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Research Paper

On Some Analytical Methods for Solving Second Order Ordinary Differential Equations with Constant Coefficients

Niti Ram Pegu
Assistant Professor
Department of Mathematics, Silapathar Town College.
Silapathar, Dhemaji, Assam, India

***Abstract**: Second- order differential equations have great significance theoretically and practically. The article begins by providing an in-depth analysis of different methods for solving homogeneous second-order ordinary differential equation(ODEs) involving constant coefficients, the concepts of solution and the different methods of obtaining these solutions are analyzed. This study reviews the classical techniques, including the characteristic equation methods, solution involving real roots, repeated roots and complex roots and presents the general solution for each case, with examples. Next, the article goes to discuss use of the method undetermined coefficients for solving non-homogeneous equations. The article also highlights the role of initial conditions in determining particular solutions. By providing a systematic overview of these approaches the researcher contributes to a deeper understanding of the structure and application of the second -order linear ODEs, offering both theoretical insights and practical problem-solving tools.*

Keywords: *Homogeneous; Second-order; Differential Operator; Characteristic Equation, Complex Roots, Solution; Constant Coefficients*

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I. INTRODUCTION

Linear differential equations are fundamental mathematical tools used to model various phenomena in science, engineering, and other disciplines. They describe relationships between a function and its derivatives, where the function and its derivatives are multiplied by coefficients that are constants or functions of the independent variable.

Linear differential equations are characterized by their linearity, meaning that the dependent variable along with its derivatives are present only to the first degree and are not multiplied together or divided. This linearity allows for the use of superposition, where the addition of any two solutions to a linear differential equation also results in a solution [Boyce & DiPrima,2017. Higher-order linear differential equations are a specific type of linear differential equation where the highest order derivative is of order n. These equations are of particular interest as they involve more complex relationships between the function and its derivatives. The highest order derivative present in a differential equation determines its order [Kreyszig,2018]. Differential equations are categorized into partial differential equations (PDE) or ordinary differential equations (ODE) based on the presence or absence of partial derivatives. The order of a differential equation is determined by the highest order derivative it contains. A solution, or particular solution, of a differential equation of order n is a function defined and differentiable n times over a domain. A solution containing arbitrary constants corresponding to the differential equations order is known as a general solution. On the other hand, a solution devoid of arbitrary constants is referred to as a particular solution [Hilbert, 2013]. Higher-order linear differential equations have great significance theoretically and practically. They are typically used in a variety of applications in Science and Engineering (Ross, 2021, p. 110). Differential equations find applications in physics, biology, economics, and many other disciplines, playing a crucial role in predicting and analyzing the behaviour of complex phenomena [Strogatz, 2014]. Most real-world equations are second-order, though higher-order ones do show up now and then. This leads to the common belief that the world operates on a "second-order" basis in modern physics. Essentially, the key results for higher-order linear ODEs are quite similar to those for second-order equations, just with "n" replacing "2" [Lebl, 2013].

This study covers both mathematical aspects of solving linear differential equations and physical interpretations and applications. We discuss the connection between the solutions and physical systems and how the solution can be used to analysis and predict the physics and engineering [8]. The techniques and methods presented in this article are fundamental to many fields of study, including mathematics, physics, and engineering. The article aims to provide a comprehensive guide for students and researchers who are interested in this topic, and it may be used as a reference for solving problems to relate fields [14].

II. PRELIMINARIES

2.1 Differential Equations

An equation involving independent and dependent variables and the derivatives or differentials of one or more dependent variables is called a differential equation.

A differential equation which involves derivatives with respect to a single variable is known as an ordinary differential equation. For example: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$.

The order of the highest derivatives involves in a differential equation is called the order of a differential equation. The degree of a differential equation is the degree of the highest order derivative present in the equation, after the differential equation has been free from the radicals and fractions as far as the derivatives concerned. For example: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$ is a second order and first degree differential equation.

A differential equation in which dependent variables and all its derivatives present occur in the first degree only and no products of dependent variables and/or derivatives occur is known as a linear differential equation. For example: $\frac{dy}{dx} = \sin x + \cos x$ is a linear equation of first order.

