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Research Paper

The sharp convex mixed Lorentz-Sobolev Inequality Corresponding and Extending to more

Rawan Moawia(1) and Shawgy Hussein(2)
(1) *Sudan University of Science and Technology.*
(2) *Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.*

Abstract

We follow the smooth entities of the screen and intensive basic needed theory of [31] to show the sharp convex mixed Lorentz-Sobolev inequality, which is the functional version of an $L_{1+\epsilon}$ Minkowski inequality. The new sharp convex mixed Lorentz-Sobolev inequality implies the improve sharp convex Lorentz-Sobolev inequality of Ludwig, Xiao and Zhang.

Keywords: Lorentz-Sobolev inequality, Isoperimetric inequality, $L_{1+\epsilon}$ Minkowski problem, Petty projection inequality, $L_{1+\epsilon}$ mixed volume.

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I. Introduction

The well-known Sobolev inequality is one of the fundamental inequalities connecting analysis and geometry. The Sobolev inequality basically improved a number of mathematical branches such as the theory of partial differential equations, geometric measure theory, algebraic geometry, convex geometry, calculus of variations and other analytic areas. The Sobolev inequality was originated and developed from the classical isoperimetric inequality. Which determine a plane figure of the largest possible area whose boundary has a given length. The solution to the isoperimetric problem is given by a circle and shine in the 19th century. The generalized isoperimetric problem used to determine a geometric object of the largest possible volume whose boundary has a fixed surface area in the Euclidean space $\mathbb{R}^{1+2\epsilon}$.

For K be a compact convex set in $\mathbb{R}^{1+2\epsilon}$. Then the surface area $S(K)$ and volume $V(K)$ of K satisfy

$$S(K)^{1+2\epsilon} \geq (1+2\epsilon)^{1+2\epsilon}\omega_{1+2\epsilon}V(K)^{2\epsilon}, \tag{1.1}$$

where $\omega_{1+2\epsilon} = \frac{\pi^{\frac{1+2\epsilon}{2}}}{\Gamma\left(\frac{3+2\epsilon}{2}\right)}$ is the volume of the unit ball in $\mathbb{R}^{1+2\epsilon}$ and $\Gamma(\cdot)$ is the Gamma function. Equality holds in (1.1) if and only if K is a ball in $\mathbb{R}^{1+2\epsilon}$.

The isoperimetric inequality (1.1) for sufficiently smooth domains is equivalent to the Sobolev inequality with optimal constant,

$$(1+2\epsilon)^{-1}\omega_{1+2\epsilon}^{-\frac{1}{1+2\epsilon}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j |\nabla f_j(x)|dx \geq \left(\int_{\mathbb{R}^{1+2\epsilon}} \sum_j |f_j(x)|^{\frac{1+2\epsilon}{2\epsilon}} dx\right)^{\frac{2\epsilon}{1+2\epsilon}} \tag{1.2}$$

for all $f_j \in W^{1,1}(\mathbb{R}^{1+2\epsilon})$, the usual Sobolev space of real-valued functions of $\mathbb{R}^{1+2\epsilon}$ with L_1 partial derivatives.

The classical sharp $L_{1+\epsilon}$ Sobolev inequality states that (see [2,8,19,25]):

For $f_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$, be the set of smooth functions with compact support on $\mathbb{R}^{1+2\epsilon}$ and for $\epsilon > 0$, we have

$$\left(\int_{\mathbb{R}^{1+2\epsilon}} \sum_j |\nabla f_j(x)|^{1+\epsilon} dx\right)^{\frac{1}{1+\epsilon}} \geq c_{1+2\epsilon,1+\epsilon} \sum_j \|f_j\|_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}, \tag{1.3}$$

where $\sum_j |\nabla f_j(x)|$ is the Euclidean norm of the gradient of f_j, $\|\cdot\|_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}$ is the usual $L_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}$ norm of f_i on $\mathbb{R}^{1+2\epsilon}$, and

$$c_{1+2\epsilon,1+\epsilon} = (1+2\epsilon)^{\frac{1}{1+\epsilon}}\left[\omega_{1+2\epsilon}\,\Gamma\left(\frac{1+2\epsilon}{1+\epsilon}\right)\Gamma\left(\frac{1+2\epsilon+2\epsilon^2}{1+\epsilon}\right)/\Gamma(1+2\epsilon)\right]^{1/(1+2\epsilon)}.$$

The extremal functions for inequality (1.2) are the characteristic functions of balls and equality holds in (1.3) when $f_j(x)$ tends to $(a + (a+\epsilon)|x - x_0|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ with $\epsilon > 0$ and $x_0 \in \mathbb{R}^{1+2\epsilon}$.

So, the sharp $L_{1+\epsilon}$ Sobolev inequality has been extended in several important directions. [29] established the sharp *affine Sobolev-Zhang inequality* which is invariant under all affine transformations of $\mathbb{R}^{1+2\epsilon}$. The affine Sobolev-Zhang inequality is significantly stronger than the classical L_1 Sobolev inequality. Moreover, the affine Sobolev-Zhang inequality is equivalent to the generalized Petty projection inequality. The affine Sobolev-Zhang inequality is a cornerstone of affine geometric analysis. [17] extended the affine Sobolev-Zhang inequality to the $L_{1+\epsilon}$ case for $\epsilon > 0$, and they proved that the sharp affine $L_{1+\epsilon}$ Sobolev inequality is the functional version of the $L_{1+\epsilon}$ affine isoperimetric inequality in [14]. A new proof of the sharp affine Sobolev type inequalities was given in [11].

[9] proved the asymmetric affine $L_{1+\epsilon}$ Sobolev inequality that is stronger than the sharp affine $L_{1+\epsilon}$ Sobolev inequality. A general case of the affine $L_{1+\epsilon}$ Sobolev inequality which constitutes a bridge between the affine $L_{1+\epsilon}$ Sobolev inequality and the asymmetric affine $L_{1+\epsilon}$ Sobolev inequality was obtained in [28]. [5] proved the affine Morrey-Sobolev inequality. An asymmetric affine Pólya-Szegö principle was established by [10], and the equality cases and stability for the affine Pólya-Szegö principle were established in [27]. Moreover, Wang extended the affine L_1 Sobolev inequality to $BV(\mathbb{R}^{1+2\epsilon})$ in [26].

Another important development with respect to the original $L_{1+\epsilon}$ Sobolev inequality (1.3) is the following Lorentz-Sobolev inequality [1,12]:

If $f_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$ and $\epsilon \geq 0$, then

$$\left\|\sum_j \nabla f_j\right\|_{1+\epsilon}^{1+\epsilon} \geq (1+\epsilon)^{-(1+\epsilon)}(\epsilon)^\epsilon(1+2\epsilon)\omega_{1+2\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}}\int_0^\infty \sum_j V\left([f_j]_t\right)^{\frac{\epsilon}{1+2\epsilon}}\,dt^{1+\epsilon}, \quad (1.4)$$

where $[f_j]_t = \{x \in \mathbb{R}^{1+2\epsilon} : |f_j(x)| \geq t\}$ is the level set of f_j.

Using Lorentz integrals of the $L_{1+\epsilon}$ convexification of level sets (see Section 5) instead of level sets, [12] obtained the following sharp convex Lorentz-Sobolev inequality:

If $f_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$ and $\epsilon \geq 0$, then

$$\left\|\sum_j \nabla f_j\right\|_{1+\epsilon}^{1+\epsilon} \geq (1+2\epsilon)\omega_{1+2\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}}\int_0^\infty \sum_j V\left(\langle f_j\rangle_t\right)^{\frac{\epsilon}{1+2\epsilon}}\,dt\ . \quad (1.5)$$

Equality holds in (1.5) as f_j tends to the characteristic function of an origin-centered ball for $\epsilon \geq 0$ equality is attained when $f_j(x)$ tends to $(a + (a+\epsilon)|x|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$, with positive constants $a, (a+\epsilon)$.

[12] showed that the inequality (1.5) is the functional analogue of the following $L_{1+\epsilon}$ isoperimetric inequality of [13],

$$S_{1+\epsilon}(K) \geq (1+2\epsilon)\omega_{1+2\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}} V(K)^{\frac{\epsilon}{1+2\epsilon}}, \quad (1.6)$$

where $K \subset \mathbb{R}^{1+2\epsilon}$ is an origin-symmetric convex body and $S_{1+\epsilon}(K)$ is the $L_{1+\epsilon}$ surface area of K for $\epsilon \geq 0$.

The authors in [31] establish a new sharp convex mixed Lorentz-Sobolev inequality for the $L_{1+\epsilon}$ convexification of level sets and the $L_{1+\epsilon}$ projection body of the $L_{1+\epsilon}$ convexification of level sets.

Sharp convex mixed Lorentz-Sobolev inequality. *Let $f_j, g_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$. If $\epsilon \geq 0$, then*

$$\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |\nabla f_j(x)\cdot\nabla g_j(y)|^{1+\epsilon}dxdy$$

$$\geq \alpha_{1+2\epsilon,1+\epsilon}\int_0^\infty \sum_j V\left(\langle f_j\rangle_t\right)^{\frac{\epsilon}{1+2\epsilon}}dt\int_0^\infty V\left(\Pi_{1+\epsilon}\langle g_j\rangle_s\right)^{\frac{1+\epsilon}{1+2\epsilon}}ds, \quad (1.7)$$

where " $\cdot$ " denotes the standard inner product and $\alpha_{1+2\epsilon,1+\epsilon} = \frac{(1+2\epsilon)^2\omega_{1+2\epsilon}\omega_{3\epsilon}}{\omega_2\omega_{2\epsilon-1}\omega_\epsilon}$. Equality holds in (1.7) as f_j and g_j tend to characteristic functions of dilates of centered polar ellipsoids for $\epsilon \geq 0$ when $f_j(x)$ tends to $(a_1 + |\psi(x-x_0)|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ and $g_j(y)$ tends to $(a_2 + |\psi^{-t}(y+x_0)|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ with $a_i > 0 (i = 1,2), x_0 \in \mathbb{R}^{1+2\epsilon}$ and $\psi \in GL(1+2\epsilon)$.

