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Research Paper

Einstein Function - Based Approach To Q-Starlike Analytic Functions and Their Geometric Properties

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ABSTRACT. Einstein functions are considered special functions with notable significance in mathematical analysis and theoretical physics, exhibit specific properties that make them suitable for defining and investigating subclasses of analytic-univalent functions. The aim of this paper is to define a new class of q-star-like function in the unit disk $\mathbb{U}$. Utilizing the principle of subordination and the framework of basic q-calculus, the sharp coefficient bounds and upper bounds for the Fekete-Szego functional for this newly defined class, were established based on the properties of Einstein functions.

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I. Introduction And Preliminaries

Let A be the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by S the subclass of A consisting of functions which are analytic, univalent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$.
A function $f(z) \in S$ of the form (1.1) is star-like in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ if it maps a unit disk onto a star-like domain. A necessary and sufficient condition for a function $f(z)$ to be star-like is that

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, (z \in \mathbb{U})$$

The class of all star-like functions is denoted by S^*.

An analytic function $f(z)$ is convex if it maps the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ conformally onto a convex domain. Equivalently, a function $f(z)$ is said to be convex if and only if it satisfies the following condition;

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ (z \in \mathbb{U}).$$

The class of all convex functions is denoted by K

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Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{U}$, then $f(z)$ is subordinate to $g(z)$ in the unit $\mathbb{U}$ written as $f(z) \prec g(z)$, if there exist a function $\omega(z)$ analytic in the unit $\mathbb{U}$ with $\omega(0)$=0, $|\omega(z)| < 1$ which is called the Schwartz function such that $f(z) = g(\omega(z))$. If the function g is univalent in the unit $\mathbb{U}$, then $f(z) \prec g(z), z \in \mathbb{U} \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
Let P be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_n z^n \tag{1.2}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $p(z) \in P$ satisfies the conditions $Re(p(z)) > 0$ and $p(0) = 1$, for $z \in \mathbb{U}$, then $p(z)$ is called a Carathéodory function or function having positive real part in the unit disk $\mathbb{U}$.

Einstein functions. In mahtematics, Einstein function is a name occasionally used for one of the functions

$$E_1(z) = \frac{z}{e^z - 1};$$
$$E_2(z) = \frac{e^z z^2}{(e^z - 1)^2};$$
$$E_3(z) = log(1 - e^{-z});$$
$$E_4(z) = \frac{z}{e^z - 1} - log(1 - e^{-z})$$

$E_1(z)$ and $E_2(z)$(convex functions) have a symmetric range along the real axis and star-like range about $E_1(0) = E_2(0) = 1$, $\mathbb{R}(E_1(0)) > 0$, and $\mathbb{R}(E_2(0)) > 0, \forall z \in \mathbb{U}$. The series expansion for $E_1(z)$ and $E_2(z)$ can be given by

$$E_1(z) = 1 + \sum_{n=1}^{\infty} \frac{B_n z^n}{n!}$$

$$E_2(z) = 1 + \sum_{n=1}^{\infty} \frac{(1-n)B_n z^n}{n!}$$

where B_n is the n^{th} Bernoulli number.
But unfortunately $E_1(z)$ and $E_2(z)$ do not satisfy the condition $E_1'(0) \not> 0$ and $E_2'(z) \not> 0$. Hence, new functions must be defined for $E_1(z)$ and $E_2(z)$ as follows

$$E(z) = E_1(z) + z = 1 + z + \sum_{n=1}^{\infty} \frac{B_n z^n}{n!} \tag{1.3}$$

and

$$\mathbb{E}(z) = E_2(z) + \frac{z}{2} = 1 + \frac{z}{2} + \sum_{n=1}^{\infty} \frac{(1-n)B_n z^n}{n!} \tag{1.4}$$

which satisfies the conditions $E'(0) > 0$ and $\mathbb{E}'(z) > 0$. Therefore, $E(z)$ and $\mathbb{E}(z) \in P$
The Bernoulli number B_n can be defined by the contour integral

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \frac{dz}{z^{n+1}},$$

where the radius of the contour encircling the origin is less than $2\pi i$.