2.2 Higher Order Linear Differential Equations

The general linear equation of nth order can be written

$$b_0(x)\frac{d^ny}{n} + b_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + b_{n-1}(x) + b_n(x)y = F(x)$$

An equation qualifies as a homogeneous linear differential equation when the function $F(x)$ equals zero for all x. If $F(x)$ is non-zero for any x, the equation is considered non-homogeneous [Rainville & Bedient, 1989].

If the solution of Eq. (1) are y_1, y_2,........, y_n and if c_1, c_2, y_2,........, c_n are constants, then

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

For a solution to a higher-order linear differential equation be valid, it must have linear independence. For a set of functions to be linearly independent, scalars v_1, v_2,........, v_n of Eq. (3) should be zero (Xu, 2011).

$$v_1y_1 + v_2y_2 + \cdots + v_ny_n$$

Linear independence is crucial in the determining the general solution to a differential equation, as it guarantees that the solution is not redundant (Coddington & Levinson, 1955).

Moreover, their Wronskian must not be equal to zero. The Wronskian of n functions $y_1(x)$, $y_2(x)$, ...,$y_n(x)$ is denoted by $W(x)$ and is defined to be the determinant

$$W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$$

If $W = 0$, then y_1, y_2,........, y_n are considered to be linearly dependent and if $W \neq 0$, it can be deduced that they are linearly independent.

2.3 Differential Operators

Let D denote the differentiation with respect to x. Then, D^k, as shown in Eq. (1), refers to differentiating k times with respect to x. This is true for positive integral k.

$$\mathrm{D^k y} = \frac{\mathrm{d^k y}}{\mathrm{dx^k}} \tag{1}$$

If $P(D)$ is a polynomial operator of order n defied by

$$P(D) = a_0 + a_1D + a_2D^2 + \cdots + a_nD^n,\ a_n \neq 0 \tag{2}$$

and y is an nth order differentiable function, then

$$P(D)y = (a_0 + a_1D + a_2D^2 + \cdots + a_nD^n)y$$

From Eq. (1), we obtain

$$P(D)y = a_0 y + a_{k-1}y^{k-1} + a_{k-2}y^{k-2} + \cdots + a_k y$$
(3)

2.4 Differential Operators Properties

For constant m and positive integral k

$$D^k e^{mx} = m^k e^{mx} \tag{4}$$

The effect of an operator upon e^{mx} can be determined. Suppose $P(D)$ be a polynomial in D

$$P(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n,\ a_n \neq 0 \tag{5}$$

Then

$$P(D)e^{mx} = a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} a_n e^{mx}$$

Therefore,

$$P(D)e^{mx} = e^{mx} P(D) \tag{6}$$

If m satisfies $P(m) = 0$, then in light of Eq. (6), we obtain

$$P(D)e^{mx} = 0$$

Eq. (7) and Eq. (8) demonstrate how the operator $(D - a)$ affects the product of a function y and e^{ax},

$$(D - a)(e^{ax}y) = D(e^{ax}y) - ae^{ax}y$$
(7)

$$= e^{ax}Dy$$
(8)

Subsequently, the use of the operator $(D - a)^{\ 2}$ is shown on Eq. (9) and (10),

$$(D - a)^2(e^{ax}) = e^{ax}D^n y$$
(9)

$$= e^{ax}D^2 y$$
(10)

By linearity of differential operators, it can be concluded that when $P(D)$ represents a polynomial in D, then

$$e^{ax}P(D)y = P(D - a)[e^{ax}y]$$
(11) 2.5 **Inverse Operation**

Let $P(D)y = F(x)$, where $P(D)$ is the polynomial operator defined in Eq. (2) and $F(x)$ is the function consisting only of such terms as b, x^k, e^{ax}, $\sin ax$, $\cos ax$ and finite number of combination of these terms, where a, b are constants and k is a positive integral. The inverse operators of $P(D)$ written as $P^{-1}(D)$ or $1/_{P(D)}$, is then defined as an operator which, when operating in $F(x)$, will give the particular integral (y_p) of $P(D)y = F(x)$ that is contains no constant multiples of a term in complementary function (y_c) .i.e.,

$$P^{-1}(D)F(x) = y_p \text{ or } y_p = \frac{1}{P(D)}F(x)$$
(12)