We show that the sharp convex mixed Lorentz-Sobolev inequality (1.7) is the functional inequality corresponding to the following $L_{1+\epsilon}$ Minkowski inequality (see [31]):

Let $K, Q \subset \mathbb{R}^{1+2\epsilon}$ be origin-symmetric convex bodies. If $\epsilon \geq 0$, then

$$V_{1+\epsilon}(K, \Pi_{1+\epsilon}Q)^{1+2\epsilon} \geq V(K)^\epsilon V(\Pi_{1+\epsilon}Q)^{1+\epsilon}. \quad (1.8)$$

Here $V_{1+\epsilon}(\cdot,\cdot)$ denotes the $L_{1+\epsilon}$ mixed volume of convex bodies.

Note that the $L_{1+\epsilon}$ isoperimetric inequality (1.6) follows from (1.8) when Q is a ball and the sharp convex Lorentz-Sobolev inequality (1.5) implies the $L_{1+\epsilon}$ isoperimetric inequality (1.6). Motivated by these facts, we prove that the sharp mixed convex Lorentz-Sobolev inequality (1.7) implies the sharp convex Lorentz-Sobolev inequality (1.5), and hence also implies the $L_{1+\epsilon}$ Sobolev inequalities (1.2) and (1.3) (see [31]).

We collect some basic concepts and facts that will be useful in the sequel. We recall some facts on the $L_{1+\epsilon}$ Petty projection body, the $L_{1+\epsilon}$ Minkowski problem that are central tools in the proofs of the main theorems. We intrduce some results on $L_{1+\epsilon}$ John ellipsoids and prove a special $L_{1+\epsilon}$ Minkowski inequality which will be used latter.

2. Preliminaries

For $\mathcal{K}^{1+2\epsilon}$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space $\mathbb{R}^{1+2\epsilon}$. We write $\mathcal{K}_o^{1+2\epsilon}$ and $\mathcal{K}_s^{1+2\epsilon}$ for the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^{1+2\epsilon}$, respectively. We assumed that all convex bodies have the origin in their interiors. Let $V(K)$ denote the $(1+2\epsilon)$-dimensional volume of the convex body K. For $B = \{x \in \mathbb{R}^{1+2\epsilon}: |x| \le 1\}$ denote the standard unit ball in $\mathbb{R}^{1+2\epsilon}$, and its volume is denoted by $V(B) = \omega_{1+2\epsilon} = \frac{\pi^{\frac{1+2\epsilon}{2}}}{\Gamma\left(\frac{3+2\epsilon}{2}\right)}$. Let $S^{2\epsilon} = \{x \in \mathbb{R}^{1+2\epsilon}: |x| = 1\}$ denote the unit sphere in $\mathbb{R}^{1+2\epsilon}$.

We write $GL(1+2\epsilon)$ for the group of general linear transformations in $\mathbb{R}^{1+2\epsilon}$. For $\varphi \in GL(1+2\epsilon)$ write φ^t and φ^{-1} for the transpose and inverse of φ respectively, and φ^{-t} for the inverse of the transpose (contragradient) of φ, and let $\det\varphi$ denote the determinant of φ. Let $SL(1+2\epsilon) = \{\varphi : |det\varphi| = 1, \varphi \in GL(1+2\epsilon)\}$.

If $K \in \mathcal{K}^{1+2\epsilon}$, then its support function, $(h_j)_K(\cdot) = h_j(K,\cdot) : \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$, is defined by

$$h_j(K,x) = max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^{1+2\epsilon}.$$

A convex body K is uniquely determined by its support function $h_j(K,\cdot)$. It is obvious that for $\lambda > 0$, the support function of the convex body $\lambda K = \{\lambda x : x \in K\}$ satisfies

$$h_j(\lambda K,\cdot) = \lambda h_j(K,\cdot).$$

For real $\epsilon \ge 0, K, L \in \mathcal{K}_o^{1+2\epsilon}$ and real $\varepsilon > 0$, the Minkowski-Firey $L_{1+\epsilon}$ combination $K +_{1+\epsilon} \varepsilon \cdot L$ is the convex body whose support function is given by

$$h_j(K +_{1+\epsilon} \varepsilon \cdot L,\cdot)^{1+\epsilon} = h_j(K,\cdot)^{1+\epsilon} + \varepsilon h_j(L,\cdot)^{1+\epsilon}.$$

The $L_{1+\epsilon}$ mixed volume $V_{1+\epsilon}(K,L)$ of convex bodies K and L is defined by

$$V_{1+\epsilon}(K,L) = \frac{1+\epsilon}{1+2\epsilon} \lim_{\varepsilon\to 0^+} \frac{V(K +_{1+\epsilon} \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

The existence of this limit was proven in [13]. In particular,

$$V_{1+\epsilon}(K,K) = V(K) \tag{2.1}$$

for $K \in \mathcal{K}_o^{1+2\epsilon}$. By [13], there exists a unique finite positive Borel measure $S_{1+\epsilon}(K,\cdot)$ on $S^{2\epsilon}$ such that

$$V_{1+\epsilon}(K,L) = \frac{1}{1+2\epsilon} \int_{S^{2\epsilon}} \sum_j h_j(L,u_j)^{1+\epsilon} dS_{1+\epsilon}(K,u_j) , \tag{2.2}$$

for $L \in \mathcal{K}_o^{1+2\epsilon}$. The measure $S_{1+\epsilon}(K,\cdot)$ is called the $L_{1+\epsilon}$ surface area measure of K. The measure $S_1(K,\cdot) = S(K,\cdot) = S_K(\cdot)$ is the classical surface area measure of K. It was shown in [13] that the measure $S_{1+\epsilon}(K,\cdot)$ is absolutely continuous with respect to $S_K(\cdot)$ and the Radon-Nikodym derivative is

$$\frac{dS_{1+\epsilon}(K,\cdot)}{dS(K,\cdot)} = h_j(K,\cdot)^{-\epsilon}.$$

If the boundary ∂K of K is C^2 with positive curvature, then the Radon-Nikodym derivative of S_K with respect to the Lebesgue measure on $S^{2\epsilon}$is the reciprocal of the Gaussian curvature of ∂K (when viewed as a function of the outer normals of ∂K).

If $K, L \in \mathcal{K}_o^{1+2\epsilon}$, then

$$V_{1+\epsilon}(tK,L) = t^{\epsilon} V_{1+\epsilon}(K,L) \quad for \ t > 0, \tag{2.3}$$

$$V_{1+\epsilon}(K,tL) = t^{1+\epsilon} V_{1+\epsilon}(K,L) \quad for \ t > 0, \tag{2.4}$$

and

$$V_{1+\epsilon}(\varphi K,L) = V_{1+\epsilon}(K,\varphi^{-1}L) \qquad for \quad \varphi \in SL(1+2\epsilon). \tag{2.5}$$

The $L_{1+\epsilon}$ Minkowski inequality was proven in [13]:

If $K, L \in \mathcal{K}_o^{1+2\epsilon}$ and $\epsilon \ge 0$, then

$$V_{1+\epsilon}(K,L)^{1+2\epsilon} \ge V(K)^{\epsilon} V(L)^{1+\epsilon}, \tag{2.6}$$

with equality if and only if K and L are homothetic when $\epsilon = 0$, and K and L are dilates when $\epsilon > 0$.

A star body M is a compact set in $\mathbb{R}^{1+2\epsilon}$ which is star shaped with respect to the origin, i.e., if $x \in M$, then the line segment joining the origin to x is contained in M. The radial function, $\rho_M(\cdot) = \rho(M,\cdot) : \mathbb{R}^{1+2\epsilon}\backslash\{0\} \to \mathbb{R}$, of M is defined for $x \neq 0$ by

$$\rho(M,x) = max\{\lambda \geq 0 : \lambda x \in M\}.$$

The radial function is positively homogeneous of -1, that is

$$\rho(M,ax) = a^{-1}\rho(M,x), \quad a > 0.$$

Let M be a star body in $\mathbb{R}^{1+2\epsilon}$. Its polar body M^* is defined by

$$M^* = \{x \in \mathbb{R}^{1+2\epsilon} : x \cdot y \leq 1 \quad for \ all \quad y \in M\}.$$

Let $K \in \mathcal{K}_s^{1+2\epsilon}$. The well-known Blaschke-Santaló inequality [24] states that:

$$V(K)V(K^*) \leq \omega_{1+2\epsilon}^2, \tag{2.7}$$

with equality if and only if K is an ellipsoid.

In particular, if $K \in \mathcal{K}_o^{1+2\epsilon}$, then

$$\rho(K^*,\cdot) = \frac{1}{h_j(K,\cdot)}, \qquad h_j(K^*,\cdot) = \frac{1}{\rho(K,\cdot)}.$$

If $K, L \in \mathcal{K}_o^{1+2\epsilon}$ and $\lambda > 0$, then

$$K \subseteq \lambda L \Longleftrightarrow K^* \supseteq \frac{1}{\lambda}L^*, \tag{2.8}$$

and

$$K = \lambda L \Longleftrightarrow K^* = \frac{1}{\lambda}L^*. \tag{2.9}$$

We will frequently apply Federer's co-area formula [7]. We state a version that is sufficient for our purposes.

If $f_j: \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$ is locally Lipschitz and $g_j: \mathbb{R}^{1+2\epsilon} \to [0,\infty)$ is measurable, then, for any Borel set $A \subseteq \mathbb{R}$,

$$\int_{\sum_j f_j^{-1}(A)\cap\{\sum_j |\nabla f_j|>0\}} g_j(x)dx = \int_A \int_{\sum_j f_j^{-1}(y)} \sum_j \frac{g_j(x)}{|\nabla f_j(x)|} d\mathcal{H}^{2\epsilon}(x)dy, \tag{2.10}$$

where $\mathcal{H}^{2\epsilon}$ is the (2ϵ)-dimensional Hausdorff measure.

Let Aff$(1+2\epsilon)$ denote the group of invertible affine transformations of $\mathbb{R}^{1+2\epsilon}$, that is, every map $\Psi \in$Aff$(1+2\epsilon)$ is a general linear transformation followed by a translation. There is a natural left action of $\mathbb{R}\backslash\{0\} \times$ Aff$(1+2\epsilon)$ on functions $f_j: \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$, given by

$$f_j \mapsto kf_j \circ \Psi^{-1}$$

for each (k, Ψ) in $\mathbb{R}\backslash\{0\} \times$ Aff$(1+2\epsilon)$. An inequality $L[f_j] \leq R[f_j]$ for a class of functions $\mathbb{R}^{1+2\epsilon} \to \mathbb{R}$ is called affine if

$$\frac{L[kf_j \circ \Psi^{-1}]}{R[kf_j \circ \Psi^{-1}]} = \frac{L[f_j]}{R[f_j]} \tag{2.11}$$

for each (k, Ψ) $\in \mathbb{R}\backslash\{0\} \times$ Aff$(1+2\epsilon)$.

