Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 9 (September 2025) pp: 16-23
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

A Modified Adomian Decomposition Method for Time-Fractional Diffusion Equations

V. D. Mathpati[1], B. B. Pandit[2]
[1](Department of Mathematics, Netaji Subhashchandra Bose ACS college, Nanded-431601, (MH), India)
[2](Department of Mathematics, Shri Datta Arts, Commerce and Science College, Hadgaon-431712, (MH), India)
Corresponding Author: V. D. Mathpati

ABSTRACT : This article deals with the fractional derivative operators related to the heat equation. Used the Adomian Decomposition Method (ADM) and fraction calculus to get the analytical solution. A modification of ADM is proposed that combines boundary conditions with initial condition to get a new initial solution at each iteration. Some illustrative examples were presented.

.

KEYWORDS: *Adomian Decomposition Method, Anomalous Diffusion, Fractional Differential and Integral Operator, Initial Boundary Value Problems, Caputo Derivative.*

Received 08 Sep., 2025; Revised 17 Sep., 2025; Accepted 19 Sep., 2025

I. INTRODUCTION

Fractional calculus has become a significant area of applied mathematics in recent years. When modelling real-world problems, fractional derivatives and fractional integrals give more realistic results as compared to classical order. In particular, fractional calculus has applications in the modelling of various physical phenomena such as, the fields of damping, robot technology, signal processing, physics, genetic algorithms, economics and finance.

Recently, considerable research has been done on Adomian decomposition method to solve initial and boundary value problems. Abdul-Majid Wazwaz *et al.* [1,3,4] modified Adomian decomposition method to accelerate the rapid convergence of the series solution and applied the modification to initial boundary value problems. Jafar Biazar *et al.* [5], used Adomian decomposition method to solve systems of differential equations and addressed convergence of the method for these systems.

The solution of fractional partial differential equations with variable coefficients using the decomposition method was given by Momani [7]. Yizheng Hu *et al* [8] used Adomian decomposition method for solving linear fractional-order differential equation with constant coefficients. A detail review of the Adomian decomposition method for boundary value problems, new developments of the method and its applications to linear or nonlinear and ordinary or partial differential equations, including fractional differential equations is given in [9,16]. Many researchers used hybrid method, combination of integral transform coupled with decomposition technique due to Adomian scheme, for solution of diffusion equation of fractional order [12-15]. Applications of Adomian decomposition method for nonlinear heat conduction problems in a straight fin are discussed in [6,11]. The linear time-fractional differential equations which presented that the convergence of the scheme provided in the research have high accuracy for solving discussed in [18,19].

In a thorough literature review it is found that many researchers used only initial or boundary conditions to solve fractional initial and boundary value problems. In this article, we propose a modification in Adomian decomposition method to obtain a new initial solution at each iteration by mixing initial and boundary conditions. This technique of constructing new successive initial solutions yields more accurate solutions.

II. SOME USEFUL DEFINITIONS

In this section basic definitions of Fractional Derivative and Fractional Integration are provided [2].

Definition 2.1: Riemann Liouville Fractional Derivative

For $\alpha \in [n-1, n)$, the α-derivativeof $f$is

$$_{a}D_t^{\alpha}(f)(t) = \frac{1}{\Gamma n-\alpha}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

Definition 2.2: Caputo Fractional Derivative

For α∈ $[n-1, n)$, the α derivative of f is

$$_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma n-\alpha}\int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx$$

Definition 2.3: Riemann Liouville Fractional integral of order $\alpha > 0$ of a function f is defined as

$$_{a}D_t^{-\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_a^x (x-t)^{(\alpha-1)} f(t)\, dt$$

III. ADOMIAN DECOMPOSITION METHOD

Applications of Adomian decomposition method (ADM) [10] for the solution of Boundary Value Problems has many applications in science and engineering. This method mainly relies on computing the Adomian polynomials for nonlinear operator. Computation of Adomian polynomials for various forms of nonlinearity is the first step required to solve non-linear problems. Consider a differential equation

$$F\big(u(x)\big) = g(x) \tag{3.1}$$

where F is a differential operator and g is arbitrary function.