Therefore, $D^{-n}F(x)$ will mean the integration of $F(x)$ n times by ignoring constants of integration.
Also, if $P(D)y = 0$, then $y_p = 0$.
Therefore, $P(D)[P^{-1}(D)F(x)] = F(x)$
(13)

III. RESEARCH OBJECTIVES

This article focuses on linear ordinary differential equations with constant coefficients, a common types of differential equation in various applications, since they very frequently in many branches of applied mathematics. Explicit methods of available for solving these equations include the variation of parameters, reduction of order, exponential shift, and undetermined coefficients. Being a relatively simple solution method requiring only skills in differentiation and algebra, this article aims to discuss the method of solutions of second-order linear differential equations with constant coefficients. More specially, it aims to:

i. Analyse the various standard methods use to solve second-ordinary ordinary differential equation with constants coefficients including the characteristic equation methods, solution involving real roots, repeated roots and complex roots and present the general solution for each case, with examples.

ii. Discuss the method of undetermined coefficients to solve a second-order differential equations.

IV. METHODOLOGY

In this section we discuss the different methods for solving second order linear differential equations (ODEs) with constant coefficients. The typical form of such equation is

$$b_0(x)\frac{d^2y}{dx^2} + b_1(x)\frac{dy}{dx} + b_2(x)y = F(x),$$
(14) Using the differential operators symbols in Eq. (14), we obtain

$$f(D)y = F(x)$$

(15) where $f(D) = b_0(x)D^2 + b_1(x)D + b_2(x)$ and b_0, $b_1$and b_2 are constants. Two forms of this equation usually presents themselves, namely, homogeneous, when the right-hand member is zero, and non-homogeneous, when the right-hand number is a function of x. We will first consider the first form and then the second.

4.1 Solution of Homogeneous Linear ODEs

Consider the equation of the form:

$$b_0(x)\frac{d^2y}{dx^2} + b_1(x)\frac{dy}{dx} + b_2(x)y = 0 \text{ or } f(D) = 0$$

(16)

Let us take $y = e^{mx}$ $(x \neq 0)$ be the nontrivial solution. The, if we put $y = Ce^{mx}$ in the left side of Eq. (16), it must satisfy the equation, i.e. we must have

$$Ce^{mx}[b_0(x)m^2 + b_1(x)m + b_2(x)] = 0$$

Since $Ce^{mx} \neq 0$, $b_0(x)m^2 + b_1(x)m + b_2(x) = 0$ or $F(m) = 0$

(17)

The Eq.(17) is called the Auxiliary Equation (A.E) or Characteristic Equation (C.E.) of Eq. (16).

Let m_1, m_2 be two roots of the equation (17).

Then, $y = c_1e^{m_1x}$ and $y = c_2e^{m_2x}$ are obviously solutions of the Eq. (16). Also, it can be easily verified by directly substitution that $y = ce^{m_1x}$, $y = ce^{m_2x}$ and $y = c_1e^{m_1x} + c_2e^{m_2x}$ satisfy the Eq. (16), and, as such are solutions of Eq. (16).

We will now consider the nature of the general solution of the Eq. (16) according as the roots of the auxiliary Eq. (17) are (i) real and distinct, (ii) real and repeated and (iii) imaginary.

Case- I: Auxiliary equation having real and distinct roots.

If m_1 and m_2 are real and distinct, then $y = c_1e^{m_1x} + c_2e^{m_2x}$is the general solution, since it is satisfy the Eq. (16), and contains two independent arbitrary constants equal in number to the order of the equation.

Example-1 Solve $(D^2 - 7D + 12)y = 0$

Solution: It has an auxiliary equation

$m^2 - 7m + 12 = 0$

Notice that this can be simplified and rewritten as

$$(m - 3)(m - 4) = 0$$

Equating each factor to 0, we obtain the roots. Therefore, the roots of the equation are $m = 3,4$

The roots are real and distinct. Thus, e^{3x} and e^{4x} are the solutions of the given equation and we can express the general solution as:

$y = c_1e^{3x} + c_2e^{4x}$, where c_1 and c_2 are arbitrary constants

Case- II: Auxiliary equation having two equal roots.