3. $L_{1+\epsilon}$ projection body and $L_{1+\epsilon}$ Minkowski problem

3.1. $L_{1+\epsilon}$ projection body

The classical projection body was introduced by Minkowski. For $K \in \mathcal{K}^{1+2\epsilon}$, the classical projection body ΠK of K is defined as the origin-symmetric convex body in $\mathbb{R}^{1+2\epsilon}$ with support function:

$$h_j(\Pi K, u_j) = V_{2\epsilon}(K|u_j^{\perp}), \quad u_j \in S^{2\epsilon}. \tag{3.1}$$

Here $V_{2\epsilon}(K|u_j^{\perp})$ is the 2ϵ dimensional volume of K projected to the hyperplane passing through the origin with the normal direction u_j.

Interest in projection bodies see [4], [20] and [23]. The fundamental inequality for projection bodies is the following Petty projection inequality [21]:

If $K \in \mathcal{K}^{1+2\epsilon}$, then

$$V(K)^{2\epsilon} V(\Pi^* K) \leq \left(\frac{\omega_{1+2\epsilon}}{\omega_{2\epsilon}}\right)^{1+2\epsilon}, \tag{3.2}$$

with equality if and only if K is an ellipsoid. Here $\Pi^* K$ denotes the polar body of the projection body ΠK.

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The Petty projection inequality (3.2) has been studied widely. In particular, Lutwak, Yang and Zhang [14] extended the Petty projection inequality (3.2) to the $L_{1+\epsilon}$-projection body. They defined the $L_{1+\epsilon}$-projection body as follows:

For $K \in \mathcal{K}_o^{1+2\epsilon}$ and $\epsilon \geq 0$, the $L_{1+\epsilon}$-projection body $\Pi_{1+\epsilon}K$ of K is the origin symmetric convex body with support function $h_j(\Pi_{1+\epsilon}K,\cdot) : \mathbb{R}^{1+2\epsilon} \to (0,\infty)$,

$$\sum_j h_j(\Pi_{1+\epsilon}K,u_j) = \left(\frac{1}{(1+2\epsilon)\omega_{1+2\epsilon}\tilde{c}_{1+2\epsilon,1+\epsilon}}\int_{S^{2\epsilon}}\sum_j |u_j\cdot v_j|^{1+\epsilon} dS_{1+\epsilon}(K,v_j)\right)^{\frac{1}{1+\epsilon}}, u_j \in S^{2\epsilon}, \quad (3.3)$$

where

$$\tilde{c}_{1+2\epsilon,1+\epsilon} = \frac{\omega_{3\epsilon}}{\omega_2\omega_{2\epsilon-1}\omega_\epsilon}. \quad (3.4)$$

The constant $\tilde{c}_{1+2\epsilon,1+\epsilon}$ is chosen such that

$$\Pi_{1+\epsilon}B = B. \quad (3.5)$$

For $\lambda > 0$ and $K \in \mathcal{K}_o^{1+2\epsilon}$, one has

$$\Pi_{1+\epsilon}(\lambda K) = \lambda^{\frac{\epsilon}{1+\epsilon}}\Pi_{1+\epsilon}K, \quad (3.6)$$

and

$$\Pi_{1+\epsilon}(\varphi K) = \varphi^{-t}\Pi_{1+\epsilon}K \quad for \quad \varphi \in SL(1+2\epsilon). \quad (3.7)$$

3.2. The $L_{1+\epsilon}$ Minkowski problem

The Minkowski problem is a central problem in integral geometry, convex geometric analysis and PDE. The classical Minkowski problem asks for necessary and sufficient conditions for a Borel measure on the unit sphere to be the surface area measure of a convex body in $\mathbb{R}^{1+2\epsilon}$. The classical Minkowski problem was solved by Minkowski when the given measure is either discrete or has a smooth density. It was extended to arbitrary measures independently by Alexandrov, and Fenchel and Jessen. The solution to the Minkowski problem states:

For each Borel measure μ on $S^{2\epsilon}$ that is not supported on a great hemisphere, there exists a unique (up to translation) convex body K so that

$$S(K,\cdot) = \mu$$

if and only if

$$\int_{S^{2\epsilon}}\sum_j v_j d\mu(v_j) = 0.$$

[13] extended the Minkowski problem to the $L_{1+\epsilon}$ version, which is a central problem in the $L_{1+\epsilon}$ Brunn-Minkowski theory.

$L_{1+\epsilon}$ Minkowski problem. For $K \in \mathcal{K}^{1+2\epsilon}$, find necessary and sufficient conditions for a finite Borel measure μ on the unit sphere $S^{2\epsilon}$ so that μ is the $L_{1+\epsilon}$-surface area measure of convex body K.

A Borel measure on $S^{2\epsilon}$ is even if for each Borel set $\omega \subset S^{2\epsilon}$ the measure of ω and the measure of $-\omega = \{-x : x \in \omega\}$ are equal. In [13], the following solution to the even case of the $L_{1+\epsilon}$ Minkowski problem was given (see [31]):

Theorem 3.1. *Suppose μ is an even positive measure on $S^{2\epsilon}$ that is not supported on a great hypersphere of $S^{2\epsilon}$. Then for real $\epsilon \geq 0$ such that $\epsilon \neq 0$ there exists a unique origin-symmetric convex body K in $\mathbb{R}^{1+2\epsilon}$ whose $L_{1+\epsilon}$ surface area measure is μ; that is,*

$$\mu = S_{1+\epsilon}(K,\cdot). \quad (3.8)$$

4. $L_{1+\epsilon}$ John ellipsoids

Ellipsoids are important objects in the Brunn-Minkowski theory and the dual-Brunn-Minkowski theory. The celebrated *John ellipsoid*, associated with each convex body K, is the unique ellipsoid JK of maximal volume contained in K. Two important results concerning the John ellipsoid are John's inclusion and Ball's volume-ratio inequality [3]. The John ellipsoid is within the classical Brunn-Minkowski theory and is extremely useful in both convex and Banach space geometry [3,22].

The authors in [15] introduced *the LYZ ellipsoid* $\Gamma_{-2}K$ whose radial function is defined by

$$\sum_j \rho_{\Gamma_{-2}K}(u_j)^{-2} = \frac{1}{V(K)}\int_{S^{2\epsilon}}\sum_j |u_j\cdot v_j|^2 dS_2(K,v_j),$$

for $K \in \mathcal{K}_o^{1+2\epsilon}$ and $u_j \in S^{2\epsilon}$.

It was proved in [15] that the properties of the LYZ ellipsoid are analogous to that of the John ellipsoid, such as the volume of the LYZ ellipsoid is dominated by the volume of K, there is an inclusion identical to John's inclusion, and a version of Ball's volume-ratio inequality for the LYZ ellipsoid also holds. Unlike the John ellipsoid, there is an analytic formulation for the LYZ ellipsoid. The domain of the LYZ ellipsoid was extended to star-shaped sets in [16]. The LYZ ellipsoid for log-concave functions was introduced in [6].

The $L_{1+\epsilon}$ curvature $(f_j)_{1+\epsilon}(K,\cdot)$ of a smooth convex body K is an important notion in convex geometry:

$$(f_j)_{1+\epsilon}(K,\cdot) = (h_j)_K^{-\epsilon} f_j(K,\cdot).$$

Here $f_j(K,\cdot)$ denotes the reciprocal of the Gaussian curvature of ∂K (when viewed as a function of the outer normals of ∂K). It is the core of the integral formula of $L_{1+\epsilon}$ mixed volume and related inequalities. In [18], the authors studied minimizing the total $L_{1+\epsilon}$-curvature of a convex body under SL$(1+2\epsilon)$-transformations.

For a smooth convex body $K \in \mathcal{K}_o^{1+2\epsilon}$, and a fixed real $\epsilon \geq 0$, find

$$\min_{\varphi \in SL(1+2\epsilon)} \int_{S^{2\epsilon}} \sum_j (f_j)_{1+\epsilon}(\varphi K, u_j) dS(u_j) .$$

By using the integral formula of $L_{1+\epsilon}$ mixed volume $V_{1+\epsilon}(K,E)$ of a convex body K and an origin-centered ellipsoid E, minimizing the total $L_{1+\epsilon}$-curvature can be formulated in the following equivalent ways [18]:

Problem $S_{1+\epsilon}$.Given a convex body $K \in \mathcal{K}_o^{1+2\epsilon}$, find an ellipsoid E, amongst all origin-centered ellipsoids, which solves the following constrained maximization problem:

$$\max \left(\frac{V(E)}{\omega_{1+2\epsilon}}\right)^{\frac{1}{1+2\epsilon}} \quad \text{subject to} \quad \left(\frac{V_{1+\epsilon}(K,E)}{V(K)}\right)^{\frac{1}{1+\epsilon}} \leq 1. \tag{4.1}$$

A maximal ellipsoid is called an $S_{1+\epsilon}$ solution for K.

The following problem is dual to $S_{1+\epsilon}$:

Problem $\overline{S_{1+\epsilon}}$ .Given a convex body $K \in \mathcal{K}_o^{1+2\epsilon}$, find an ellipsoid E, amongst all origin-centered ellipsoids, which solves the following constrained minimization problem

$$\min \left(\frac{V_{1+\epsilon}(K,E)}{V(K)}\right)^{\frac{1}{1+\epsilon}} \quad \text{subject to} \quad \left(\frac{V(E)}{\omega_{1+2\epsilon}}\right)^{\frac{1}{1+2\epsilon}} \geq 1. \tag{4.2}$$

A minimal ellipsoid is called an $\bar{S}_{1+\epsilon}$ solution for K.

The existence of solutions of problem $S_{1+\epsilon}$ and problem $\bar{S}_{1+\epsilon}$ was given by Lutwak, Yang and Zhang.

