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Research Paper

η^*-REGULAR AND η^*-NORMAL SPACES IN TOPOLOGY

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Abstract: *In this paper, we introduce and study two new classes of generalized regular and normal spaces are called η^*-regular and η^*-normal spaces which are weaker than regularity and normality respectively. The relationships among strongly rg-regular, g-regular, regular, Q^*-regular, η^*-regular, almost regular and softly regular spaces are investigated. Some of basic properties and characterizations of η^*-regular spaces in the terms of some other separation axioms are obtained.*

Key words: *regular open, η^*-open sets; softly regular, almost-regular, Q^*-regular, η^*-regular spaces*
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I. Introduction

In 1925, Urysohn [**17**] introduced and studied a new type of separation axiom, called Urysohn space. In 1926, Cartan [**1**] introduced and studied the concept of symmetric space. In topology, an R_0-space is also known as a symmetric space. In 1937, Stone [**16**] introduced the notion of semi-regular spaces and obtained their characterizations. In 1958, Kuratowski [**7**] introduced a new type of generalized sets, called regularly-open and regularly-closed sets in general topology. In 1963, Levine [**8**] introduced the concept of generalized closed sets and obtained their properties. In 1969, Singal and Arya [**14**] introduced a new class of separation axiom (namely almost regular space) in topological spaces and investigated some basic properties with other separation axioms such as T_0, T_1, semi-regular, Hausdorff and k-spaces. In 1986, Munshi [**10**] introduced and studied some new class of separation axioms (named g-regular and g-normal spaces etc.) in topological spaces which are stronger than regularity and normality. In 1993, Palaniappan [**12**] introduced the concept of generalized closed sets, called regular generalized closed which is a weaker form of closed and g-closed sets and studied some properties. In 2010, Murugalingam and Lalitha [**11**] introduced and studied the concept of Q^*-open sets and obtained some properties of Q^*-open sets. In 2011, Gnanachandra and Thangavelu [**3**] introduced and studied the concepts of strongly rg-regular and strongly rg-normal spaces in topological spaces which are stronger than regularity and normality. In 2018, Kumar and Sharma [**4**] introduced and studied the concept of softly regular spaces in topological spaces which is a weak form of regularity and obtained some characterizations with regular, strongly rg-regular, weakly regular, almost regular, π-normal and quasi normal spaces. Recently, Hamant Kumar [**6**] introduced and studied the concept of Q^*-regular spaces which is a weaker form of regularity and obtained some characterizations.

II. Preliminaries

Throughout this paper, spaces $(X, \Im)$, (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by **cl(A)** and **int(A)** respectively. A subset A of a topological space $(X, \Im)$ is said to be **regularly-open** [7] if it is the interior of its own closure or, equivalently, if it is the interior of some closed sets or equivalently, A = int(cl(A)). A subset A is said to be **regularly-closed** [7] if it is the closure of its own interior or, equivalently, if it is the closure of some open sets or equivalently, A = cl(int(A)). Clearly, a set is regularly-open iff its complement is regularly-closed. The finite union of regularly open sets is said to be **π–open**. The complement of a π-open set is said to be **π-closed**. Every regularly open (resp. regularly closed) set is π–open (resp. π-closed).

2.1 Definition. A subset A of a space (X, ℑ) is said to be
(i) **generalized closed** (briefly **g-closed**) **[8]** if cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(ii) **regular generalized closed** (briefly r**g-closed**) [**12]** if cl(A) ⊂ U whenever A ⊂ U and U is regular-open in X.

The complement of a is g-closed (resp. rg-closed) set is said to be **g-open** (resp**. rg-open**).
The family of all Q^*-closed (resp. Q^*-open) sets of a space X is denoted by Q^*-C(X) (resp. Q^*-O(X)).

The **generalized closure** of A is defined as the intersection of all g-closed sets in X containing A and is denoted by **cl*(A)**. The **generalized interior** of A is defined as the union of all g-open sets in X contained in A and is denoted by **int*(A)**.