If the auxiliary equation has two equal roots, the method of the preceding paragraph does not lead to the general solution. For, if $m_1 = m_2 = \alpha$ say, then the solution of the preceding paragraph assumes the form

$$y = (c_1 + c_2)e^{\alpha x} = ce^{\alpha x} \text{ , where } c_1 + c_2 = c,$$

which is not the general solution, since it involves only one independent constant and the equation is of the second order.

A method will now be devised for finding the general solution in the case under discussion. Since the auxiliary solution (17) has two equal roots each being equal to, it follows from the Eq. (16) assumes the form

$$b_0(x)m^2 + b_1(x)m + b_2(x) = 0 \quad (18)$$

Let $= e^{\alpha x}v$, where v is a function of x, be a trial solution of this equation. Substituting this value of y in the left side of the above equation, we obtain

$$e^{\alpha x}b_0(x)\frac{d^2y}{dx^2} = 0, \text{ i.e., } \frac{d^2y}{dx^2} = 0, \text{ since } e^{\alpha x} \neq 0$$

Now, integrating this twice, we get $v = (c_1 + cx)\, e^{\alpha x}$.

Hence, the solution of the Eq. (17) in this case is $y = (c_1 + c_2x)\, e^{\alpha x}$.

This is the general solution of Eq. (17), since it satisfies the Eq. (17), and contains two independent arbitrary constants.

Example-2. Solve $(16D^2 - 24D + 9)y = 0$

Solution: It has an auxiliary equation

$16m^2 - 24m + 9 = 0$

Notice that this can be simplified and rewritten as

$(4m - 3)^2 = 0$

Equating each factor to 0, we obtain the roots. Therefore, the roots of the equation are $m = \frac{3}{4}, \frac{3}{4}$

The roots are real and equal. Therefore, $e^{\frac{3}{4}x}$ and $e^{\frac{3}{4}x}$ are the solutions of the given equation. Therefore, the general solution is:
$y = (c_1 + c_2x)e^{\frac{3}{4}x}$, where c_1 and c_2 are arbitrary constants

Case- III: Auxiliary equation having a pair of complex roots.
Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution of Eq. (17) is
$y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}$
The above solution, by adjusting the arbitrary constants, can be put in a more convenient form not involving imaginary expression, thus we have

$$\begin{aligned} y &= e^{\alpha x}\left[c_1e^{i\beta x} + c_2e^{-i\beta x}\right] \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\ &= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x] \\ &= e^{\alpha x}[A\cos\ \beta x + iB\sin\ \beta x] \end{aligned}$$

where $A = c + c_2$ and $B = i(c_1 - c_2)$ are the arbitrary constants, which may be given any real values we like.
Example-3 Solve $(D^2 - 2D + 5)y = 0$
Solution: It has an auxiliary equation
$m^2 - 2m + 5 = 0$
By the quadratic formula, the roots of the auxiliary equation are

$$m = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Here, the roots are complex numbers $\alpha \pm i\beta$, where $\alpha = 1$ and $\beta = 2$.
Therefore, the general solution to this differential equation is

$$y = e^x(c_1\cos 2x + c_2\sin 2x),\text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants}$$

4.2 Solution of Non-Homogeneous Linear ODEs
Consider the equation of the form

$$b_0(x)\frac{d^2y}{dx^2} + b_1(x)\frac{dy}{dx} + b_2(x)y = F(x),$$

(19) where the coefficients b_0 , b_1 and b_2 are constants but the non-homogeneous term $F(x)$ in general a non-constant and function of x.
The general solution of the Eq. (19) is may be written,$y = y_c + y_p$, where y_c is the general solution of the corresponding homogeneous Eq. (16) with $F(x)$ replacing by zero and y_p is called the particular integral and it is any specific solution to the non-homogeneous ODE that contains no arbitrary constant. The complete general solution to the ODE is the sum of the complementary function and the particular integral.

