Research Paper

Low Regularity Application of Elliptic Problems in Certain Besovand Sobolev–Triebel–Lizorkin Spaces

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Abstract
Elliptic problems with additional unknown distributions in boundary conditions are investigated in Besov and Sobolev–Triebel–Lizorkin spaces of low regularity, specifically of an arbitrary negative order.I.S. Chepurukhina, A.A. Murach [15] find that the problems induce Fredholm bounded operators on appropriate pairs of these spaces. We add another show of an application on the way of [15].

Keywords: *elliptic problem, Besov space, Sobolev space, Triebel–Lizorkin space, Fredholmoperator.*

I.   Introduction

Here the modern theory of elliptic boundary-value problems. Thus, these problems are considered.These problem generate Fredholm bounded operators on appropriate pairs of the indicated spaces provided that their order of regularity is large enough (see, e.g., [1, Section 5.5.2; 2, Section 4.3.3]). The case where these spaces are of low (specifically, negative) regularity is more intricate and less studied. But it is naturally to use distribution spaces of low regularity if we investigate elliptic problems whose right-hand sides contain functions with power singularities, delta-functions and their derivatives, distributions induced by some stochastic processes (e.g., the Lévy white noise), and other irregular distributions. The main question arising in this case concerns the interpretation of traces of low regularity distributions. Imposing some conditions on the right-hand side of an arbitrary elliptic differential equation, we may define the traces of its solutions with the help of passage to the limit on a class of smooth functions. Such an approach was elaborated by J.-L. Lions and E. Magenes [3, 4] for Sobolev spaces and some Besov spaces and recently was developed in [5; 6, Sections 4.4 and 4.5; 7] for Sobolev spaces and some of their generalizations in the class of Hilbert spaces. This approach gives theorems on the Fredholmproperty of regular elliptic problems in distribution spaces of arbitrarily low order. Anotherapproach is due to Ya. A. Roitberg [8] and works with normed spaces which are not formed by distributions given in the domain of the elliptic equation. A relation between these two approaches is considered in [5; 9, p. 85].

I.S. Chepurukhina, A.A. Murach [15] extend the Lions-Magenes approach to general Besov and Triebel–Lizorkin reflexive normed spaces of low regularity.They investigate elliptic problems with additional unknown distributions in boundary conditions. The class of such problems is closed with respect to the transition to the formally adjoint problem, which allows us to describe explicitly the ranges of operators induced by the problems under investigation. Note that classical elliptic problems (both regular and nonregular) belong to this class. The main results are four theorems on the Fredholm property of the induced operators on appropriate pairs of Besov and Triebel–Lizorkin spaces of low regularity, specifically of an arbitrary negative order. We impose weak enough conditions on the right-hand sides of elliptic differential equations in terms of some nonweightedor weighted spaces. These theorems allows us to derive the maximal regularity of solutionsto the elliptic problems from the given regularity of the data. The results are also new forgeneral Sobolev spaces.

II.   Statement of the Problem

For Ω be a bounded domain in the Euclidean space $\mathbf{R}^{2+\epsilon}$, with $\epsilon \geq 0$, and let Γ denote the boundary of Ω. Suppose that $\Gamma$is an infinitely smooth closed manifold of dimension $\epsilon + 1$ and that the C^∞-structure on Γ is induced by $\mathbf{R}^{2+\epsilon}$.

Let integers $\epsilon \geq 0$, and $m_1,\ldots,m_{(1+\epsilon)+(1+\epsilon)} \leq 2\epsilon - 1$. If $\epsilon \geq 0$, let $(1 + 2\epsilon)_1,\ldots,(1 + 2\epsilon)_{1+\epsilon}$be arbitrary integers. We consider the following boundary-value problem:

$$A\left(\sum_{L} u^L\right) = f^L \text{in}\Omega, \tag{1}$$

$$B_j u^L + \sum_{\epsilon=0}^{1+2\epsilon} \sum_{L} C_{j,1+\epsilon} v_{1+\epsilon}^L = g_j^L \text{on}\Gamma, \qquad j = 1,\ldots,(1+\epsilon) + (1+\epsilon). \tag{2}$$