2.2 Definition. A subset A of a space (X, ℑ) is said to be **Q^*-closed** [**11**] if int(A) = ϕ and A is closed. The complement of a Q^*-closed set is said to be **Q^*-open**.

2.3 Definition. The **δ-interior** of a subset A of X is the union of all regular open sets of X contained in A and is denoted by **δ-int(A)**. The subset A is called **δ-open** [**18**] if δ-int(A) = A. i.e. a set is δ-open if it is the union of regular open sets, the complement of a δ-open set is called **δ-closed**. Alternatively, a set A ⊂ X is δ-closed if A = δ-cl(A), where δ-cl(A) is the intersection of all regular closed sets of (X, ℑ) containing A.

2.4 Definition. Let (X, ℑ) be a topological space. A subset A of (X, ℑ) is called **regular*-open** [**13**] (or r*-open) if A = int(cl*(A)). The complement of a regular*-open set is called **regular*-closed**. The union of all regular*-open sets of X contained in A is called **regular*-interior** and is denoted by **r*-int(A)**. The intersection of all regular*-closed sets of X containing A is called **regular*-closure** and is denoted by **r*-cl(A)**.

2.5 Definition. A subset A of a topological space (X, ℑ) is called **η*-open** [**9**] set if it is a union of regular*-open sets (r*-open sets).The complement of a η*-open set is called **η*-closed**. A subset A of a topological space (X, ℑ) is called **η*-Interior** of A is the union of all η*-open sets of X contained in A and is denoted by **η*-int(A)**. The intersection of all η*-closed sets of X containing A is called as the **η*-closure** of A and is denoted by **η*-cl(A)**.

2.6 Remark.
(i) regular closed ⇒ π-closed ⇒ δ-closed ⇒ η*-closed ⇒ closed ⇒ g-closed ⇒ rg-closed

2.7 Remark. For every subset U of X,
(i) g-cl(U) ⊂ cl(U)) ⊂ η*-cl(U) ⊂ δ-cl(U) ⊂ π-cl(U) ⊂ r-cl(U).

2.8 Remark. We have the following implications for the properties of subsets:

	regular closed	⇒	π-closed	⇒	η∗-closed	⇒	closed	
	⇓		⇓		⇓		⇓	
Q^*-closed ⇒	closed	⇒	g-closed	⇒	πg-closed	⇒	rg-closed	

Where none of the implications is reversible as can be seen from the following examples:

2.9 Example. Let X = {a, b, c} and ℑ = {ϕ, X, {a}, {b}, {a, b}}. Then
(i) regular closed sets are : ϕ, X, {a, c}, {b, c}.
(ii) π-closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(iii) closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(iv) Q^*-closed sets are : ϕ, {c}.
(v) g-closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(vi) πg-closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(vii) rg-closed sets are : ϕ, X, {c}, {a, b}, {a, c}, {b, c}.
(viii) η∗-closed sets are : ϕ, X, {c}, {a, c}, {b, c}.

2.10 Example. In R with usual metric, finite sets are Q^*-closed but not regular closed. [0, 1] is regular closed but not Q^*-closed. Hence regular closed and Q^*-closed sets are independent of each other.

2.11 Definition. A space X is said to be a **Urysohn space** [**17**] if for every pair of distinct points x and y, there exist open sets U and V such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \phi$.

2.12 Definition. A space X is said to **symmetric space** (or **R_0-space**) [**1**] if for any two distinct points x and y of X, $x \in cl(\{y\})$ implies that $y \in cl(\{x\})$.

2.13 Definition. A topological space X is called **$T_{\eta*}$-space** if every $\eta*$-closed set in it is closed set.

2.14 Definition. A space X is said to be **η∗-normal** if for every pair of disjoint $\eta*$-closed subsets A, B of X, there exist disjoint open sets U, V of X such that $A \subset U$ and $B \subset V$.

2.15 Definition. A space X is said to be **g-normal** [**10**] (resp. **strongly rg-normal** [**3**], **Q^*-normal** [**5**]) if for every pair of disjoint g-closed (resp. rg-closed, Q^*-closed) subsets A, B of X, there exist disjoint open sets U, V of X such that $A \subset U$ and $B \subset V$.