Theorem 4.1. ([18, Theorem 2.2]). *Suppose real $\epsilon \geq 0$ and $K \in \mathcal{K}_o^{1+2\epsilon}$. Then $S_{1+\epsilon}$ as well as $\bar{S}_{1+\epsilon}$ has a unique solution.*

By Theorem 4.1, Lutwak, Yang and Zhang define $L_{1+\epsilon}$ John ellipsoids [18]:

Definition 4.1. Suppose $K \in \mathcal{K}_o^{1+2\epsilon}$ and $0 \leq \epsilon \leq \infty$. Amongst all origin-centered ellipsoids, the unique ellipsoid that solves the constrained maximization problem

$$\max \left(\frac{V(E)}{\omega_{1+2\epsilon}}\right)^{\frac{1}{1+2\epsilon}} \quad \text{subject to} \quad \left(\frac{V_{1+\epsilon}(K,E)}{V(K)}\right)^{\frac{1}{1+\epsilon}} \leq 1. \tag{4.3}$$

is called the $L_{1+\epsilon}$ John ellipsoid of K and is denoted by $E_{1+\epsilon}K$. Amongst all origin-centered ellipsoids, the unique ellipsoid that solves the constrained minimization problem

$$\min \left(\frac{V_{1+\epsilon}(K,E)}{V(K)}\right)^{\frac{1}{1+\epsilon}} \quad \text{subject to} \quad \left(\frac{V(E)}{\omega_{1+2\epsilon}}\right)^{\frac{1}{1+2\epsilon}} \geq 1. \tag{4.4}$$

is called the normalized $L_{1+\epsilon}$ John ellipsoid of K and is denoted by $\bar{E}_{1+\epsilon}K$.

$L_{1+\epsilon}$ John ellipsoids provide a unified treatment for several fundamental objects in convex geometry. If the John point of K, the center of John ellipsoid JK, is at the origin, then $E_\infty K$ is precisely the classical John ellipsoid JK. The L_2 John ellipsoid E_2K is the LYZ ellipsoid $\Gamma_{-2}K$. The L_1 ellipsoid E_1K is the Petty ellipsoid.

If $K \in \mathcal{K}_o^{1+2\epsilon}$ and $0 \leq \epsilon \leq \infty$, then for $\varphi \in GL(1+2\epsilon)$,

$$E_{1+\epsilon}\varphi K = \varphi E_{1+\epsilon}K. \tag{4.5}$$

This means that if E is an ellipsoid centered at the origin, then

$$E_{1+\epsilon}E = E. \tag{4.6}$$

If $K \in \mathcal{K}_o^{1+2\epsilon}$ and real $\epsilon \geq 0$, the star body $\Gamma_{-(1+\epsilon)}K$ is defined as the body whose radial function, for $u_j \in S^{2\epsilon}$ is given by [18]:

$$\sum_j \rho_{\Gamma_{-(1+\epsilon)}K}(u_j)^{-(1+\epsilon)} = \frac{1}{V(K)} \int_{S^{2\epsilon}} \sum_j |u_j \cdot v_j|^{1+\epsilon} dS_{1+\epsilon}(K, v_j) . \tag{4.7}$$

We will need the following results (see [31]):

Lemma 4.1. ([18, Corollary 4.5]). *If* $K \in \mathcal{K}_o^{1+2\epsilon}$, *then*

$$E_{1+\epsilon}K \supseteq \Gamma_{-(1+\epsilon)}K \supseteq (1+2\epsilon)^{\frac{\epsilon-1}{1+\epsilon}} E_{2+\epsilon}K \quad \text{when} \quad 0 \leq \epsilon \leq 1,$$

$$E_{2+\epsilon}K \subseteq \Gamma_{-(2+\epsilon)}K \subseteq (1+2\epsilon)^{\frac{\epsilon}{2+\epsilon}} E_{2+\epsilon}K \quad \text{when} \quad 0 \leq \epsilon \leq \infty.$$

Lemma 4.2. ([18, Theorem 5.2]). *If* $K \in \mathcal{K}_o^{1+2\epsilon}$ *and* $0 \leq \epsilon \leq \infty$, *then*

$$V(E_{1+\epsilon}K) \leq V(K),$$

with equality for $\epsilon > 0$ *if and only if* K *is an ellipsoid centered at the origin, and with equality for* $\epsilon = 0$ *if and only if* K *is an ellipsoid.*

We can prove the following $L_{1+\epsilon}$ Minkowski inequalities, where $\tilde{c}_{1+2\epsilon,1+\epsilon}$ is the constant defined in (3.4).

Lemma 4.3 (see [31]). *Let* $K, L \in \mathcal{K}_o^{1+2\epsilon}$.

(i) *When* $0 \leq \epsilon \leq 1$, *then*

$$V_{1+\epsilon}(K, \Pi_{1+\epsilon}L) \geq \frac{\omega_{1+2\epsilon}^{\frac{1}{1+2\epsilon}}}{(1+2\epsilon)\tilde{c}_{1+2\epsilon,1+\epsilon}} V(K)^{\frac{\epsilon}{1+2\epsilon}} V(L)^{\frac{\epsilon}{1+2\epsilon}}, \tag{4.8}$$

with equality when $\epsilon = 1$ *and* K *and* L *are dilates of polar ellipsoids centered at the origin.*

(ii) *When* $0 \leq \epsilon \leq \infty$, *then*

$$V_{2+\epsilon}(K, \Pi_{2+\epsilon}L) \geq \frac{\omega_{1+2\epsilon}^{\frac{1}{1+2\epsilon}}}{(1+2\epsilon)^{\frac{2+\epsilon}{2}} \tilde{c}_{1+2\epsilon,2+\epsilon}} V(K)^{\frac{\epsilon-1}{1+2\epsilon}} V(L)^{\frac{\epsilon-1}{1+2\epsilon}}, \tag{4.9}$$

with equality when $\epsilon = 0$ *and* K *and* L *are dilates of polar ellipsoids centered at the origin.*

Proof. Let $\Gamma^*_{-(2+\epsilon)}L$ denote the polar body of $\Gamma_{-(2+\epsilon)}L$. By (3.3) and (4.7), we have

$$\Gamma^*_{-(2+\epsilon)}L = \left(\frac{(1+2\epsilon)\tilde{c}_{1+2\epsilon,2+\epsilon}\omega_{1+2\epsilon}}{V(K)}\right)^{\frac{1}{2+\epsilon}} \Pi_{2+\epsilon}L. \tag{4.10}$$

(i) For $0 \leq \epsilon \leq 1$, by (2.9), (4.10) and Lemma4.1, we obtain

$$\left(\frac{(1+2\epsilon)\tilde{c}_{1+2\epsilon,1+\epsilon}\omega_{1+2\epsilon}}{V(L)}\right)^{-\frac{1}{1+\epsilon}} \Pi^*_{1+\epsilon}L = \Gamma_{-(1+\epsilon)}L \subseteq E_{1+\epsilon}L.$$

Then by (2.8) we have

$$\left(\frac{(1+2\epsilon)\tilde{c}_{1+2\epsilon,1+\epsilon}\omega_{1+2\epsilon}}{V(L)}\right)^{\frac{1}{1+\epsilon}} \Pi_{1+\epsilon}L \supseteq E^*_{1+\epsilon}L. \tag{4.11}$$

By (2.2), (4.11), the Blaschke-Santaló inequality (2.7) and Lemma4.2, we have

$$\begin{aligned} V_{1+\epsilon}(K, \Pi_{1+\epsilon}L) &\geq \frac{V(L)}{(1+2\epsilon)\tilde{c}_{1+2\epsilon,1+\epsilon}\omega_{1+2\epsilon}} V_{1+\epsilon}(K, E^*_{1+\epsilon}L) \\ &\geq \frac{V(L)}{(1+2\epsilon)\tilde{c}_{1+2\epsilon,1+\epsilon}\omega_{1+2\epsilon}} V(K)^{\frac{\epsilon}{1+2\epsilon}} V(E^*_{1+\epsilon}L)^{\frac{1+\epsilon}{1+2\epsilon}} \\ &= \frac{V(L)\omega_{1+2\epsilon}^{\frac{1}{1+2\epsilon}}}{(1+2\epsilon)\tilde{c}_{1+2\epsilon,1+\epsilon}} V(K)^{\frac{\epsilon}{1+2\epsilon}} V(E_{1+\epsilon}L)^{-\frac{1+\epsilon}{1+2\epsilon}} \\ &\geq \frac{\omega_{1+2\epsilon}^{\frac{1}{1+2\epsilon}}}{(1+2\epsilon)\tilde{c}_{1+2\epsilon,1+\epsilon}} V(K)^{\frac{\epsilon}{1+2\epsilon}} V(L)^{\frac{\epsilon}{1+2\epsilon}}. \end{aligned} \tag{4.12}$$

When $\epsilon = 1, \Gamma_{-2}L = E_2L$, there is an equality in (4.11). By the equality condition of $L_{1+\epsilon}$ Minkowski inequality (2.6) and Lemma 4.2, we conclude that equality holds in (4.8) when $\epsilon = 0$ and K and L are dilates of polar ellipsoids centered at the origin.

(ii) For $\epsilon \geq 0$, by (2.9), (4.10) and Lemma4.1, we have

$$\left(\frac{(1+2\epsilon)\tilde{c}_{1+2\epsilon,2+\epsilon}\omega_{1+2\epsilon}}{V(L)}\right)^{-\frac{1}{2+\epsilon}} \Pi^*_{2+\epsilon}L = \Gamma_{-(2+\epsilon)}L \subseteq (1+2\epsilon)^{\frac{\epsilon}{2+\epsilon}} E_{2+\epsilon}L,$$

which implies

$$(1+2\epsilon)^{\frac{\epsilon}{2+\epsilon}} \left(\frac{(1+2\epsilon)\tilde{c}_{1+2\epsilon,2+\epsilon}\omega_{1+2\epsilon}}{V(L)}\right)^{\frac{1}{2+\epsilon}} \Pi_{2+\epsilon}L \supseteq E^*_{2+\epsilon}L. \tag{4.13}$$

By (2.2), (4.13), the Blaschke-Santaló inequality (2.7) and Lemma 4.2, we have

$$\begin{aligned} V_{2+\epsilon}(K, \Pi_{2+\epsilon}L) &\geq \frac{V(L)}{(1+2\epsilon)^{\frac{2+\epsilon}{2}}\tilde{c}_{1+2\epsilon,2+\epsilon}\omega_{1+2\epsilon}} V_{2+\epsilon}(K, E^*_{2+\epsilon}L) \\ &\geq \frac{V(L)}{(1+2\epsilon)^{\frac{2+\epsilon}{2}}\tilde{c}_{1+2\epsilon,2+\epsilon}\omega_{1+2\epsilon}} V(K)^{\frac{\epsilon-1}{1+2\epsilon}} V(E^*_{2+\epsilon}L)^{\frac{2+\epsilon}{1+2\epsilon}} \\ &= \frac{V(L)\omega_{1+2\epsilon}^{\frac{3+\epsilon}{1+2\epsilon}}}{(1+2\epsilon)^{\frac{2+\epsilon}{2}}\tilde{c}_{1+2\epsilon,2+\epsilon}} V(K)^{\frac{\epsilon-1}{1+2\epsilon}} V(E_{2+\epsilon}L)^{-\frac{2+\epsilon}{1+2\epsilon}} \\ &\geq \frac{\omega_{1+2\epsilon}^{\frac{3+\epsilon}{1+2\epsilon}}}{(1+2\epsilon)^{\frac{2+\epsilon}{2}}\tilde{c}_{1+2\epsilon,2+\epsilon}} V(K)^{\frac{\epsilon-1}{1+2\epsilon}} V(L)^{\frac{\epsilon-1}{1+2\epsilon}}. \end{aligned}$$

When $\epsilon = 0$, since $\Gamma_{-2}L = E_2L$, then equality holds in (4.13). By the equality conditions of the $L_{2+\epsilon}$ Minkowski inequality (2.6) and Lemma 4.2, equality holds in (4.9) when $\epsilon = 0$ and K and L are dilates of polar ellipsoids centered at the origin.