The linear terms on left hand side of (3.1) are decomposed into $Lu + Ru$, where $L$is the highest order derivative and R is the remainder of the linear operator. Equation (3.1) can be written as

$$Lu + Ru + Nu = g \tag{3.2}$$

where, Nu contains the nonlinear terms. Operating the inverse operator L^{-1} on (3.2) gives

$$u = \varphi + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu) \tag{3.3}$$

where φ is integration constant satisfying the condition $L\varphi = 0$. The solution u can be represented as an infinite series of the form

$$u = \sum_{n=0}^{\infty} u_n \tag{3.4}$$

Also assume that the nonlinear term Nu can be written as an infinite series using Adomian polynomials A_n of the form

$$Nu = \sum_{n=0}^{\infty} A_n \tag{3.5}$$

where the Adomian polynomials A_n are evaluated using the formula

$$A_n = \frac{1}{n!}\frac{d^n}{dx^n} N\left(\sum_{i=0}^{\infty}(\lambda^i u_i)\right)\Bigg|_{\lambda=0}, n = 0,1,2, \dots \tag{3.6}$$

Then using (3.4) and (3.5) into (3.3) gives

$$\sum_{n=0}^{\infty} u_n = \varphi + L^{-1}(g) - L^{-1}\left(R\sum_{n=0}^{\infty} u_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{3.7}$$

Each term of (3.4) is given by the recurrence relation

$$u_0 = \varphi + L^{-1}(g),$$

$$u_{n+1} = \varphi + L^{-1}(Ru_n) - L^{-1}(A_n), n = 0,1,2, \dots \tag{3.8}$$

The convergence of the decomposition series has investigated by several authors [16].

IV. ADM FOR SOLVING INITIAL BOUNDARY VALUE PROBLEMS

A fractional heat equation is obtained from the classical diffusion equation by replacing the first-order time derivative term by a fractional derivative of order $\alpha > 0$ [10].

In this article, the fractional heat equations of the following form are considered

$$\frac{\partial^\alpha T}{\partial t^\alpha} = f(x,y)T_{xx} + g(x,y)T_{yy} \tag{4.1}$$

with the boundary conditions $0 < x < a,\ 0 < y < b,\quad t > 0$

$$T(0,y,t) = f_0(y,t), T(a,y,t) = f_0(y,t),$$

$$T(x,0,t) = g_0(x,t), T(a,0,t) = g_0(x,t) \tag{4.2}$$

and initial condition

$$T(x,y,0) = \varphi(x,y) \tag{4.3}$$

We propose successive initial solution $T_n{}^*$ at every iteration for (4.1) by adopting a new technique

$$T_n{}^*(x,y,t) = T_n(x,y,t) + (1-x)[T(0,y,t) - T_n(0,y,t)] + x[T(a,y,t) - T_n(a,y,t)] + (1-y)[T(x,0,t) - T_n(x,0,t)] + y[T(b,y,t) - T_n(b,y,t)] \tag{4.4}$$

where $n = 0,1,2, \dots$,

clearly the new successive initial solution $T_n{}^*$of (4.1) satisfy initial and boundary conditions.

The first approximation is $T_0(x,y,t) = T(x,y,0)$.

In operator form equation (4.1) can be expressed as

$$D_t^\alpha T = f(x,y)T_{xx} + g(x,y)T_{yy} \tag{4.5}$$

where the fractional differential operator

$$D_t^\alpha T = \frac{\partial^\alpha}{\partial t^\alpha}$$

Apply $D_t^{-\alpha}$ on both side of equation (4.5) we get

$$T(x,y,t) = T(x,y,0) + D_t^{-\alpha}[f(x,y)T_{xx} + g(x,y)T_{yy}]$$

The Adomian's decomposition method assumes a series solution for $T(x,y,t)$ given by an infinite sum of components

$$T(x,y,t) = \sum_{n=0}^{\infty} T_n(x,y,t)$$

One can obtain

$$T_0(x,y,t) = T(x,y,0)$$

$$T_{n+1}(x,y,t) = D_t^{-\alpha}\left[(T_n^*)_{xx} + (T_n^*)_{yy}\right]$$

where $n = 0,1,2, ...,$

V. APPLICATIONS

Example 1: Consider the fractional heat equation in one-dimension

$$\frac{\partial^\alpha T}{\partial t^\alpha} - \frac{x^2}{2}\frac{\partial^2 T}{\partial x^2} = 0, \quad 0 < x < 1,\ 0 < \alpha \leq 1\ \ t > 0, \tag{5.1}$$