4.3 Determination of the particular integral (P.I)
Case-I: When $F(x) = bx^k$ and $P(D) = D - a_o, a_o \neq 0$.
(20)
Then P.I. $=y_p = \frac{1}{P(D)}b = \frac{b}{a_0}, a_0 \neq 0.$
Example-4: Solve $(D^2 - 2D - 3)y = 5$
Solution: The A.E. is: $m^2 - 2m - 3 = 5$
After solving, the roots of the equation are m=-1, 3
Thus, C.F. $=y_c = c_1e^{-x} + c_2e^{3x}$, where c_1 and c_2 are arbitrary constants
Now, from the Eq. (20) the particular integral can be expressed as
P.I. $=P(D) = D^2 - 2D - 3$, with $a_0 = -3$ and $b = 5$
Hence P.I. $=y_p=\frac{b}{P(D)} = \frac{b}{a_0} = -\frac{5}{3}$
Therefore, the complete solution is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} - \frac{5}{3}$$

Case-II: When $F(x) = bx^k$ and $P(D) = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D$. (21)
Then, from Eq. (13), we get, P.I.$=y_p = \frac{1}{D^r(a_nD^n+a_{n-1}D^{n-1}+\cdots+a_1D+a_r)}bx^k$
Example-5: Solve $(D^2 + 4)y = x^2$
Solution: The auxiliary equation is $m^2 + 4 = 0$ has roots $m = \pm 2i$.
The complementary function $(y_c) = c_1\cos 2x + c_2\sin 2x$
Now, using Eq. (21), wee obtain
Particular Integral $(y_p) = \frac{1}{D^2+4}x^2 = \frac{1}{4(1+\frac{1}{4}D^2)}x^2 = \frac{1}{4}(1 + \frac{1}{4}D^2)^{-1}x^2$

$$= \frac{1}{4}(1+\frac{1}{4}D^2+\frac{1}{16}D^4-\cdots)x^2 = \frac{1}{4}(x^2-\frac{1}{2})$$

Therefore, the complete solution is

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}(x^2-\frac{1}{2})$$

Case-III: When $F(x) = be^{ax}$

In this case $P(D)y = F(x)$ becomes $P(D) = be^{ax}$.

The P.I. here is

$$y_p = \frac{1}{P(D)}be^{ax} = \frac{be^{ax}}{P(a)}, \quad P(a) \neq 0 \tag{22}$$

Example-6: Solve $(D^2-2D+5)y = e^{-x}$

Solution: The A. E. is: $m^2-2m+5=0$

The roots are m= $-1,\ \pm 2i$

C. F. $=e^{-x}(c_1\cos 2x + c_2\sin 2x)$

Now, using the Eq. (22) the particular integral is

$$\text{P.I.} = \frac{1}{D^2-2D+5}e^{-x} = \frac{1}{(-1)-2(-1)+5} = \frac{1}{8}e^{-x}$$

The complete solution is

$$y = y_c + y_p = e^{-x}(c_1\cos 2x + c_2\sin 2x) + \frac{1}{8}e^{-x}$$

Case-IV: When $F(x) = \cos ax$ or $\sin bx$ and $P(-a^2) \neq 0$. (23)

Then, Particular Integral $(y)_p = \frac{1}{P(D^2)}\sin ax = \frac{1}{P(-a^2)}\sin ax$

Also, $y_p = \frac{1}{P(D^2)}\cos ax = \frac{1}{P(-a^2)}\cos ax$

If $y_p = \frac{1}{P(D)}\sin ax$, then we should put $-a^2$ for D^2, $-a^4$ for D^4, $-a^6$ for D^6 and so on, in $P(D)$ to calculate y_p

The above method fails when $P(-a^2) = 0$. In this situation, we proceed as follows:

We know that

$$e^{iax} = \cos ax + \sin ax \tag{24}$$

From this relation, we obtain

$$\frac{1}{D^2+a^2}\sin ax = \text{Im}\frac{1}{D^2+a^2}e^{iax}$$

and

$$\frac{1}{D^2+a^2}\cos ax = \text{Re}\frac{1}{D^2+a^2}e^{iax}$$

Now,

$$\frac{1}{D^2+a^2}\cos ax = \frac{1}{(D-ia)(D+ia)}e^{iax} = \frac{1}{(D-ia)}\frac{e^{iax}}{2ia} = \frac{x}{2ia}e^{iax}$$

$$= \frac{x}{2ia}(\cos ax + \sin ax) = \frac{x}{2a}(\sin ax - \cos ax)$$

Equating the real and imaginary parts, we obtain

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a} \tag{25}$$

and

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax$$

(26)

Example-7: Solve $(D^2+4)y = \cos 2x$

Solution: The A.E. m^2+4=0 gives $m = \pm 2i$

Therefore, $y_c = c_1\cos 2x + c_2\sin 2x$

Now, by using the Eq. (25) we can find the particular integral as

$$y_p = \frac{1}{D^2+a^2}\cos 2x = \frac{1}{D^2+4}\cos 2x = \frac{x}{2(2)}\sin 2x$$

Hence, complete solution is

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{x}{4}\sin 2x$$

Case-V: When $F(x) = xV$, where V is any function of x.

Here, $y_p = \frac{1}{P(D)}(xV)$

$$= x\frac{1}{P(D)}V - \frac{P'(D)}{[P(D)]^2}V \tag{27}$$

Example-8: Solve $(D^2-2D+1)y = xe^x\sin x$

Solution: The A.E. m^2-2m+1=0 gives $m = 1, 1$

Therefore, $y_c = (c_1+c_2x)e^x$

Now, by using the Eq. (27) we can find the particular integral as

$$y_p = \frac{1}{D^2-2D+1}xe^x\sin x = \frac{1}{(D+1)^2-2(D+1)+1}x\sin x = e^xD^{-1}(x\sin x)$$

Integrating by parts, we get

$$y_p = e^{-s}(-x\sin x - 2\cos x)$$

Hence, complete solution is

$$y = y_c + y_p = (c_1 + c_2 x)e^x - e^{-s}(x\sin x + 2\cos x)$$

4.4 Method of Undetermined Coefficients

This method is yet another method of finding a particular integral of non-homogeneous linear ODEs

$$f(D)y = F(x) \quad (28)$$

In this method we first find the complementary function of Eq. (28) as in Eq. (16). This method is useful only when $F(x)$ contains terms in some special forms given in the **Table-1**. The method of undetermined coefficients consists in making a guess of the **trial solution** $\boldsymbol{y}^*$ from the form of $F(x)$**.** Then, we substitute the trial solution $\boldsymbol{y}^*$ in Eq. (28) and determine constants by comparing like terms on both sides of the equation $f(D)\boldsymbol{y}^* = F(x)$. Finally, the general solution is given by

$$y = C.F. + \boldsymbol{y}^*$$

Table-1 shows the general form of right-hand member $F(x)$ along with their corresponing prediction for the particular solution. The specific functions that this method can handle are those that have a finite family of constants occurring in first column are known and the constants second column are determined by substituting into the trial solution in the given equation i.e., they are obtained from the resulting identity, $f(D)\boldsymbol{y}^* = F(x)$.

Table 1. Functions suitable for method of undetermined coefficients

Special form of $F(x)$	Trial solution y^* for P.I.
x^n or $a^x x^n$ or $a_0 + a_1 + \cdots + ax^n$	$A_0 + A_1 x + \cdots + A_n x^n$
e^{ax} or pe^x	Ae^{ax}
$q\cos ax$	$A\sin ax + B\cos ax$
$p\sin ax$	$A\sin ax + B\cos ax$
$p\sin ax + q\cos ax$	$A\sin ax + B\cos$
n^{th} degree polynomial	$A_0 + A_1 x + \cdots + A_n x^n$
Where n is a positive integer and A, B, p, q are arbitrary constants	

Case-I: When $F(x)$ is an exponential function

Example: 9 Solve $(D^2 - 2D - 3)y = 2e^{4x}$

(29)