Here, $A := A(x, D)$ is a linear partial differential operator (PDO) on $\overline{\Omega} := \Omega \cup \Gamma$of the even order $2(1 + \epsilon)$. Besides, each $B_j := B_j(x, D)$ is a linear boundary PDO on Γ, and each $C_{j,1+\epsilon} := C_{j,1+\epsilon}(x, D_\tau)$ is a linear tangent PDO on Γ. They orders satisfy the conditions ord$B_j \leq m_j$ and ord$C_{j,1+\epsilon} \leq m_j + (1 + 2\epsilon)_{1+\epsilon}$. (As usual, PDOs of negative order are defined tobe zero operators.) We suppose that all coefficients of the above PDOs are infinitely smooth functions on $\overline{\Omega}$ or Γ respectively. The distribution u^L on Ω and the distributions $v_1^L,\ldots,v_{1+\epsilon}^L$on Γ are unknown in this problem. Of course, if $(1 + \epsilon) = 0$, the problem does not contain any $v_{1+\epsilon}^L$. In this case the boundary conditions (2) become $B_j u^L = g_j^L$ on Γ, with $j = 1,\ldots,(1 + \epsilon)$.

All functions and distributions are supposed to be complex-valued; we therefore use complex distribution/function spaces. We interpret distributions as *anti* linear functionals on a relevant space of test functions.

Let $m := \max\{m_1,\ldots,m_{(1+\epsilon)+(1+2\epsilon)}\}$. We assume that $m \geq -(1 + 2\epsilon)_{1+\epsilon}$ whenever $\epsilon \geq 0$. Our assumption is natural; indeed, if $m + (1 + 2\epsilon)_{1+\epsilon} < 0$ for some $(1 + \epsilon)$, then $C_{1,1+\epsilon} = \cdots = C_{(1+\epsilon)+(1+2\epsilon),1+\epsilon} = 0$, i.e.the unknown $v_{1+\epsilon}^L$ will be absent in the boundary conditions (2).

We assume that the boundary-value problem (1), (2) is elliptic in   as a problem with additional unknown distributions $v_1^L,\ldots,v_{1+2\epsilon}^L$ on Γ (see, e.g., [10, Definition 3.1.2]). This means that the PDO A is properly elliptic on $\overline{\Omega}$ and that the system of boundary conditions (2) covers A on Γ. Certainly, if $\epsilon = -1$, our assumption becomes the usual ellipticity condition for the problem with respect to the single unknown u^L. The problem (1), (2) induces the linear mapping

$$\Lambda: (u^L, v_1^L,\ldots,v_{1+2\epsilon}^L) \to (f^L, g_1^L,\ldots,g_{(1+\epsilon)+(1+2\epsilon)}^L),$$
$$\text{where}u^L \in C^\infty(\overline{\Omega}) \text{ and } v_1^L,\ldots,v_{1+2\epsilon}^L \in C^\infty(\Gamma). \tag{3}$$

Here, of course, the functions f^L and $g_1^L,\ldots,g_{(1+\epsilon)+(1+2\epsilon)}^L$ are defined by formulas (1) and (2). We study extensions of this mapping on appropriate pairs of normed distribution spaces.

for$N$ denote the linear space of all solutions $(u^L, v_1^L,\ldots,v_{(1+2\epsilon)}^L) \in C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^{1+2\epsilon}$to the problem (1), (2) in which $f^L = 0$ on Ω and all $g_j^L = 0$ on Γ. Analogously, let N^+stand for the linear space of all solutions $(w^L, h_1^L,\ldots,h_{(1+\epsilon)+(1+2\epsilon)}^L) \in C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^{(1+\epsilon)+(1+2\epsilon)}$to the formally adjointproblem in which all right-hand sides are zeros. The last problem is explicitly written, contains $(1 + \epsilon) + (1 + 2\epsilon)$ additional unknown distributions $h_1^L,\ldots,h_{(1+\epsilon)+(1+2\epsilon)}^L$ on Γ, and is also elliptic [10, Theorem 3.1.2]. The spaces N and $N +$ are finite-dimensional [10, Lemma 3.4.2].

III.   Relevant Distribution Spaces

We recall the definitions of the normed Besovand Tribel–Lizorkin spaces over $\mathbf{R}^{2+\epsilon}$, we mainly following [2, Section 2.3]. As usual, $S'(\mathbf{R}^{2+\epsilon})$stands for the linear topological space of all tempered distributions in $\mathbf{R}^{2+\epsilon}$, whereas F andF^{-1} denote the direct and inverse Fourier transforms on $S'(\mathbf{R}^{2+\epsilon})$. If Q is a normed linearspace, then $\| \cdot , Q \|$designates the norm in Q.