Subjected to the initial condition and boundary conditions

$$T(0,t) = 0, \qquad T(1,t) = e^t, \qquad T(x,0) = x^2 \tag{5.2}$$

By applying the new approximation T_n^*

$$T_n{}^*(x,t) = T_n(x,,t) + (1-x)[T(0,t) - T_n(0,t)] + x[T(1,t) - T_n(1,t)] \tag{5.3}$$

where $n = 0,1,2, ...,$ The initial approximation is

$$T_0(x,t) = T(x,0) = x^2 \tag{5.4}$$

The first approximation $T_0{}^*$

$$T_0{}^*(x,t) = x^2 + x(e^t - 1) \tag{5.5}$$

In an operator form equation (5.1) can be expressed as

$$D_t^\alpha T = \tfrac{1}{2}x^2 T_{xx} \tag{5.6}$$

Apply $D_t^{-\alpha}$ on both side of equation (5.6) we get

$$T(x,t) = x^2 + D_t^{-\alpha}\left[\frac{1}{2}x^2 T_{xx}\right]$$

$$T_0(x,t) = x^2$$

$$T_{n+1}(x,t) = D_t^{-\alpha}\left[\frac{1}{2}x^2 (T_n^*)_{xx}\right]$$

$$T_1(x,t) = D_t^{-\alpha}\left[\frac{1}{2}x^2 (T_0^*)_{xx}\right] = x^2\frac{t^\alpha}{\Gamma(1+\gamma)}$$

$$T_1{}^*(x,t) = x^2\frac{t^\alpha}{\Gamma(1+\alpha)} + x[e^t - \frac{1}{\Gamma(1+\alpha)}]$$

$$T_2(x,t) = D_t^{-\alpha}\left[\frac{1}{2}x^2 (T_1^*)_{xx}\right] = x^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

On continuing in this way, we get,

$$T(x,t) = x^2 + D_t^{-\alpha}\left[\frac{1}{2}x^2 T_{xx}\right]$$

$$T_n(x,t) = D_t^{-\alpha}\left[\frac{1}{2}x^2 (T_{n-1}^*)_{xx}\right] = x^2\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$$

$$T(x,t) = x^2 + x^2\frac{t^\alpha}{\Gamma(1+\gamma)} + x^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots$$

For $\alpha = 1$

$$T(x,t) = x^2 e^t$$

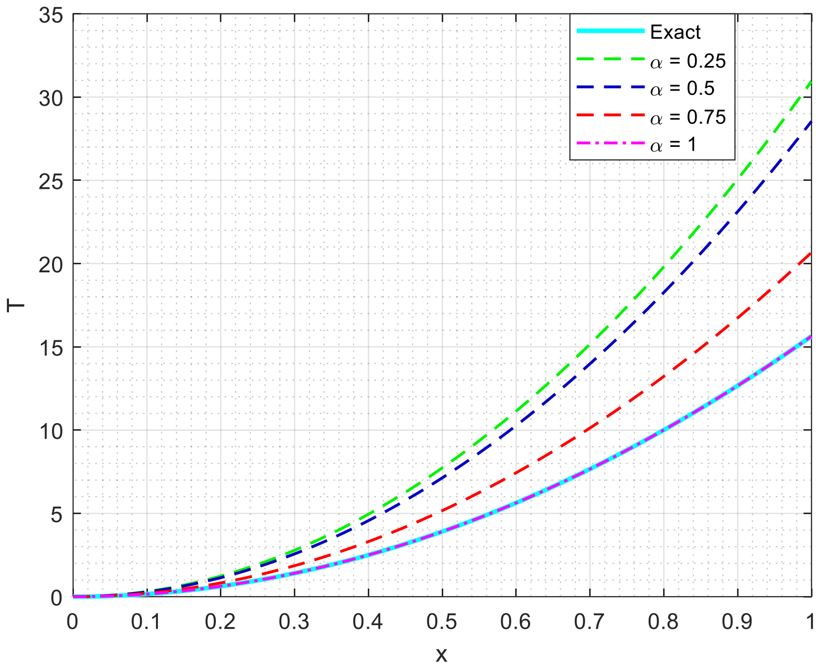

Figure 1: Comparison of exact solution of the Problem (5.1) with ADM results for $t = 5$, various values of α