Solution: The A. E. is: $m^2 - 2m - 3 = 0$

The roots are $m = 3, -1$

The C.F. is written is $y_c = c_1 e^{3x} + c_2 e^{-x}$

Now by undetermined coefficient method- Trial solution for $\boldsymbol{y}^*$ is :

$$\boldsymbol{y}^* = Ae^{4x} \quad (30)$$

Since Eq. (30) is second-order differential equation, $\boldsymbol{y}^*$ must be differentiated two times, we get

$$D\boldsymbol{y}^* = 4Ae^{4x} \quad \text{and} \quad D^2\boldsymbol{y}^* = D(Dy_p) = D(4Ae^{4x}) = 16Ae^{4x}$$

Now, since Eq. (30) is the trial solution of the Eq. (29), so it must satisfy the Eq. (29). Substituting these values of $\boldsymbol{y}^*$, $D\boldsymbol{y}^*$ and $D^2\boldsymbol{y}^*$ in the Eq. (29), which yields,

$$D^2\boldsymbol{y}^* - 2D\boldsymbol{y}^*_p - 3\boldsymbol{y}^* = 2e^{4x}$$

$$16Ae^{4x} - 8Ae^{4x} - 3Ae^{4x} = 2e^{4x}$$

which will simplify to

$5Ae^{4x} = 2e^{4x}$

Equating the coefficient of e^{4x} on the both sides :

$$5A = 2 \text{ i.e., } A = \frac{2}{5}.$$

Thus $\quad \boldsymbol{y}^* = \frac{2}{5}e^{4x}$

Since

$y = y_c + \boldsymbol{y}^*$

the general solution of Eq. (29) is the linearly independent

$$y = c_1 e^{3x} + c_2 e^{-x} + \frac{2}{5}e^{4x}$$

Case-II: If $F(x)$ is a polynomial

Example:10 Solve $(D^2 + 4)y = x^2$ (31)

Solution: The A. E. is: $m^2 + 4 = 0$

The roots are $m = \pm 2i$

The C.F. is written is $y_c = c_1 \cos 2x + c_2 \sin 2x$
Now by undetermined coefficient method- Trial solution for $\boldsymbol{y}^*$ is :
$$\boldsymbol{y}^* = A_0 + A_1 x + A_2 x^2 \qquad (32)$$
Since $\boldsymbol{y}^*$ must satisfy Eq. (32), we get
$$(D^2 + 4)\boldsymbol{y}^* = x^2 \text{ or } D^2\boldsymbol{y}^* + 4\boldsymbol{y}^* = x^2 \qquad (33)$$
Since Eq. (31) is second-order differential equation, $\boldsymbol{y}^*$ must be differentiated two times, we get
$$D\boldsymbol{y}^* = A_1 + 2A_2 x \quad \text{and} \quad D^2\boldsymbol{y}^* = 2A_2$$
Substituting these values of $\boldsymbol{y}^*$, $D\boldsymbol{y}^*$ and $D^2\boldsymbol{y}^*$ in the Eq. (33), which yields,
$$2A_2 + 4(A_0 + A_1 x + A_2 x^2) = x^2$$
which will simplify to
$$2A_2 + 4A_0 + 4A_1 x + 4A_2 x^2 = x^2$$
Equating the coefficients of the like terms:
$$x^2: \ 4A_2 = 1 \text{ i.e., } A_2 = \frac{1}{4}$$
$$x: \ 4A_1 = 0 \text{ i.e. } A_1 = 0$$
$$x^0: \ 2A_2 + 4A_0 = 0 \text{ i.e., } A_0 = -\frac{1}{8}$$
Substituting the obtained values of A_0, A_1 and A_2 into Eq. (32),
$$\boldsymbol{y}^* = -\frac{1}{8} + \frac{x^2}{4}$$
The general solution is
$$y = y_c + \boldsymbol{y}^* = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{8} + \frac{x^2}{4}$$

Case-III: When $F(x)$ contains $\sin ax$ or $\cos ax$

Example:11 Solve $(D^2 + 3D + 2)y = x + \cos x$
(34)