III. Q^*-regular spaces

3.1 Definition. A space $(X, \Im)$ is said to be **η^*-regular** if for every η^*-closed set A and a point $x \notin A$, there exist disjoint open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

3.2 Definition. A space $(X, \Im)$ is said to be **Q^*-regular [6]** (resp. **softly regular** [**4**], **almost regular** [**14**], **g-regular** [**10**], **strongly rg-regular** [**3**]) if for every Q^*-closed (resp. π–closed, regularly closed, g-closed, rg-closed) set A and a point $x \notin A$, there exist disjoint open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

3.3 Definition. A space $(X, \Im)$ is said to be **semi-regular** [**16**] if for each point x of the space and each open set U containing x, there is an open set V such that $x \in V \subset int(cl(V)) \subset U$.

3.4 Theorem. Every regular space is η^*-regular.
Proof. Let X be a regular space. Let F be any η^*-closed set in X and a point $x \in X$ such that $x \notin F$. Since we know that every η^*-closed set is closed. So, F is closed and $x \notin F$. Since X is a regular space, there exists a pair of disjoint open sets G and H such that $F \subset G$ and $x \in H$. Hence X is a η^*-regular space.

3.5 Theorem [10]. Every g-regular space is regular hence η^*-regular.

3.6 Theorem [3]. Every strongly rg-regular space is regular hence η^*-regular.

By the definitions and results stated above, we have the following diagram:

		Q^*-regular				
		⇑				
strongly rg-regular	⇒	g-regular	⇒	regular	⇒	η^*-regular
				⇓		⇓
				softly regular	⇒	almost regular

Where none of the implications is reversible as can be seen from the following examples:
3.7 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b, c}, X}. Consider the closed set {b, c} and a point 'a' such that $a \notin$ {b, c}. Then {b, c} and {a} are disjoint open sets such that {b, c} $\subset$ {b, c}, a $\in$ {a} and {b, c} $\cap$ {a} = ϕ. Similarly, for the closed set {a} and a point 'c' such that c $\notin$ {a}. Then there exist open sets {a} and {b, c} such that {a} $\subset$ {a}, c $\in$ {b, c} and {a} $\cap$ {b, c} = ϕ. It follows that $(X, \Im)$ is regular as well as softly regular space.

3.8 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b}, {a, b}, X}. If we take a point 'a' and an open set V = {a}, then cl(V) = {a, c} and a regularly-open set U = X. So by the definition of weakly regular space $x \in V \subset cl(V) \subset U$, where V be an open set and U be a regularly-open set such that a $\in$ {a} $\subset$ {a, c} $\subset$ X. Hence $(X, \Im)$ is weakly regular. If we take a point 'a' and a regularly-closed set A = {b, c} does not containing the point 'a', there donot exist disjoint open sets containing the point 'a' and a regularly-closed set A = {b, c}. Hence (X, $\Im$) is not almost-regular.

3.9 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b}, {a, b}, X}. Then (X, $\Im$) is weakly regular but not partly-regular. If we take a point 'a' and an open set V = {a}, then cl(V) = {a, c}. Let U = {a, b} be any π–open set. So by the definition of partly-regular space $x \in V \subset cl(V) \subset U$, where V be an open set and U be a π-open set such that $a \in \{a\} \subset \{a, c\} \not\subset \{a, b\}$. Hence (X, $\Im$) is not partly-regular.

3.10 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b}, {a, b}, X}. Then (X, $\Im$) is weakly regular but not softly-regular. Let A = {c} be any π–closed set doesnot containing a point 'a' i.e. $a \notin \{c\}$, there do not exist disjoint open sets containing the point 'a' and the π–closed set A = {c}. Hence (X, $\Im$) is not softly-regular.