We arbitrarily choose a function $\varphi_0 \in C^\infty(\mathbf{R}^{2+\epsilon})$ such that $\varphi_0(y) = 1$ whenever $|y| \leq 1$and that $\varphi_0(y) = 0$ whenever $|y| \geq 2$, with $y \in \mathbf{R}^{2+\epsilon}$. Given $1 + \epsilon \in \mathbf{N}$, we define a function $\varphi_{1+\epsilon}(y) := \varphi_0(2^{-(1+\epsilon)}y) - \varphi_0(2^{-\epsilon}y)$ of $y \in \mathbf{R}^{2+\epsilon}$. The functions $\varphi_{1+\epsilon}$, where $0 \leq 1 + \epsilon \in \mathbf{Z}$, form an infinitely smooth resolution of unity on $\mathbf{R}^{2+\epsilon}$.

Let $1 + \epsilon \in \mathbf{R}$ and $0 \leq \epsilon < \infty$. By definition, the Besov space $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$consists of all distributions $w^L \in S'(\mathbf{R}^{2+\epsilon})$ such that

$$\left\|w^L, B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})\right\|^{1+2\epsilon} := \sum_{1+\epsilon=0}^{\infty} 2^{(1+\epsilon)3} \left(\int_{\mathbf{R}^{2+\epsilon}} \sum_{L} |F^{-1}[\varphi_{1+\epsilon}Fw^L]|^{1+\epsilon}(x)dx\right)^{1+2\epsilon/1+\epsilon} < \infty.$$

By definition, the Triebel–Lizorkin space $F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$ consists of all distributions $w^L \in S'(\mathbf{R}^{2+\epsilon})$such that

$$\left\|w^L, F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})\right\|^p := \int_{\mathbf{R}^{2+\epsilon}} \left(\sum_{(1+\epsilon)=0}^{\infty} \sum_L 2^{(1+\epsilon)^3} |F^{-1}[\varphi_{1+\epsilon} F w^L]|^{1+2\epsilon}(x) \right)^{1+\epsilon/1+2\epsilon} dx < \infty.$$

These spaces are complete (i.e. Banach) and separable (with respect to the norms $\left\|\cdot, B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})\right\|$and $\left\|\cdot, F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})\right\|$of course). They do not depend up to equivalence of norms on the choice of φ_0. The space $F_{1+\epsilon,2}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$ coincides (up toequivalence of norms) with the Sobolev space $H_{1+\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$ of regularity order $1 + \epsilon$. Note that $B_{1+\epsilon,1+\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon}) = F_{1+\epsilon,1+\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$ with equality of norms.

Being based on the above spaces, we introduce their versions for Ωin a way common to Besov and Triebel–Lizorkin spaces. We follow the approach used, e.g., in [4, Chapter 1]for Sobolev spaces. Let E denote either B or F in our designations of distribution spaces. Thus, e.g., $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$ means either $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$ or $F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})$ in the considerations given below.
If $\epsilon \geq -1$, we put

$$E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega) := \{w^L|_\Omega : w^L \in E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})\}, \tag{4}$$

$$\left\|u^L, E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\right\| := \inf\{\left\|w^L, E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})\right\| : w^L \in E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon}), u^L = w^L|_\Omega\}. \tag{5}$$

(Of course, $w^L|_\Omega$ stands for the restriction of a distribution $w^L \in S'(\mathbf{R}^{2+\epsilon})$ to Ω.) The space $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ is Banach. Let $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon,0}(\Omega)$ denote the completion of the linear space $\{u^L \in C^\infty(\Omega) : \text{supp} u^L \subset \Omega\}$with respect to the norm in $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$. Here, as usual, $\text{supp} u^L$ is the largest closed subset of $\mathbf{R}^{2+\epsilon}$ outside which $u^L = 0$.