Solution: The A. E. is: $m^2 + 3m + 2 = 0$
The roots are $m = -1, -2$
The C.F. is written is $y_c = c_1 e^{-x} + c_2 e^{-2x}$
Corresponding to special form x of R.H.S. of Eq.(34), we choose trial solution for P.I. as $A_0 + A_1 x$ and corresponding to special form $\cos x$ of R.H.S. of (34), we predict trial solution for P.I. as $A_2 \cos x + A_3 \sin x$. Combining these, we attempt a trial solution for particular solution as:
$$\boldsymbol{y}^* = A_0 + A_1 x + A_2 \cos x + A_3 \sin x. \qquad (35)$$
where A_0, A_1, A_2 and A_3 are constants to be determined.
Since $\boldsymbol{y}^*$ must satisfy Eq. (34), we get
$$D^2\boldsymbol{y}^* + 3D\boldsymbol{y}^* + 2\boldsymbol{y}^* = x + \cos x \qquad (36)$$
Since Eq. (34) is second-order differential equation, $\boldsymbol{y}^*$ must be differentiated two times, we get
$$D\boldsymbol{y}^* = A_1 - A_2 \sin x + A_3 \cos x$$
and
$$D^2\boldsymbol{y}^* = -A_2 \cos x - A_3 \sin x$$
Substituting these values of $\boldsymbol{y}^*$, $D\boldsymbol{y}^*$ and $D^2\boldsymbol{y}^*$ in the Eq. (36), which yields,
$$-A_1 \cos x - A_3 \sin x + 3(A_1 - A_2 \sin x + \sin x + A_3 \cos x) + 2(A_0 + A_1 x + A_2 \cos x + A_3 \sin x) = x + \cos x$$
which will simplify to
$$3A_1 + 2A_0 + 2A_1 x + (A_2 + 3A_1)\cos x + (A_3 - A_1)\sin x = x + \cos x$$
which an identity and so equating the coefficients of the like terms:
$$x^0: \ 3A_1 + 2A_0$$
$$x: \ 2A_0 = 1$$
$$\cos x: A_2 + 3A_1 = 1$$
and
$$\sin x: A_3 - A_1 = 0$$
After solving these, we obtain
$$A_0 = -\frac{3}{4}, A_1 = \frac{1}{2}, \ A_2 = \frac{1}{10}, \text{ and } A_3 = \frac{3}{10}$$
Substituting the obtained values of A_0, A_1, A_2 and A_3 into Eq. (35), we get
$$\boldsymbol{y}^* = -\frac{3}{4} + \frac{x}{2} + \frac{1}{10}(\cos x + 3\sin x)$$
The general solution is
$$y = y_c + \boldsymbol{y}^* = c_1 e^{-x} + c_2 e^{-2x} - \frac{3}{4} + \frac{x}{2} + \frac{1}{10}(\cos x + 3\sin x)$$

V. RESULTS AND DISCUSSION

This study examined the analytical methods such as characteristic equation method and method of undetermined for solving second order linear ordinary differential equations with constant coefficients. The characteristic equation method provides a direct and elegant solution for equations with constants coefficients, but becomes less convenient when non-homogeneous terms are complex. Equations that result in separate and actual solutions are represented as combinations of exponential functions, which can be described in clear and analytical form. The method of undetermined coefficients is a simple method, only requiring skills in differentiation and algebra. However, its use is limited because it required that the non-homogeneous term has functions that have a finite family of derivatives. This is only applicable to algebraic, sinusoidal and exponential functions. Additionally, when predicting for particular solution, one may encounter difficulties when assuming its appropriate form. This happens while non-homogeneous term is a solution of the differential equations itself.

VI. CONCLUSION

In conclusion, the solution of linear ODEs with constant coefficients of second order is an important topic, since they occur very frequently in many branches of applied mathematics. This class of equations arises in many fields of study and has a wide range of applications in physical system and engineering. While the method of undetermined coefficients provides a systematic and efficient technique for solving non-homogeneous linear differential equations with constant coefficients, its effectiveness depends on the existence of finite families of derivatives of non-homogeneous term. This method remains an essential tool in applied mathematics, physical system, and engineering, as it not only simplifies the solution but also deepens the understanding of the behavior of differential systems. Higher-order differential equations involve more complex algebra and differentiation. Therefore, determining if the method of undermined coefficients is suitable for a given differential equation is crucial for accuracy of solution.

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