Example 2: Consider the fractional heat equation in two-dimension

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \quad ,0 < x,y < 2\pi\,, 0 < \alpha \le 1\,, t > 0, \tag{5.7}$$

with the boundary conditions

$$T(0,y,t) = 0, \qquad T(2\pi,y,t) = 0,$$
$$T(x,0,t) = 0, \qquad T(x,2\pi,t) = 0 \tag{5.8}$$

and initial condition

$$T(x,y,0) = sinx\ siny \tag{5.9}$$

By applying new technique

$$T_n{}^*(x,y,t) = T_n(x,y,t) + (1-x)[T(0,y,t) - T_n(0,y,t)] + x[T(2\pi,y,t) - T_n(2\pi,y,t)]$$
$$+ (1-y)[T(x,0,t) - T_n(x,0,t)] + y[T(2\pi,y,t) - T_n(2\pi,y,t)]$$

where $n = 0,1,2,\ldots,$

The initial approximation is

$$T_0(x,y,t) = T(x,y,0) = sinx\ siny$$

The first approximation $T_0{}^*$

$$T_0{}^*(x,y,t) = sinx\ siny$$

In an operator form equation (5.7) can be expressed as

$$D_t^{\alpha} T = T_{xx} + T_{yy} \tag{5.10}$$

Apply $D_t^{-\alpha}$ on both side of equation (5.10) we get

$$T(x,y,t) = sinx\ siny + D_t^{-\alpha}[T_{xx} + T_{yy}]$$

$$T_0(x,y,t) = sinx\ siny$$

$$T_{n+1}(x,y,t) = D_t^{-\alpha}[(T_n^*)_{xx} + (T_n^*)_{yy}]$$

$$T_1(x,y,t) = D_t^{-\alpha}[(T_0^*)_{xx} + (T_0^*)_{yy}]$$

$$T_1(x,y,t) = D_t^{-\alpha}[-2sinx\ siny] = -2sinx\ siny \frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$T_1^*(x,y,t) = -2sinx\ siny\ \frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$T_2(x,y,t) = 4sinx\ siny\ \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

By continuing in this way, we obtain

$$T(x,y,t) = sinx\ siny - 2sinx\ siny \frac{t^{\alpha}}{\Gamma(1+\alpha)} + 4sinx\ siny \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots$$

For $\alpha = 1$,

$$T(x,y,t) = e^{-2t} sinx\ siny$$

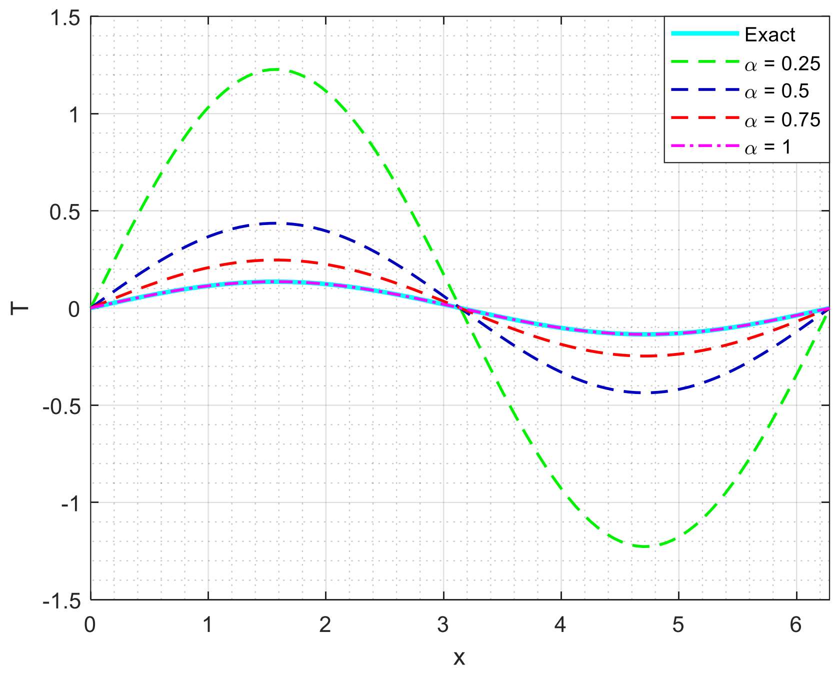

Figure 2: Comparison of exact solution of the Problem (5.7) with ADM results for $t = 3$, $x = \pi/2$ and various values of α