If $\epsilon < 1$, then $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ is defined to be the antidual Banach space to $E_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon),0}(\Omega)$. Here, thenumbers $0 < \epsilon < \infty)$ satisfy the conditions $\epsilon = 1$ and $\epsilon = 1/2$. In this case, we have the following description:

$$E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega) = \{w^L|_\Omega : w^L \in E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon}), \text{supp} w^L \subseteq \overline{\Omega}\},$$

$$\left\|u^L, E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\right\| \asymp \inf\{\left\|w^L, E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon})\right\| : w^L \in E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{2+\epsilon}), \text{supp} w^L \subseteq \overline{\Omega}, u^L = w^L|_\Omega\},$$

with $\asymp$meaning equivalence of norms. Besides, if negative $(1 + \epsilon) \notin \{-(1 + \epsilon) + 1/(1 + \epsilon) : (1 + \epsilon) \in \mathbf{N}\}$, then we may omit the condition $\text{supp} w^L \subseteq \overline{\Omega}$ in this description, which gives us the equivalent definition (4), (5) of $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ for such $1 + \epsilon$.

Whatever $1 + \epsilon \in \mathbf{R}$, the Banach space $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ is separable and continuously embedded in thetopological space $D'(\Omega)$ of all distributions in Ω. Besides, the set $C^\infty(\overline{\Omega})$ is dense in $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$.We have the dense continuous embeddings$E_{(1+\epsilon)+\delta,(1+2\epsilon)}^{1+2\epsilon}(\Omega) \subset E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ and $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega) \subset E_{1+\epsilon,1+2\epsilon+\theta}^{1+\epsilon}(\Omega)$whenever $\varepsilon \geq 0, \delta \geq 0$, and $\theta > 0$. The first embedding is compact when $\varepsilon > 0$. If thenumbers $1 + \epsilon, t \in \mathbf{R}$ and $0 < \epsilon < \infty)$ satisfy the condition $(1 + \epsilon) - t = (2 + \epsilon)(1/(1 + \epsilon) - 1/(1 + 2\epsilon)) > 0$, then thedense continuous embedding $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega) \subset E_{1+2\epsilon,1+2\epsilon}^{t}(\Omega)$ holds true.

Given a positive function $\rho \in C^\infty(\Omega)$, we put

$$\rho E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega) := \{\rho u^L : u^L \in E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\},$$

$$\left\|f^L, \rho E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\right\| := \left\|\rho^{-1} f^L, E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\right\|,$$

where $f^L \in \rho E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$. The space $\rho E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega\rho)$ is Banach, separable, and continuously embedded in $D'(\Omega)$.

Thus, we have introduced the spaces $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ and $F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$and their weighted analogs$\rho B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ and $\rho F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$. Note that

$$B_{1+\epsilon,\min\{1+\epsilon,1+2\epsilon\}}^{1+\epsilon}(\Omega) \subset F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega) \subset B_{1+\epsilon,\max\{1+\epsilon,1+2\epsilon\}}^{1+\epsilon}(\Omega),$$

the embeddings being continuous and dense.

We also need Besov spaces over the boundary Γ of Ω. Briefly saying, the space $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Gamma)$,where $1 + \epsilon \in \mathbf{R}$, consists of all distributions $h^L \in D'(\Gamma)$ that yield elements of $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\mathbf{R}^{1+\epsilon})$in local charts on Γ. The detailed definition is given, e.g., in Γ [2, Section 3.2.2]. The set $C^\infty(\Gamma)$is dense in this space. The above embeddings are also true for the Besov spaces over Γ, with $1 + \epsilon$ being taking instead of $2 + \epsilon$ in the condition for $t, 1 + \epsilon$, and $(1 + 2\epsilon)$.

If $(1 + \epsilon)^2 > 1$, then $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon - \frac{1}{1+\epsilon}}(\Gamma)$ is the space of traces of distributions from $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$. This means that the mapping $u^L \mapsto u^L \upharpoonright \Gamma$, where $u^L \in C^\infty(\overline{\Omega})$, extends uniquely (by continuity) to a bounded linear operator from $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ onto $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon - \frac{1}{1+\epsilon}}(\Gamma)$ [2, Theorem 3.3.3]. Besides, if $(1 + \epsilon)^2 > 1$,then $B_{1+\epsilon,1+\epsilon}^{1+\epsilon - \frac{1}{1+\epsilon}}(\Gamma)$ is the space of traces of distributions from $F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ whenever $0 < \epsilon < \infty$. This explains why we do not need Triebel–Lizorkin spaces over Γ in the next section.

All the normed spaces just considered are reflexive.