5. Sharp convex mixed Lorentz-Sobolev inequality

The $L_{1+\epsilon}$ convexification was introduced in [29] for $\epsilon = 0$ and in [12] for $\epsilon > 0$. Refer to [5] and [17] for more detailed information on $L_{1+\epsilon}$ convexification.

Given any measurable function $f_j: \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$, the level set $[f_j]_t$ of f_j is defined by:

$$[f_j]_t = \{x \in \mathbb{R}^{1+2\epsilon}: |f_j(x)| \geq t\}, \qquad t > 0. \tag{5.1}$$

In this paper, we always assume that all functions are such that the level sets $[f_j]_t$ are compact for all $t > 0$.

Assume $\epsilon \geq 0$. Suppose $f_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$, by Sard's Lemma, for a.e. $t > 0$,

$$\{|f_j| > t\} \text{ is a bounded open set with a } C^1 \text{ boundary,} \tag{5.2}$$

$$\partial\{|f_j| > t\} = \{|f_j| = t\}, \tag{5.3}$$

and

$$\nabla f_j(x) \neq 0, \quad for \quad x \in \{|f_j| = t\}. \tag{5.4}$$

Suppose $t > 0$ and f_j satisfies (5.2), (5.3) and (5.4). Let $\lambda_t(f_j,\cdot)$ be the even positive Borel measure on $S^{2\epsilon}$ such that

$$\int_{S^{2\epsilon}} \sum_j \varphi(v_j) d\lambda_t(f_j, v_j) = \int_{\{\sum_j |f_j|=t\}} \sum_j \varphi\left(v_j(x)\right) |\nabla f_j(x)|^{\epsilon} d\mathcal{H}^{2\epsilon}(x) \tag{5.5}$$

for every even Borel function $\varphi : S^{2\epsilon} \to \mathbb{R}$, where $v_j(x) = \frac{\nabla f_j(x)}{|\nabla f_j(x)|}$. Since for fixed $u_j \in S^{2\epsilon}$,

$$\mathcal{H}^{2\epsilon}(\{x : |f_j(x)| = t \text{ and } u_j \cdot v_j(x) \neq 0\}) > 0,$$

we have

$$\int_{\{\sum_j |f_j|=t\}} \sum_j |u_j \cdot v_j(x)| |\nabla f_j(x)|^{\epsilon} d\mathcal{H}^{2\epsilon}(x) > 0. \tag{5.6}$$

Hence

$$\int_{S^{2\epsilon}} \sum_j |u_j \cdot v_j| d\lambda_t(f_j, v_j) > 0, \tag{5.7}$$

for $u_j \in S^{2\epsilon}$. Hence, the measure $\lambda_t(f_j,\cdot)$ is not supported in the intersection of $S^{2\epsilon}$ with any subspace. By the solution to the even $L_{1+\epsilon}$ Minkowski problem (Theorem 3.1), there exists a unique origin-symmetric convex body $\langle f_j \rangle_t$ such that

$$S(\langle f_j \rangle_t, \cdot) h_j(\langle f_j \rangle_t, \cdot)^{-\epsilon} = \lambda_t(f_j, \cdot). \tag{5.8}$$

We remark that the convex body $\langle f_j \rangle_t$ is called the $L_{1+\epsilon}$ *convexification* of the level set $[f_j]_t$. For our aims, some properties of $\langle f_j \rangle_t$ will be listed.

Lemma 5.1. ([12, Lemma 8]). *If $K \in \mathcal{K}_s^{1+2\epsilon}$ and $f_j(x) = \phi(1/\rho_K(x))$, where $\phi \in C^1(0, \infty)$ is strictly decreasing, then for $t > 0$ and $\epsilon \geq 0$, the convex bodies of the $L_{1+\epsilon}$ convexification of the level sets of f_j are dilates of K, that is*

$$\langle f_j \rangle_t = c_{1+\epsilon}(t) K,$$

and $c_{1+\epsilon}(t)^{\epsilon} = |\phi'(s)|^{\epsilon} s^{2\epsilon}$, where $t = \phi(s)$.

Lemma 5.2. ([12, Lemma 13]). *Let $\epsilon \geq 0$. If $f_j \in C_0^{\infty}(\mathbb{R}^{1+2\epsilon})$ and $\nabla f_j(x) \neq 0$ on $\partial [f_j]_t$ for $t > 0$, then*

$$\langle k f_j \circ \Psi^{-1} \rangle_t = k^{\frac{1+\epsilon}{\epsilon}} \psi \langle f_j \rangle_t,$$

for $k > 0$ and $\Psi \in$Aff$(1 + 2\epsilon)$ given by $\Psi(x) = \psi x + y$ where $\psi \in GL(1 + 2\epsilon)$ and $y \in \mathbb{R}^{1+2\epsilon}$.

The next result has been proved in [17].

Lemma 5.3. *Let $f_j \in C_0^{\infty}(\mathbb{R}^{1+2\epsilon})$. If $\epsilon \geq 0$, then*

$$\int_0^{\infty} \sum_j V(\langle f_j \rangle_t)^{\frac{\epsilon}{1+2\epsilon}} dt \geq (1 + 2\epsilon)^{-\frac{\epsilon}{1+2\epsilon}} \hat{c}_{1+2\epsilon, 1+\epsilon}^{1+\epsilon} \sum_j \|f_j\|_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}^{1+\epsilon}. \tag{5.9}$$

Here

$$\hat{c}_{1+2\epsilon,1+\epsilon} = ((1 + 2\epsilon)\omega_{1+2\epsilon})^{-\frac{1}{1+2\epsilon}} c_{1+2\epsilon,1+\epsilon}.$$

The following results will be used later.

Lemma 5.4 (see [31]). *Let* $f_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$.

(i) *If* f_j *tends to the characteristic function* χ_{aB} *of the ball* aB *with* $a > 0$, *then* $\langle f_j \rangle_t$ *converges to* aB *(up to translation) in Hausdorff metric when* $0 < t < 1$, *and the surface area measure of* $\langle f_j \rangle_t$ *converges weakly to zero when* $t \geq 1$.

(ii) *If* $\epsilon > 0$ *and* $f_j(x) = \left(a + (a+\epsilon)|x|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{\epsilon}{1+\epsilon}}$ *with* $\epsilon > 0$, *then*

$$\langle f_j \rangle_t = \alpha(t)B, \tag{5.10}$$

where

$$\alpha(t) = \left(\frac{1 - at^{\frac{1+\epsilon}{\epsilon}}}{(a+\epsilon)^{\frac{\epsilon}{1+2\epsilon}}}\right)^{\frac{1+2\epsilon}{1+\epsilon}},$$

and $t \in (0, +\infty)$ *such that* $\alpha(t)$ *is meaningful depending on* a *and* $(a+\epsilon)$.

Proof. (i) We will prove this statement by approximation argument in Zhang [29]. We set

$$(f_j)_\varepsilon(x) = \begin{cases} 0 & dist(x, aB) \geq \varepsilon, \\ 1 - \dfrac{dist(x, aB)}{\varepsilon} & dist(x, aB) < \varepsilon, \end{cases}$$

for small $\varepsilon > 0$ and $a > 0$. Here $dist(x, aB) = \min_{y \in aB} |x - y|, x \in \mathbb{R}^{1+2\epsilon}$. It is clear that $(f_j)_\varepsilon$ tends to the characteristic function χ_{aB} of the ball aB as $\varepsilon \to 0$.

If ε is small and $0 < dist(x, aB) < \varepsilon$, then there exists a unique $x' \in \partial(aB)$ such that

$$dist(x, aB) = |x' - x|.$$

Let

$$\nu(x') = \frac{x' - x}{|x' - x|}.$$

From the definition of level sets (5.1) yields

$$[(f_j)_\varepsilon]_t = \begin{cases} \{x \in \mathbb{R}^{1+2\epsilon} : dist(x, aB) \leq (1-t)\varepsilon\} & 0 < t < 1, \\ \emptyset & t \geq 1. \end{cases}$$

Therefore, from (5.5) we have

$$\int_{S^{2\epsilon}} \sum_j \varphi(v_j) d\lambda_t((f_j)_\varepsilon, v_j) = \int_{\{\Sigma_j |(f_j)_\varepsilon| = t\}} \varphi\left(v_j(x)\right) d\mathcal{H}^{2\epsilon}(x) = 0,$$

for every even Borel function $\varphi : S^{2\epsilon} \to \mathbb{R}$ and $t \geq 1$. This deduces that the surface area measure of $\langle (f_j)_\varepsilon \rangle_t$ converges weakly to zero when $t \geq 1$.