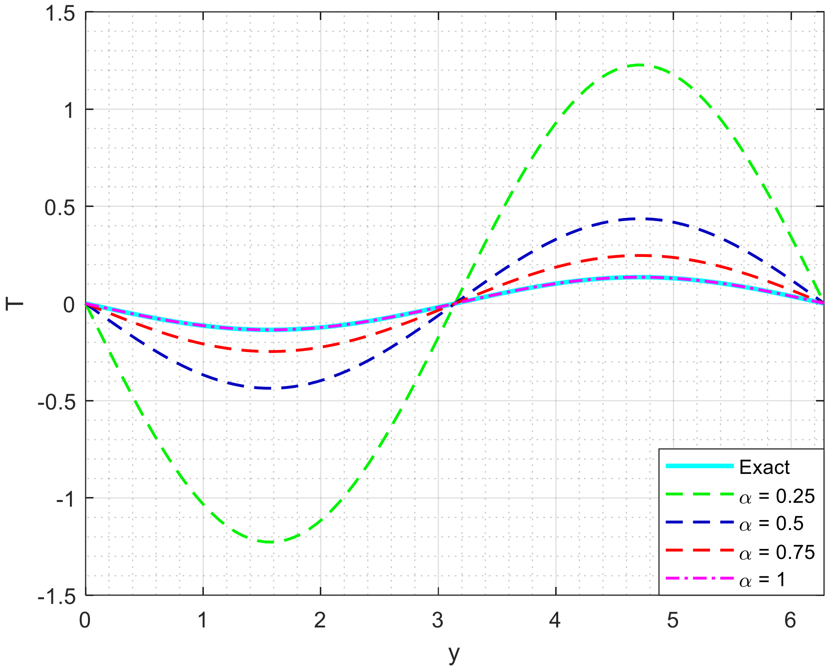

Figure 3: Comparison of exact solution of the Problem (5.7) with ADM results for $t = 3$, $y = 3\pi/2$ and various values of α

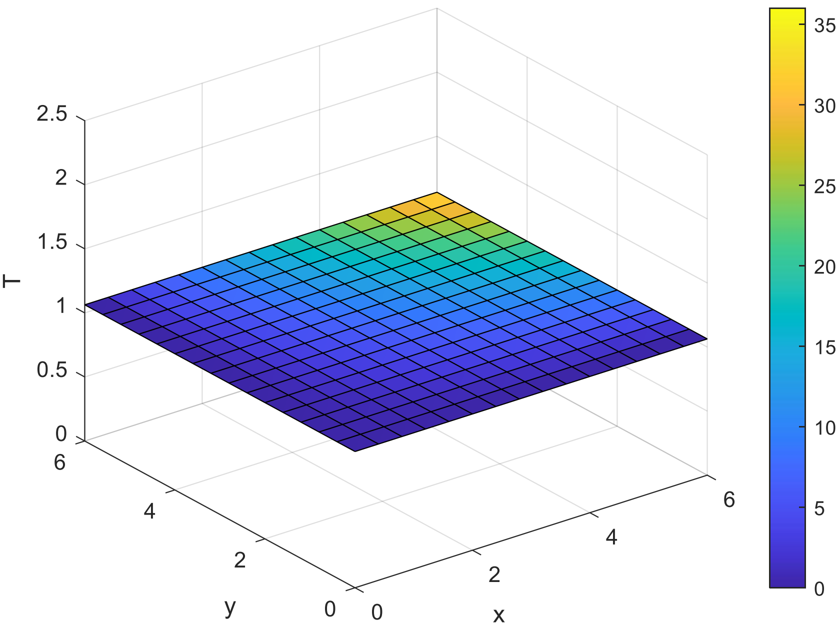

Figure 3:ADM results of the Problem (5.7) for $t = 3$

VI. CONCLUSION

In this article, we applied a modification of the Adomian Decomposition Method (ADM) to find an approximate solution to the fractional heat equation. This ADM to fractional-order heat equation gives a more realistic series solution that converges to the desired solution very quickly. It is worth noting that ADM is computationally less expensive and takes minimal time to process fractional-order PDEs. The procedure of ADM is accurate as observed in the comparison of approximate solution with the exact solutions of problems. This result suggests that it can be applied to other fractional-order boundary value problems.