3.11 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {a, b}, {a, b, c}, {a, b, d}, X}. Then the space (X, $\Im$) is almost regular but not strongly rg-regular. If we take a point 'a' and F = {b} be any rg-closed set. Then there do not exist disjoint open sets containing the point 'a' and rg-closed set F = {b}. Hence (X, $\Im$) is not strongly rg-regular.

IV. Properties of η^*-regular spaces

4.1 Theorem. For a topological space (X, $\Im$), the following properties are equivalent:
(a) (X, $\Im$) is η^*-regular.
(b) For every $x \in X$ and every η^*-open set U containing x, there exists an open set V such that $x \in V \subset cl(V) \subset U$.
(c) For every η^*-closed set A, the intersection of all the closed neighbourhood of A is A.
(d) For every set A and a η^*-open set B such that $A \cap B \neq \phi$, there exists an open set F such that $A \cap F \neq \phi$ and $cl(F) \subset B$.
(e) For every nonempty set A and η^*-closed set B such that $A \cap B = \phi$, there exist disjoint open sets L and M such that $A \cap L \neq \phi$ and $B \subset M$.

Proof.
(a) $\Rightarrow$ (b). Suppose (X, $\Im$) is η^*-regular. Let $x \in X$ and U be a η^*-open set containing x so that X – U is η^*-closed. Since (X, $\Im$) is η^*-regular, there exist open sets V_1 and V_2 such that $V_1 \cap V_2 = \phi$ and $x \in V_1$, $X - U \subset V_2$. Take $V = V_1$. Since $V_1 \cap V_2 = \phi$, $V \subset X - V_2 \subset U$ that implies $cl(V) \subset cl(X - V_2) = X - V_2 \subset U$. Therefore $x \in V \subset cl(V) \subset U$.

(b) $\Rightarrow$ (c). Let A be η^*-closed set and $x \notin A$. Since A is η^*-closed, X – A is η^*-open and $x \in X - A$. Therefore by (b) there exists an open set V such that $x \in V \subset cl(V) \subset X - A$. Thus $A \subset X - cl(V) \subset X - V$ and $x \notin X - V$. Consequently X – V is a closed neighborhood of A.

(c) $\Rightarrow$ (d). Let $A \cap B \neq \phi$ and B be η^*-open. Let $x \in A \cap B$. Since B is η^*-open, X – B is η^*-closed and $x \notin X - B$. By our assumption, there exists a closed neighborhood V of X – B such that $x \notin V$. Let $X - B \subset U \subset V$, where U is open. Then F = X – V is open such that $x \in F$ and $A \cap F \neq \varnothing$. Also X – U is closed and $cl(F) = cl(X - V) \subset X - U \subset B$. This shows that $cl(F) \subset B$.

(d) $\Rightarrow$ (e). Suppose $A \cap B = \phi$, where A is non-empty and B is η^*-closed. Then X – B is η^*-open and $A \cap (X - B) \neq \phi$. By (d), there exists an open set L such that $A \cap L \neq \phi$, and $L \subset cl(L) \subset X - B$. Put M = X – cl(L). Then $B \subset M$ and L, M are open sets such that $M = X - cl(L) \subset (X - L)$.

(e) $\Rightarrow$ (a). Let B be η^*-closed and $x \notin B$. Then $B \cap \{x\} = \phi$. By (e), there exist disjoint open sets L and M such that $L \cap \{x\} \neq \phi$ and $B \subset M$. Since $L \cap \{x\} \neq \phi$, $x \in L$. This proves that (X, $\Im$) is η^*-regular.

4.2 Theorem. A topological space (X, $\Im$) is η^*-regular if and only if for each η^*-closed set F of (X, $\Im$) and each $x \in X - F$, there exist open sets U and V of (X, $\Im$) such that $x \in U$ and $F \subset V$ and $cl(U) \cap cl(V) = \phi$.
Proof: Let F be a η^*-closed set in (X, $\Im$) and $x \notin F$. Then there exist open sets U_x and V such that $x \in U_x$, $F \subset V$ and $U_x \cap V = \phi$. This Implies that $U_x \cap cl(V) = \phi$. Since cl(V) is closed and $x \notin cl(V)$. Since (X, $\Im$) is η^*-regular, there exist open sets G and H of (X, $\Im$) such that $x \in G$, $cl(V) \subset H$ and $G \cap H = \phi$. This implies $cl(G) \cap$

H = ϕ. Take U = U_x ∩ G. Then U and V are open sets of (X, ℑ) such that x ∈ U and F ⊂ V and cl(U) ∩ cl(V) = ϕ, since cl(U) ∩ cl(V) ⊂ cl(G) ∩ H = ϕ.