We assume that $0 < t < 1$. We note that

$$\nabla (f_j)_\varepsilon(x) = \varepsilon^{-1} \nu(x')$$

for $x \in [(f_j)_\varepsilon]_t$. By (5.5), we have

$$\int_{S^{2\epsilon}} \sum_j \varphi(v_j) d\lambda_t((f_j)_\varepsilon, v_j) = \int_{\{\sum_j |(f_j)_\varepsilon| = t\}} \varphi\left(v_j(x)\right) d\mathcal{H}^{2\epsilon}(x)$$
$$= \int_{\{x\in\mathbb{R}^{1+2\epsilon}, x\prime\in\partial(aB):\, |x\prime - x| = (1-t)\varepsilon\}} \varphi\left(v_j(x')\right) d\mathcal{H}^{2\epsilon}(x)$$

for every even Borel function $\varphi : S^{2\epsilon} \to \mathbb{R}$. Therefore, as $\varepsilon \to 0$, we have

$$\int_{S^{2\epsilon}} \sum_j \varphi(v_j) d\lambda_t((f_j)_\varepsilon, v_j) \to \int_{\partial(aB)} \varphi\left(v_j(x')\right) d\mathcal{H}^{2\epsilon}(x')$$

for every even Borel function $\varphi : S^{2\epsilon} \to \mathbb{R}$. This means that the measure $\lambda_t((f_j)_\varepsilon, \cdot)$ converges weakly to the surface area measure of the ball aB as $\varepsilon \to 0$. By the continuity of the solution to the classical Minkowski problem (see, e.g., [30]), we conclude that $\langle f_j \rangle_t$ converges to aB (up to translation) in Hausdorff metric as f_j tends to the characteristic function χ_{aB} of the ball aB.

(iii) When $\epsilon > 0$ and $f_j(x) = \left(a + (a+\epsilon)|x|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{\epsilon}{1+\epsilon}}$ with $\epsilon > 0$, by a direct calculation, we have

$$\nabla f_j(x)| = -(a+\epsilon)\left(a + (a+\epsilon)|x|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{1+2\epsilon}{1+\epsilon}} |x|^{\frac{1}{\epsilon}} \frac{x}{|x|}. \tag{5.11}$$

Let $\psi(t) = \left(\frac{t^{\frac{1+\epsilon}{\epsilon}} - a}{(a+\epsilon)}\right)^{\frac{\epsilon}{1+\epsilon}}$, where $t \in (0, +\infty)$ such that $\alpha(t)$ is meaningful. Then, by (5.5) and (5.11) we have

$$\int_{S^{2\epsilon}} \sum_j \varphi(v_j) d\lambda_t(f_j, v_j)$$
$$= \int_{\{\sum_j |f_j| = t\}} \sum_j \varphi\left(v_j(x)\right) |\nabla f_j(x)|^\epsilon d\mathcal{H}^{2\epsilon}(x)$$
$$= (a+\epsilon)^\epsilon \int_{\{|x| = \psi(t)\}} \varphi\left(\frac{x}{|x|}\right) (a + (a+\epsilon)|x|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{(1+2\epsilon)(\epsilon)}{1+\epsilon}} |x| d\mathcal{H}^{2\epsilon}(x)$$

for every even Borel function $\varphi : S^{2\epsilon} \to \mathbb{R}$. Let $x = \psi(t)u_j$ for $u_j \in S^{2\epsilon}$. Then

$$\int_{S^{2\epsilon}} \sum_j \varphi(v_j) d\lambda_t(f_j, v_j) = (a+\epsilon)^\epsilon \left(\frac{1 - at^{\frac{1+\epsilon}{\epsilon}}}{a+\epsilon}\right)^{\frac{(1+2\epsilon)(\epsilon)}{1+\epsilon}} \int_{S^{2\epsilon}} \sum_j \varphi(u_j) dS(u_j).$$

Therefore

$$\lambda_t(f_j, \cdot) = (a+\epsilon)^\epsilon \left(\frac{1 - at^{\frac{1+\epsilon}{\epsilon}}}{a+\epsilon}\right)^{\frac{(1+2\epsilon)(\epsilon)}{1+\epsilon}} S(\cdot).$$

Combining with (5.8) and the uniqueness of the $L_{1+\epsilon}$ Minkowski problem, we deduce that

$$\langle f_j \rangle_t = \alpha(t)B$$

with $\alpha(t) = (a+\epsilon)\left(\frac{1-at^{\frac{1+\epsilon}{\epsilon}}}{a+\epsilon}\right)^{\frac{1+2\epsilon}{1+\epsilon}}$.

We are now in the position to prove our sharp convex mixed Lorentz-Sobolev inequality.

Theorem 5.1 (see [31]). *Suppose* $\epsilon \geq 0$. *If* $f_j, g_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$, *then*

$$\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j \left|\nabla f_j(x)\cdot \nabla g_j(y)\right|^{1+\epsilon} dxdy \geq \alpha_{1+2\epsilon,1+\epsilon}\int_0^\infty \sum_j V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}}dt\int_0^\infty V(\Pi_{1+\epsilon}\langle g_j\rangle_s)^{\frac{1+\epsilon}{1+2\epsilon}}ds, \tag{5.12}$$

where $\alpha_{1+2\epsilon,1+\epsilon} = (1+2\epsilon)^2\omega_{1+2\epsilon}\tilde{c}_{1+2\epsilon,1+\epsilon}$. *For* $\epsilon = 0$, *equality holds as* f_j *and* g_j *tend to the characteristic functions of dilates of centered polar ellipsoids. For* $\epsilon > 0$ *equality holds when* $f_j(x)$ *tends to* $\left(a_1 + |\psi(x-x_0)|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{\epsilon}{1+\epsilon}}$ *and* $g_j(y)$ *tends to* $(a_2 + |\psi^{-t}(y+x_0)|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ *with* $a_i > 0$ $(i = 1,2)$, $x_0 \in \mathbb{R}^{1+2\epsilon}$ *and* $\psi \in GL(1+2\epsilon)$.

Proof. The proof consists of several steps.

Step 1. Inequality.

Let $f_j, g_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$. By the co-area formula (2.10), (5.4), $v_j(x) = \frac{\nabla f_j(x)}{|\nabla f_j(x)|}$, the Fubini theorem, co-area formula (2.10), $u_j(x) = \frac{\nabla g_j(x)}{|\nabla g_j(x)|}$, (5.5), (5.8) and (3.3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j \left|\nabla f_j(x)\cdot \nabla g_j(y)\right|^{1+\epsilon} dxdy \\
&= \int_{\mathbb{R}^{1+2\epsilon}}\int_0^\infty \int_{\{\sum_j |f_j|=t\}}\sum_j \left|\nabla f_j(x)\cdot\nabla g_j(y)\right|^{1+\epsilon}\left|\nabla f_j(x)\right|^{-1}d\mathcal{H}^{2\epsilon}(x)dtdy \\
&= \int_{\mathbb{R}^{1+2\epsilon}}\int_0^\infty \int_{\{\sum_j |f_j|=t\}}\sum_j \left|v_j(x)\cdot\nabla g_j(y)\right|^{1+\epsilon}\left|\nabla f_j(x)\right|^{\epsilon}d\mathcal{H}^{2\epsilon}(x)dtdy \\
&= \int_0^\infty \int_{\{\sum_j |f_j|=t\}}\sum_j \left|\nabla f_j(x)\right|^{\epsilon}\int_0^\infty\int_{\{|g_j|=s\}}\sum_j \left|v_j(x)\cdot u_j(y)\right|^{1+\epsilon}\left|\nabla g_j(y)\right|^{\epsilon}d\mathcal{H}^{2\epsilon}(y)dsd\mathcal{H}^{2\epsilon}(x)dt \\
&= \int_0^\infty \int_{\{\sum_j |f_j|=t\}}\sum_j \left|\nabla f_j(x)\right|^{\epsilon}\int_0^\infty\int_{S^{2\epsilon}}\sum_j \left|v_j(x)\cdot u_j\right|^{1+\epsilon}d\lambda_s(g_j,u_j)dsd\mathcal{H}^{2\epsilon}(x)dt \\
&= \int_0^\infty \int_{\{\sum_j |f_j|=t\}}\sum_j \left|\nabla f_j(x)\right|^{\epsilon}\int_0^\infty\int_{S^{2\epsilon}}\sum_j \left|v_j(x)\cdot u_j\right|^{1+\epsilon}h_j(\langle g_j\rangle_s,u_j)^{-\epsilon}dS(\langle g_j\rangle_s,u_j)dsd\mathcal{H}^{2\epsilon}(x)dt \\
&= (1+2\epsilon)\omega_{1+2\epsilon}\tilde{c}_{1+2\epsilon,1+\epsilon}\int_0^\infty\int_{\{\sum_j |f_j|=t\}}\int_0^\infty\sum_j h_j\left(\Pi_{1+\epsilon}\langle g_j\rangle_s, v_j(x)\right)^{1+\epsilon}\left|\nabla f_j(x)\right|^{\epsilon}dsd\mathcal{H}^{2\epsilon}(x))dt.
\end{aligned} \tag{5.13}$$

By the Fubini theorem, (5.5), (5.8) and (2.2), we have

$$\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |\nabla f_j(x)\cdot \nabla g_j(y)|^{1+\epsilon}dxdy$$
$$= (1+2\epsilon)\omega_{1+2\epsilon}\tilde{c}_{1+2\epsilon,1+\epsilon}\int_0^\infty\int_0^\infty\int_{S^{2\epsilon}}\sum_j h_j(\Pi_{1+\epsilon}\langle g_j\rangle_s, v_j)^{1+\epsilon}d\lambda_t(f_j,v_j)dsdt$$
$$= (1+2\epsilon)\omega_{1+2\epsilon}\tilde{c}_{1+2\epsilon,1+\epsilon}\int_0^\infty\int_0^\infty\int_{S^{2\epsilon}}\sum_j h_j(\Pi_{1+\epsilon}\langle g_j\rangle_s, v_j)^{1+\epsilon}h_j(\langle f_j\rangle_t, v_j)^{-\epsilon}dS(\langle f_j\rangle_t, v_j)\,dsdt$$
$$= (1+2\epsilon)^2\omega_{1+2\epsilon}\tilde{c}_{1+2\epsilon,1+\epsilon}\int_0^\infty\int_0^\infty\sum_j V_{1+\epsilon}(\langle f_j\rangle_t\,,\Pi_{1+\epsilon}\langle g_j\rangle_s)dsdt \tag{5.14}$$

By (2.6) and (5.14), we have

$$\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |\nabla f_j(x)\cdot \nabla g_j(y)|^{1+\epsilon}dxdy \geq \alpha_{1+2\epsilon,1+\epsilon}\int_0^\infty\int_0^\infty\sum_j V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}}V(\Pi_{1+\epsilon}\langle g_j\rangle_s)^{\frac{1+\epsilon}{1+2\epsilon}}dsdt.$$

Step 2. The inequality (5.12) is affine, that is

$$\frac{\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |\nabla(kf_j\circ\Psi^t)\cdot\nabla(kg_j\circ\Psi^{-1})|^{1+\epsilon}dxdy}{\int_0^\infty\sum_j V(\langle kf_j\circ\Psi^t\rangle_t)^{\frac{\epsilon}{1+2\epsilon}}dt\int_0^\infty V(\Pi_{1+\epsilon}\langle kg_j\circ\Psi^{-1}\rangle_t)^{\frac{1+\epsilon}{1+2\epsilon}}ds}$$
$$=\frac{\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |\nabla f_j(x)\cdot \nabla g_j(y)|^{1+\epsilon}dxdy}{\int_0^\infty\sum_j V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}}dt\int_0^\infty V(\Pi_{1+\epsilon}\langle g_j\rangle_s)^{(1+\epsilon)/(1+2\epsilon)}ds}$$

for $k>0$, $\Psi\in\mathrm{Aff}(1+2\epsilon)$, $\Psi(x)=\psi x+z, \psi\in GL(1+2\epsilon)$ *and* $z\in\mathbb{R}^{1+2\epsilon}$.