The q-derivative (or q-difference) of a function $f(z) \in A$ given by (1.1) is defined by

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & if z \neq 0 \\ f'(0), & if z = 0 \end{cases}$$

When $q \to 1^-$,

$$D_q f(z) = f'(z)$$

provided $f'(z)$ exists. Also, for analytic functions $f(z) \in A$ given by (1.1),

$$D_q f(z) = 1 + \sum_{n=0}^{\infty} [n]_q a_n z^{n-1} \tag{1.5}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Lemma 1.1. *Let $p \in P$ defined by (1.2), then*

$$|c_n| \leq 2,$$

for all $n \geq 1$. This result is sharp and equality holds for the Möobius function $M_o(z)$. [3]

Lemma 1.2. *Let the function $p \in P$ given by (1.2) with $Re(p(z)) > 0, p(0) = 1$ and has the power series representation*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then, $2p_2 = p_1^2 + x(4 - p_1^2)$, for some $x : |x| \leq 1$. [6]

Definition 1.3. A function $f \in A$ of the form (1.1) belong to the class $\c{R}(q, E)$ if

$$\frac{zD_q f(z)}{f(z)} \prec E(z) \tag{1.6}$$

$z \in \mathbb{U}, q \in (0, 1)$ and $E(z)$ is the modified Einstein function of the first kind.

Ma and Minda [9], studied the geometric properties (such as distortion, growth and covering theorems) for a class of star-like functions by means of subordination principle, given by:

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{z} \prec \phi(z), \phi \in p, z \in \mathbb{U} \right\} \tag{1.7}$$

Janowski and Sokol *et al* also established some geometric properties of star-like functions defined in the unit disk $\mathbb{U}$, for more information on this, see [8] and [14].

Remark 1.4. When $q \to 1^-$ and $E(z)$ is replaced with $\phi(z)$, Ma-Minda star-like function in equation (1.7) is obtained from equation (1.6).

The researchers in [7] were the first to introduce a subclass of q-starlike functions in the theory of univalent functions. Subsequently, few other researchers has worked on q-satrlike functions and established their geometric properties. For further studies on subclasses of q-starlike and q-convex functions and their geometric properties, see[1, 12].
El-Qadeem *et al* [4, 5] and Rossdy *et al* [13] established the coefficient estimates and Fekete-Szego functionals for new subclasses of bi-univaelnt functions using Einstein first and second kinds.

Motivated by [4, 5, 13], the present work introduces a novel subclass of univalent function related to q-calculus by means of subordination involving Einstein function and discuss the first two coefficient bounds and the upper bound for the Fekete - Szego functional.

2. Main results

2.1. Coefficient bounds for the Class Ŗ(q, E).

Theorem 2.1. *If $f(z) \in A$ belong to the class Ŗ(q, E). then*

$$|a_2| \leq \frac{1}{2q}$$

$$|a_3| \leq \frac{1}{2q(1+q)} + \frac{3-5q}{12q(q^2+q+1)}$$

$q \in (0,1), z \in \mathbb{U}$.

Proof. Let $f \in A$ be in the subclass Ŗ(q, E), then from the definition of Ŗ(q, E) in (1.6), which states that

$$\frac{zD_q f(z)}{f(z)} \prec E(z) \tag{2.1}$$

where $E_1(z)$ is the modified Einstein function of the first kind. Also, from the definition of subordination, (2.1) can be written as

$$\frac{zD_q f(z)}{f(z)} = E(\omega(z)) \tag{2.2}$$

$$\frac{zD_q f(z)}{f(z)} = 1 + \frac{\omega(z)}{2} + \frac{(\omega(z))^2}{12} + ...$$

Since $\omega(z)$ is a Schwartz function, then $p(z)$ can be expressed as

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + ...$$

$$w(z) = \frac{p(z)-1}{p(z)+1}$$

$$w(z) = \left\{p_1(z) + p_2 z^2 + ...\right\}\left\{2 + p_1 z + p_2 z^2 + ...\right\}^{-1} \tag{2.3}$$

On further simplification of (2.3) using binomial expansion,

$$\omega(z) = \frac{p_1 z}{2} + \frac{1}{2}\left\{p_2 - \frac{p_1^2}{2}\right\} z^2 + ...$$

So,

$$E(w(z)) = 1 + \frac{w(z)}{2} + \frac{(w(z))^2}{12} + ...$$

$$= 1 + \frac{1}{2}\left\{\frac{p_1 z}{2} + \frac{1}{2} l p_2 - \frac{p_1^2}{2}\right\} z^2 + \frac{1}{12}\left[\frac{p_1 z}{2} + \frac{1}{2}\left\{p_2 - \frac{p_1^2}{2}\right\}\right]^2 + ...$$

$$E(w(z)) = 1 + \frac{p_1 z}{4} + \frac{1}{4}\left\{p_2 - \frac{5p_1^5}{12}\right\} z^2 + ... \tag{2.4}$$

Next is to present the series expansion of q-starlike function introduced in (2.1) as;

$$\frac{zD_q f(z)}{f(z)}$$

where

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

$$\frac{zD_q f(z)}{f(z)} = \left(1 + [2] q a_2 z + [3]_q a_3 z^2 + ...\right)\left(1 + a_2 z + a_3 z^2 + ...\right)^{-1}$$

$$\frac{zD_q f(z)}{f(z)} = 1 + q a_2 z + \left\{a_3(q + q^2) - q a_2^2\right\} z^2 + ... \tag{2.5}$$

comparing the coefficients of z and z^2 in (2.4) and (2.5);

$$a_2 = \frac{p_1}{4q} \tag{2.6}$$

$$a_3 = \frac{1}{q^2 + q}\left\{\frac{p_2}{4} + \frac{(3 - 5q)p_1^2}{48q}\right\} \tag{2.7}$$

