x}), \forall \mathbf{x} \in \Omega$$

for all $\mathbf{f_j} \in {}_{\mathrm{v}}\mathbf{A}(\Omega)$.

(ii) There exist functions $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3 : \Omega \times \Omega \to \mathbb{R}^3$ such that $\mathbf{C}_1(\cdot, \mathbf{x}), \mathbf{C}_2(\cdot, \mathbf{x}), \mathbf{C}_3(\cdot, \mathbf{x}) \in {}_{\mathrm{v}}\mathbf{A}(\Omega)$, for each $\mathbf{x} \in \Omega$ and

$$\int_\Omega \sum_j \big\langle \mathbf{C}_i(\mathbf{x}+\boldsymbol{\epsilon}, \mathbf{x}), \mathbf{f_j}(\mathbf{x}+\boldsymbol{\epsilon}) \big\rangle d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})} = (f_j)_i(\mathbf{x}),$$

for all $\mathbf{f_j} \in {}_{\mathbf{v}}\mathbf{A}(\Omega)$, where $\mathbf{f_j} = \sum_{i=0}^{3} e_i (f_j)_i$ and $(f_j)_i \in C^1(\Omega, \mathbb{R})$. Therefore,

$$\mathbf{f_j}(\mathbf{x}) = \int_\Omega \sum_{i=0}^{3} \sum_j e_i \langle \mathbf{C}_i(\mathbf{x}+\boldsymbol{\epsilon}, \mathbf{x}), \mathbf{f_j}(\mathbf{x}+\boldsymbol{\epsilon}) \rangle d\mu_{(\mathbf{x}+\boldsymbol{\epsilon})}$$

Proof.(i) It is a consequence from the contention ${}_v\mathbf{A}(\Omega) \subset {}_{\mathbf{v}}\mathcal{A}(\Omega)$ and the properties of the Bergman kernel of $\mathbf{v}\mathcal{A}(\Omega)$.

(ii) As $\mathbf{v_v A}(\Omega)$ is a real Hilbert space, the functional $\Pi_{\mathbf{x}}^i[f_j] = (f_j)_i(\mathbf{x}), \mathbf{x} \in \Omega$ with $\mathbf{f_j} = \sum_{i=1}^{3} \sum_j e_i (f_j)_i$ and $(f_j)_i : \Omega \to \mathbb{R}$ for all i, is bounded for $i = 1,2,3$ since the valuation functional is bounded on ${}_{\mathbf{v}}\mathcal{A}(\Omega)$. Therefore, there exist reproducing functions $\mathbf{C}_i$ for $i = 1,2,3$.

Property (ii) shows us that the "appropriate" L_2-space is $\hat{L}_2(\Omega, \mathbb{R}^3)$ in definition of ${}_{\mathbf{v}}\mathbf{A}(\Omega)$.

Proposition 4.8 (see [29]). If $\Xi, \Omega \subset \mathbb{R}^3$ be conformal equivalent domains and set T given by (2.5) such that $\Omega = T(\Xi)$. Consider $\mathbf{v}, \mathbf{u_j} \in \mathbb{R}^3$ and define

$$P[\mathbf{f_j}] = \sum_j \frac{1}{2}\left(t^{(u_j,x)} M \circ ({}^{C_T}M - M^{C_T}) \circ W_T[\mathbf{f_j}],\right.$$

$$Q[\mathbf{f_j}] = \sum_j \frac{1}{2} t^{(u_j,x)} M \circ ({}^{C_T}M + M^{\tilde{C}_T}) \circ W_T[\mathbf{f_j}].$$

Therefore, $\mathbf{f_j} \in C^1(\Omega, \mathbb{R}^3)$ satisfies that $\mathrm{div}\mathbf{f_j} = -\langle \delta_T, \mathbf{f_j} \rangle$, $\mathrm{rot}\mathbf{f_j} = -[\delta_T, \mathbf{f_j}]$ on Ω and

$$\int_\Omega \sum_j |\mathbf{f_j}|^2 d\mu < \infty$$

if and only if

$$\mathrm{div}Q[\mathbf{f_j}] = -\langle \mathbf{v}, Q[\mathbf{f_j}] \rangle$$
$$\mathrm{grad}P[\mathbf{f_j}] + \mathrm{rot}Q[\mathbf{f_j}] = -P[\mathbf{f_j}]\mathbf{v} - [\mathbf{v}, Q[\mathbf{f_j}]], \text{ on } \Xi$$

and

$$\int_\Omega \sum_j \left(|P[\mathbf{f_j}]|^2 + |Q[\mathbf{f_j}]|^2\right) \gamma_T d\mu < \infty.$$

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