Conversely, suppose for each η^*-closed set F of (X, ℑ) and each x ∈ X – F, there exist open sets U and V of (X, ℑ) such that x ∈ U, F ⊂ V and and cl(U) ∩ cl(V) = ϕ. Now U ∩ V ⊂ cl(U) ∩ cl(V) = ϕ. Therefore U ∩ V = ϕ. Thus (X, ℑ) is η^*-regular.

4.3 Theorem. In a η^*-regular space X, every pair consisting of a compact set A and a disjoint η^*-closed set B can be separated by open sets.
Proof. Let X be η^*-regular space and let A be a compact set, B be a η^*-closed set with A ∩ B = ϕ. Since X is η^*-regular space, for each x ∈ A, there exist disjoint open sets U_x and V_x such that x ∈ U_x, B ⊂ V_x. Clearly {U_x : x ∈ A} is an open covering of the compact set A. Since A is compact, there exists a finite subfamily {U_{xi} : i = 1, 2, 3, ……. , n} which covers A. It follows that A ⊂ ∪ {U_{xi} : i = 1, 2, 3, ……. , n} and B ⊂ ∩ {V_{xi} : i = 1, 2, 3, ……. , n}. Put U = ∪ {U_{xi} : i = 1, 2, 3, …., n} and V = ∩ {V_{xi} : i = 1, 2, 3, ……. , n}, then U ∩ V = ϕ. For, if x ∈ U ∩ V ⇒ x ∈ U_{xj} for some j and x ∈ V_{xi} for every i. This implies that x ∈ U_{xj} ∩ V_{xi}, which is a contradiction to U_{xj} ∩ V_{xi} = ϕ. Thus U and V are disjoint open sets containing A and B respectively.

V. Relationships of η^*-regular spaces with Some Separation Axioms

5.1 Theorem [14]. Every almost regular, semi-regular space is regular.

5.2 Corollary. Every almost regular, semi-regular space is η^*-regular.
Proof. Using the fact that every regular space is η^*-regular.

5.3 Corollary. Every softly regular, semi-regular space is η^*-regular.
Proof. Using the fact that every softly regular space is almost regular.

5.4 Theorem [1]. Every normal, symmetric space is regular.

5.5 Corollary. Every normal, symmetric space is η^*-regular.
Proof. Using the fact that every regular space is η^*-regular.

5.6 Corollary. Every g-normal, symmetric space is η^*-regular.
Proof. Using the fact that every g-normal space is normal.

5.7 Corollary. Every rg-normal, symmetric space is η^*-regular.
Proof. Using the fact that every rg-normal space is normal.

5.8 Theorem [15]. Every compact Hausdorff space is regular.
5.9 Corollary. Every compact Hausdorff space is η^*-regular.
Proof. Using the fact that every regular space is η^*-regular.

5.10 Corollary. Every compact Urysohn space is η^*-regular.
Proof. Using the fact that every Urysohn space is Hausdorff.

VI. Conclusion.

In this paper, we introduce and study a new class of generalized regular space is called η^*-regular space which is weaker than regularity. The relationships among strongly rg-regular, g-regular, regular, Q^*-regular, η^*-regular, almost regular and softly regular spaces are investigated. Some of basic properties and characterizations of η^*-regular spaces in the terms of other separation and countability axioms such as semi-regular, Hausdorff, separable, second countable and Lindelof spaces are obtained. This idea can be extended to topological ordered, bitopological, bitopological ordered and fuzzy topological spaces etc.

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