REFERENCES

[1]. Abdul-Majid Wazwaz, A reliable modification of Adomian decomposition method, *Applied Mathematics and Computation*, 102, (1999), 77-86. DOI:10.1016/S0096-3003(98)10024-3
[2]. P. Igor, *Fractional differential equations*, New York: Academics Press, 1999.
[3]. A. M. Wazwaz, the decomposition method for solving higher dimensional initial boundary value problems of variable coefficients, *International Journal of Computer Mathematics*, 76, (2000), 159-172. DOI:10.1080/00207160008805017
[4]. Abdul-Majid Wazwaz and Salah M. El-Sayed, A new modification of the decomposition method for linear and nonlinear operators, *Applied Mathematics and Computation* 122, (2001), 393-405. DOI: 10.1016/S0096-3003(00)00060-6
[5]. Jafar Biazar, E. Babolian and R. Islam, Solution of the system of ordinary differential equations by Adomian decomposition method, *Journal of Computational and Applied Mathematics*, 147, (2004), 713–719. DOI: 10.1016/S0096-3003(02)00806-8
[6]. Cihat Arslanturk, A decomposition method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity, *International Communications in Heat and Mass Transfer*, 32, (2005), 831–841. DOI:10.1016/j.icheatmasstransfer.2004.10.006
[7]. S. Momani, "Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method", *Applied Mathematics and Computation*, 165, (2005), 459–472. DOI:10.1016/j.amc.2004.06.025
[8]. Yizheng Hu, Yong Luoa and Zhengyi Lua, Analytical solution of the linear fractional differential equation by Adomian decomposition method, *Journal of Computational and Applied Mathematics,* 215, (2008), 220 – 229. DOI: 10.1016/j.cam.2007.04.005
[9]. Jun-Sheng Duan, Randolph Rach, Dumitru Baleanu and Abdul-Majid Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, *Communications in Fractional Calculus*, 3 (2), (2012),73 – 99. https://www.researchgate.net/publication/256841837
[10]. Elaf Jaafar Ali, A new technique of initial boundary value problems using Adomian decomposition method, *International Mathematical Forum*, 7, (2012), 799-814. http://www.m-hikari.com/imf/imf-2012/17-20-2012/index.html
[11]. Pranab Kanti Roy, Hiranmoy Mondal and Ashis Mallick, A decomposition method for convective–radiative fin with heat generation, *Ain Shams Engineering Journal*, 6, (2015), 307-313. DOI: 10.1016/j.asej.2014.10.003
[12]. R. I. Nuruddeenand Bashir Danladi Garba, Analytical technique for (2+1) fractional diffusion equation with nonlocal boundary conditions,*Open Journal of Mathematical Science*, 2, (2018), 287-300. DOI: 10.30538/oms2018.0035
[13]. Rasool Shah, Hassan Khan, Saima Mustafa, Poom Kumam and Muhammad Arif, Analytical Solutions of Fractional-Order Diffusion Equations by Natural Transform Decomposition Method, *Entropy*, 21, (2019), 557. DOI:10.3390/e21060557
[14]. Haji Gul, Hussam Alrabaiah , Sajjad Ali, Kamal Shah and Shakoor Muhammad, Computation of solution to fractional order partial reaction diffusion equations, *Journal of Advanced Research*, 25, (2020), 31-38. DOI: 10.1016/j.jare.2020.04.021
[15]. R. K. Bairwal and Karan Singh, Analytical Solution of Time-Fractional Klien-Gordon Equation by using Laplace-Adomian Decomposition Method, *Annals of Pure and Applied Mathematics,*24, (2021), 27-35. DOI:10.22457/apam. v24n1a04836
[16]. Mukesh Kumar and Umesh, Recent Development of Adomian Decomposition Method for Ordinary and Partial Differential Equations, *International Journal of Applied and Computational Mathematics*, 8, (2022),81. DOI:10.1007/s40819-022-01285-6
[17]. Saadia Masood, Hajira, Hassan Khan *et al.,* A New Modified Technique of Adomian Decomposition Method for Fractional Diffusion Equations with Initial Boundary Condition, *Hindawi Journal of Function Spaces,* Volume 2022, DOI: https://doi.org/10.1155/2022/6890517
[18]. Razaq Adekola Oderinu, Johnson Adekunle Owolabi, Musilimu Taiwo, Approximate solutions of linear time-fractional differential equations, *Journal of Mathematics & computer Science, 29(2023), 60-72, DOI:*10.22436/jmcs.029.01.06