Using triangle inequality and lemma 1.1 in (2.6) and (2.7), the bounds on a_2 and a_3 can be obtained as follows;

$$|a_2| \le \frac{1}{2q}$$

$$|a_3| \le \frac{1}{2q(1+q)} + \frac{3 - 5q}{12q(q^2 + q + 1)}$$

□

Corollary 2.2. *Let $f(z) \in \c{R}(q, E)$ and $q \to 1^-$. Then,*

$$|a_2| \le \frac{1}{2}$$

$$|a_3| \le \frac{7}{36}$$

The next result is the upper bound for the Fekete-Szego functional of the class $\c{R}(q, E)$

Theorem 2.3. *For $q \in (0,1)$ and $z \in \mathbb{U}, f(z) \in A$ given by (1.1) belongs to the class $\mathfrak{R}(q, E)$ if*

$$|a_3 - a_2^2| \leq \frac{1}{2q(q+1)} \tag{2.8}$$

Proof. If (2.6) and (2.7) are substituted into the Fekete-Szego functional defined by

$$|a_3 - \sigma a_2^2|$$

with $\sigma = 1$, then

$$|a_3 - a_2^2| \leq \left| \frac{p_2}{4q(q+1)} - \frac{p_1^2}{6q(q+1)} \right| \tag{2.9}$$

using lemma 1.2 in (2.9) and on application of triangle inequality, it yields

$$|a_3 - a_2^2| \leq \left| \frac{x(4-p_1^2)}{8q(q+1)} \right| + \left| \frac{p_1^2}{24q(q+1)} \right| \tag{2.10}$$

Suppose $p_1 = p : p \in [0,2]$ and $\xi = |x| \leq 1$

$$|a_3 - a_2^2| \leq \frac{\xi(4-p_1^2)}{4q(q+1)} + \frac{p_1^2}{6q(q+1)} \tag{2.11}$$

Setting $|a_3 - a_2^2| = \phi_q(p, \xi)$, (2.11) can be written as

$$\phi_q(p, \xi) = \frac{\xi(4-p_1^2)}{4q(q+1)} + \frac{p_1^2}{6q(q+1)} \tag{2.12}$$

To maximize the function $\phi_q(p, \xi)$ on the closed region $[0,2] \times [0,1]$, we find the first partial derivative of $\phi_q(p, \xi)$ with respect to ξ

$$\frac{\partial \phi_q}{\partial \xi} = \frac{4-p_1^2}{4q(q+1)} \geq 0$$

Therefore $\phi_q(p, \xi)$ becomes an increasing function of ξ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,1] \times [0,2]$. Moreover, for a fixed $p \in [0,2]$

$$\max_{0 \leq \xi \leq 1} \phi_q(p, \xi) = \phi_q(p, 1) = \frac{4-p_1^2}{8q(q+1)} + \frac{p_1^2}{24q(q+1)} = \phi_q(p) \tag{2.13}$$

On further simplification of (2.13);

$$\phi_q(p) = \frac{1}{2q(q+1)} - \frac{p_1^2}{6q(q+1)}$$

Obviusly, the function $\phi_q(p)$ has a maximum value at $p = 0$.
Hence,

$$\max\{\phi_q(p) : p \in [0,2]\} = \phi_q(0)$$

where

$$\phi_q(0) = \frac{1}{2q(q+1)}$$

Therefore,

$$|a_3 - a_2^2| \leq \frac{1}{2q(q+1)}$$

When $q \to 1^-$, the following corollary can be deduced from theorem 2.3

Corollary 2.4. *Let* $f(z) \in$. *Then*

$$|a_3 - a_2^2| \leq \frac{1}{4}$$

Conclusion

Researchers in geometric function theory have used quite a number of Special functions such as, but not limited to Chebyshev polynomials, Bessel functions, Einstein functions, in order to study the geometric properties of various subclasses of analytic and univalent functions defined in the unit disk $\mathbb{U}$. The coefficient bounds and the upper bound for the Fekete-Szego functional obtained in this paper are new the geometric properties for the class $\c{R}(q, E)$ defined in the unit disk $\mathbb{U}$. The results established not only contributes to knowledge in theoretical advancement of q-calculus and special functions in geometric function theory but open up potential opportunity for applications in signal processing, fluid dynamics and engineering fields

Declarations

Ethics approval and consent to participate

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