By Lemma5.2, (3.6) and (3.7), we have

$$\langle kf_j\circ\Psi^t\rangle_t = k^{\frac{1+\epsilon}{\epsilon}}\psi^{-t}\langle f_j\rangle_t, \quad \text{and}$$
$$\Pi_{1+\epsilon}\langle kg_j\circ\Psi^{-1}\rangle_t = k|\det\psi|^{\frac{1}{1+\epsilon}}\psi^{-t}\Pi_{1+\epsilon}\langle g_j\rangle_t. \tag{5.15}$$

By (5.15), (2.3), (2.4) and (2.5) we obtain

$$V_{1+\epsilon}(\langle kf_j\circ\Psi^t\rangle_t,\Pi_{1+\epsilon}\langle kg_j\circ\Psi^{-1}\rangle_s) = k^{2(1+\epsilon)}V_{1+\epsilon}(\langle f_j\rangle_t,\Pi_{1+\epsilon}\langle g_j\rangle_s),$$
$$V(\langle kf_j\circ\Psi^t\rangle_t)^{\frac{\epsilon}{1+2\epsilon}} = k^{1+\epsilon}|\det\psi|^{-\frac{\epsilon}{1+2\epsilon}}V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}},$$
$$V(\Pi_{1+\epsilon}\langle kg_j\circ\Psi^{-1}\rangle_s)^{\frac{1+\epsilon}{1+2\epsilon}} = k^{1+\epsilon}|\det\psi|^{\frac{\epsilon}{1+2\epsilon}}V(\Pi_{1+\epsilon}\langle f_j\rangle_t)^{\frac{1+\epsilon}{1+2\epsilon}}.$$

The desired property follows from the above three formulas and (5.14).

Step 3. Equality conditions of (5.12).

If $\epsilon=0$, and f_j and g_j tend to the characteristic functions of the unit ball, by Lemma 5.4 (i), we infer that $\langle f_j\rangle_t$ and $\langle g_j\rangle_s$ converge to the unit ball when $0<t,s<1$, and their surface area measures converge weakly to zero when $t,s\geq 1$. Therefore

$$\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |\nabla f_j(x)\cdot \nabla g_j(y)|\,dxdy \to (1+2\epsilon)^2\omega_{1+2\epsilon}^2\tilde{c}_{1+2\epsilon,1}$$

and

$$(1+2\epsilon)^2\omega_{1+2\epsilon}\tilde{c}_{1+2\epsilon,1}\int_0^\infty \sum_j V(\langle f_j\rangle_t)^{\frac{2\epsilon}{1+2\epsilon}}\,dt\int_0^\infty V(\Pi\,\langle g_j\rangle_s)^{\frac{1}{1+2\epsilon}}ds \to (1+2\epsilon)^2\omega_{1+2\epsilon}^2\tilde{c}_{1+2\epsilon,1}.$$

By the affine invariance of inequality (5.12), we conclude that equality holds in (5.12) if f_j and g_j tend to $\chi_{\psi B-x_0}$ and $\chi_{\psi^{-t}B+x_0}$ with $x_0\in\mathbb{R}^{1+2\epsilon}$ and $\psi\in GL(1+2\epsilon)$, respectively.

If $\epsilon>0$ and $f_j(x)=\left(a_1+(a_1+\epsilon)|x|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{\epsilon}{1+\epsilon}}, g_j(y)=(a_2+(a_2+\epsilon)|y|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ with $\epsilon>0,(i=1,2)$, by (ii) in Lemma 5.4 $\langle f_j\rangle_t$ and $\langle g_j\rangle_s$ are balls for every $t,s>0$. Then

$$V_{1+\epsilon}(\langle f_j\rangle_t,\Pi_{1+\epsilon}\langle g_j\rangle_s)=V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}}V(\Pi_{1+\epsilon}\langle g_j\rangle_s)^{\frac{1+\epsilon}{1+2\epsilon}}.$$

By the affine invariance of inequality (5.12), equality holds in (5.12) for $f_j(x)$ tends to $(a_1+|\psi(x-x_0)|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ and $g_j(y)$ tends to $(a_2+|\psi^{-t}(y+x_0)|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ with $a_i>0$ $(i=1,2), x_0\in\mathbb{R}^{1+2\epsilon}$ and $\psi\in GL(1+2\epsilon)$.

Next, we will show that the analytic inequality (5.12) implies a special case of the $L_{1+\epsilon}$ Minkowski inequality.

Remark 5.1 (see [31]). The analytic inequality (5.12) implies the following geometric inequality

$$V_{1+\epsilon}(K,\Pi_{1+\epsilon}Q)^{1+2\epsilon}\ \geq\ V(K)^{\epsilon}V(\Pi_{1+\epsilon}Q)^{1+\epsilon}, \tag{5.16}$$

where $K,Q\in\mathcal{K}_s^{1+2\epsilon}$, and $\epsilon\geq 0$.

Proof .Let $\phi\in C^1(0,\infty)$ be strictly decreasing and

$$f_j(x)=\phi(1/\rho_K(x)) \quad \text{and} \quad g_j(y)=\phi(1/\rho_Q(y)), \tag{5.17}$$

for $K,Q\in\mathcal{K}_s^{1+2\epsilon}$, and $\epsilon\geq 0$. By (5.14), Lemma 5.1, (3.6) and (2.3), the left hand side of (5.12) is

$$\int_{\mathbb{R}^{1+2\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |\nabla f_j(x)\cdot\nabla g_j(y)|^{1+\epsilon}\,dxdy$$

$$=\alpha_{1+2\epsilon,1+\epsilon}\int_0^\infty\int_0^\infty\sum_j V_{1+\epsilon}(\langle f_j\rangle_t,\Pi_{1+\epsilon}\langle g_j\rangle_s)\,ds\,dt$$

$$=\alpha_{1+2\epsilon,1+\epsilon}V_{1+\epsilon}(K,\Pi_{1+\epsilon}Q)\int_0^\infty|\phi'(s)|^{1+\epsilon}s^{2\epsilon}ds\int_0^\infty|\phi'(t)|^{1+\epsilon}t^{2\epsilon}dt. \tag{5.18}$$

The right hand side of (5.12) is

$$\alpha_{1+2\epsilon,1+\epsilon}\int_0^\infty\int_0^\infty\sum_j V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}}V(\Pi_{1+\epsilon}\langle g_j\rangle_s)^{\frac{1+\epsilon}{1+2\epsilon}}dsdt \tag{5.19}$$

$$=\alpha_{1+2\epsilon,1+\epsilon}V(K)^{\frac{\epsilon}{1+2\epsilon}}V(\Pi_{1+\epsilon}Q)^{\frac{1+\epsilon}{1+2\epsilon}}\int_0^\infty|\phi'(s)|^{1+\epsilon}s^{2\epsilon}ds\int_0^\infty|\phi'(t)|^{1+\epsilon}t^{2\epsilon}dt.$$

Therefore the analytic inequality (5.12) implies the geometric inequality (5.16).

The sharp convex mixed Lorentz-Sobolev inequality (5.12) implies the sharp convex Lorentz-Sobolev inequality (1.5) of [12].

Corollary 5.1 (see [31]). *If* $f_j\in C_0^\infty(\mathbb{R}^{1+2\epsilon})$ *and* $\epsilon\geq 0$, *then*

$$\sum_j \|\nabla f_j\|_{1+\epsilon}^{1+\epsilon} \geq (1+2\epsilon)\omega_{1+2\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}} \int_0^\infty \sum_j V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}} dt, \tag{5.20}$$

with equality as f_j tends to the characteristic function of a ball for $\epsilon \geq 0$ equality is attained when $f_j(x)$ tends to $(a + (a+\epsilon)|x|^{\frac{1+\epsilon}{\epsilon}})^{-\frac{\epsilon}{1+\epsilon}}$ with $\epsilon > 0$.

Proof. If $\epsilon = 0$ and g_j tends to the characteristic function of the unit ball B, then by (i) of Lemma 5.4, (3.5), (5.14) and the definition of $\langle f_j\rangle_t$, we have

$$\int_{\mathbb{R}^{1+2\epsilon}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j |\nabla f_j(x) \cdot \nabla g_j(y)|\, dxdy \to (1+2\epsilon)^2 \omega_{1+2\epsilon} \tilde{c}_{1+2\epsilon,1} \int_0^\infty \sum_j V_1(\langle f_j\rangle_t, B)\, dt$$

$$= (1+2\epsilon)\, \omega_{1+2\epsilon} \tilde{c}_{1+2\epsilon,1} \int_0^\infty \sum_j S(\langle f_j\rangle_t)\, dt$$

$$= (1+2\epsilon)\, \omega_{1+2\epsilon} \tilde{c}_{1+2\epsilon,1} \int_0^\infty \int_{\{\sum_j |f_j|=t\}} d\mathcal{H}^{2\epsilon}(x) dt$$

$$= (1+2\epsilon)\, \omega_{1+2\epsilon} \tilde{c}_{1+2\epsilon,1} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j |\nabla f_j(x)|\, dx,$$

and

$$(1+2\epsilon)^2 \omega_{1+2\epsilon} \tilde{c}_{1+2\epsilon,1} \int_0^\infty \sum_j V(\langle f_j\rangle_t)^{\frac{2\epsilon}{1+2\epsilon}} dt \int_0^\infty V(\Pi \langle g_j\rangle_s)^{\frac{1}{1+2\epsilon}} ds$$

$$\to (1+2\epsilon)^2 \omega_{1+2\epsilon}^{\frac{2(1+\epsilon)}{1+2\epsilon}} \tilde{c}_{1+2\epsilon,1} \int_0^\infty \sum_j V(\langle f_j\rangle_t)^{\frac{2\epsilon}{1+2\epsilon}} dt$$

It follows that (5.12) implies (5.20) when $\epsilon = 0$.

For $\epsilon > 0$, let $\phi(t) = \left(1 + t^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{\epsilon}{1+\epsilon}}$ and $g_j(y) = \phi(1/\rho_B(y))$. Then

$$\nabla g_j(y) = \nabla\left(1 + |y|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{\epsilon}{1+\epsilon}} = -\left(1 + |y|^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{1+2\epsilon}{1+\epsilon}} |y|^{\frac{1}{\epsilon}} \frac{y}{|y|}.$$

The left hand side of (5.12) can be written as

$$\int_{\mathbb{R}^{1+2\epsilon}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j |\nabla f_j(x) \cdot \nabla g_j(y)|^{1+\epsilon} dxdy$$

$$= \int_0^\infty |\phi'(t)|^{1+\epsilon} t^{2\epsilon} dt \int_{\mathbb{R}^{1+2\epsilon}} \int_{S^{2\epsilon}} \sum_j |\nabla f_j(x) \cdot u_j|^{1+\epsilon} dudx. \tag{5.21}$$

From Lemma 5.1, we have

$$\langle g_j\rangle_s = c_{1+\epsilon}(s) B,$$

where $c_{1+\epsilon}(s)^\epsilon = |\phi'(t)|^\epsilon t^{2\epsilon}$ and $s = \phi(t) = \left(1 + t^{\frac{1+\epsilon}{\epsilon}}\right)^{-\frac{\epsilon}{1+\epsilon}}$. Therefore, the right-side of (5.12) is

$$\alpha_{1+2\epsilon,1+\epsilon} \int_0^\infty \int_0^\infty \sum_j V(\langle f_j\rangle_t)^{\frac{\epsilon}{1+2\epsilon}} V(\Pi_{1+\epsilon}\langle g_j\rangle_s)^{\frac{1+\epsilon}{1+2\epsilon}} dsdt$$

$$= \alpha_{1+2\epsilon,1+\epsilon} \omega_{1+2\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}} \int_0^\infty |\phi'(t)|^{1+\epsilon} t^{2\epsilon} dt \int_0^\infty \sum_j V(\langle f_j\rangle_{\tilde{t}})^{\frac{\epsilon}{1+2\epsilon}} d\tilde{t} \tag{5.22}$$

From Theorem 5.1, (5.21) and (5.22), we have

$$\int_{\mathbb{R}^{1+2\epsilon}} \int_{S^{2\epsilon}} \sum_j \left|\nabla f_j(x) \cdot u_j\right|^{1+\epsilon} du dx \geq \alpha_{1+2\epsilon,1+\epsilon} \omega_{1+2\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}} \int_0^\infty \sum_j V(\langle f_j \rangle_t)^{\frac{\epsilon}{1+2\epsilon}} dt. \quad (5.23)$$

By a direct calculation, one has

$$\int_{\mathbb{R}^{1+2\epsilon}} \int_{S^{2\epsilon}} \sum_j \left|\nabla f_j(x) \cdot u_j\right|^{1+\epsilon} du dx = \int_{\mathbb{R}^{1+2\epsilon}} \int_{S^{2\epsilon}} \sum_j \left|(v_j)_0 \cdot u_j\right|^{1+\epsilon} du \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \left|\nabla f_j(x)\right|^{1+\epsilon} dx$$
$$= \frac{(1+2\epsilon)\omega_{1+2\epsilon}\omega_{3\epsilon}}{\omega_2 \omega_{2\epsilon-1}\omega_\epsilon} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \left|\nabla f_j(x)\right|^{1+\epsilon} dx. \quad (5.24)$$

Inequality (5.20) follows from (5.23) and (5.24).

The equality condition of the sharp convex Lorentz-Sobolev inequality (5.20) comes from Theorem 5.1.

By Lemma 4.3and (5.14), we obtain the following results.

Theorem 5.2 (see [31]). *Let $f_j, g_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$ and $\epsilon \geq 0$.*

(i) *If $\epsilon \geq 0$, then*

$$\int_{\mathbb{R}^{1+2\epsilon}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \left|\nabla f_j(x) \cdot \nabla g_j(y)\right|^{1+\epsilon} dx dy$$
$$\geq (1+2\epsilon)\omega_{1+2\epsilon}^{\frac{2(1+\epsilon)}{1+2\epsilon}} \int_0^\infty \sum_j V(\langle f_j \rangle_t)^{\frac{\epsilon}{1+2\epsilon}} dt \int_0^\infty V(\langle g_j \rangle_s)^{\frac{\epsilon}{1+2\epsilon}} ds. \quad (5.25)$$

(ii) *If $\epsilon > 0$, then*

$$\int_{\mathbb{R}^{2(1+\epsilon)}} \int_{\mathbb{R}^{2(1+\epsilon)}} \sum_j \left|\nabla f_j(x) \cdot \nabla g_j(y)\right|^{2+\epsilon} dx dy$$
$$\geq \left(2(1+\epsilon)\right)^{\frac{2-\epsilon}{2}} \omega_{2(1+\epsilon)}^{\frac{2+\epsilon}{1+\epsilon}} \int_0^\infty \sum_j V(\langle f_j \rangle_t)^{\frac{\epsilon}{2(1+\epsilon)}} dt \int_0^\infty V(\langle g_j \rangle_s)^{\frac{\epsilon}{2(1+\epsilon)}} ds. \quad (5.26)$$

Each equality holds when $\epsilon = 0$ and $f_j(x)$ tends to $(a_1 + |\psi(x - x_0)|^{\frac{2+\epsilon}{1+\epsilon}})^{-\frac{\epsilon}{2+\epsilon}}, g_j(y)$ tends to $(a_2 + |\psi^{-t}(y + x_0)|^{\frac{2+\epsilon}{1+\epsilon}})^{-\frac{\epsilon}{2+\epsilon}}$ with $a_i > 0$ $(i = 1, 2), x_0 \in \mathbb{R}^{2(1+\epsilon)}$ and $\psi \in GL(2(1+\epsilon))$.

Proof. (i) For $0 \leq \epsilon \leq 1$, by (5.14) and Lemma 4.3(i), we have

$$\int_{\mathbb{R}^{1+2\epsilon}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \left|\nabla f_j(x) \cdot \nabla g_j(y)\right|^{1+\epsilon} dx dy$$
$$= (1+2\epsilon)^2 \omega_{1+2\epsilon} \tilde{c}_{1+2\epsilon,1+\epsilon} \int_0^\infty \int_0^\infty \sum_j V_{1+\epsilon}(\langle f_j \rangle_t, \Pi_{1+\epsilon} \langle g_j \rangle_s) \, ds \, dt$$
$$\geq (1+2\epsilon)\omega_{1+2\epsilon}^{\frac{2(1+\epsilon)}{1+2\epsilon}} \int_0^\infty \sum_j V(\langle f_j \rangle_t)^{\frac{\epsilon}{1+2\epsilon}} dt \int_0^\infty V(\langle g_j \rangle_s)^{\frac{\epsilon}{1+2\epsilon}} ds.$$

Similar to the proof in Theorem 5.1, the equality condition follows immediately from the equality condition of Lemma 4.3.

(iii) For $\epsilon \geq 0$, by (5.14) and Lemma 4.3 (ii), we have

$$\int_{\mathbb{R}^{2(1+\epsilon)}} \int_{\mathbb{R}^{2(1+\epsilon)}} \sum_j \left|\nabla f_j(x) \cdot \nabla g_j(y)\right|^{2+\epsilon} dx dy$$

$$= (2(1+\epsilon))^2 \omega_{2(1+\epsilon)} \tilde{c}_{2(1+\epsilon),2+\epsilon} \int_0^\infty \int_0^\infty \sum_j V_{2+\epsilon}(\langle f_j \rangle_t, \Pi_{2+\epsilon} \langle g_j \rangle_s)\, ds\, dt$$

$$\geq (2(1+\epsilon))^{\frac{2-\epsilon}{2}} \omega_{2(1+\epsilon)}^{\frac{2(2+\epsilon)}{2(1+\epsilon)}} \int_0^\infty \sum_j V(\langle f_j \rangle_t)^{\frac{\epsilon}{2(1+\epsilon)}} dt \int_0^\infty V(\langle g_j \rangle_s)^{\frac{\epsilon}{2(1+\epsilon)}} ds.$$

Similar to the proof in Theorem 5.1, the equality condition follows immediately from the equality condition of Lemma4.3.

Remark 5.2. The functional inequality (5.25) implies the $L_{2+\epsilon}$ Minkowski inequality (4.8) and the functional inequality (5.26) implies the $L_{2+\epsilon}$ Minkowski inequality (4.9).

The following analytic inequalities are direct consequences of (5.9) and Theorem5.2.

Corollary 5.2 (see [31]). *Suppose* $\epsilon \geq 0$. *Let* $f_j, g_j \in C_0^\infty(\mathbb{R}^{1+2\epsilon})$.

(i) *If* $0 \leq \epsilon \leq 1$, *then*

$$\int_{\mathbb{R}^{1+2\epsilon}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j |\nabla f_j(x) \cdot \nabla g_j(y)|^{1+\epsilon} dxdy \geq (1+2\epsilon)^{-1} c_{1+2\epsilon,1+\epsilon}^{2(1+\epsilon)} \sum_j \|f_j\|_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}^{1+\epsilon} \|g_j\|_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}^{1+\epsilon}. \quad (5.27)$$

(ii) *If* $\epsilon \geq 0$, *then*

$$\int_{\mathbb{R}^{2(1+\epsilon)}} \int_{\mathbb{R}^{2(1+\epsilon)}} \sum_j |\nabla f_j(x) \cdot \nabla g_j(y)|^{2+\epsilon} dxdy \geq (2(1+\epsilon))^{-\frac{2+\epsilon}{2}} c_{2(1+\epsilon),2+\epsilon}^{2(2+\epsilon)} \sum_j \|f_j\|_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}^{2+\epsilon} \|g_j\|_{\frac{(1+\epsilon)(1+2\epsilon)}{\epsilon}}^{2+\epsilon}. \quad (5.28)$$

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