IV.   Results

We arbitrarily choose $0 < \epsilon < \infty$. We investigate the elliptic problem (1), (2) in the case where solutions of the elliptic equation (1) belong to one of the spaces $B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ and $F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ of regularity order $\epsilon^2 + \epsilon + 1 \geq 0$. If $(1 + \epsilon)$ satisfies theinverse inequality, the following result is known:

The mapping (3) extends uniquely (by continuity) to a bounded linear operator

$$\Lambda: B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega) \oplus \bigoplus_{1+\epsilon=1}^{1+2\epsilon} B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)+(1+2\epsilon)_{1+\epsilon}-\frac{1}{1+\epsilon}}(\Gamma)$$

$$\to B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)-2(1+\epsilon)}(\Omega) \oplus \bigoplus_{k=1}^{(1+\epsilon)+(1+2\epsilon)} B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)-m_j-\frac{1}{1+\epsilon}}(\Gamma) \qquad (6)$$

whenever $0 > \epsilon^2 + \epsilon + 1$. This operator is Fredholm. Its kernel is N, and its range consists of all vectors $(f^L, g^L) := (f^L, g_1^L, \ldots, g_{(1+\epsilon)+(1+2\epsilon)}^L)$ that belong to the target space in (6) and satisfy the condition

$$(f^L, w^L)_\Omega + \sum_{j=1}^{(1+\epsilon)+(1+2\epsilon)} (g_j^L, h_j^L)_\Gamma = 0 \quad \text{for all}(w^L, h_1^L, \ldots, h_{(1+\epsilon)+(1+2\epsilon)}^L) \in N^+. \quad (7)$$

Thus, the index of the operator (6) is equal to $\alpha := \dim \ N - \dim \ N^+$ and does not dependon $(1 + \epsilon)$ and $(1 + 2\epsilon)$. This statement remains true if we replace $B_{1+\epsilon,1+2\epsilon}^t(\Omega)$ with $F_{1+\epsilon,1+2\epsilon}^t(\Omega)$ and $B_{1+\epsilon,1+2\epsilon}^t(\Gamma)$with $B_{1+\epsilon,1+\epsilon}^t(\Gamma)$ for all mentioned values of t.

This result is contained in [11, Theorem 5.2] (see also [12, Corollary 5.5] in the case of
Sobolev spaces $H_{1+\epsilon}^t(\Omega) = F_{1+\epsilon,2}^t(\Omega)$), condition (7) following from [10, Theorem 4.1.4]. If $(1 + 2\epsilon) = 0$and if the elliptic problem is regular, this result is proved in [2, Theorem 4.3.3] (under the assumption that $N = \{0\}$and $N^+ = \{0\}$) and in [13, Theorem 15] (without this assumption).

Recall that a bounded linear operator $T: Q_1 \to Q_2$, where Q_1 and Q_2 are Banach spaces, is said to be Fredholm if its kernel $\ker T := \{x \in Q_1: Tx = 0\}$and co-kernel $Q_2/T(Q_1)$ are finite-dimensional. In this case, the range $T(Q_1)$ is closed in Q_2, and the operator has the finite index, which is equal by definition to $\dim \ker T - \dim Q_2/T(Q_1)$.

Generally, the bounded linear operator (6) is not well defined in the case of $\epsilon^2 + \epsilon + 1 \geq 0$,which follows from [2, Section 2.7.2, Remark 4]. The same is true for the above-mentioned analog of (6) for Triebel–Lizorkin spaces. Considering this case, we have to take a narrower space than $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ as a domain of solutions u^L to the elliptic equation (1). (Recall that Emeans either B or F.) This space is defined with the help of some conditions imposed on Au. We separately consider two approaches. The first of them demands that Au^L belongs to the space $E_{1+\epsilon,1+2\epsilon}^{-1+\frac{1}{1+\epsilon}}(\Omega)$, whose regularity order is limiting in (6) where $\epsilon^2 + \epsilon + 1 < 0$.The second uses some weighted spaces $\rho E_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)-2(1+\epsilon)}(\Omega)$ for Au^L.

Dealing with the first approach, we put

$$E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\Omega) := \{u^L \in E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega): Au^L \in E_{1+\epsilon,1+2\epsilon}^{-1+\frac{1}{1+\epsilon}}(\Omega)\},$$

$$\|u^L, E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\Omega)\| := \|u^L, E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\| + \left\|Au^L, E_{1+\epsilon,1+2\epsilon}^{-1+\frac{1}{1+\epsilon}}(\Omega)\right\|.$$

Here and below, we understand Au^L in the theory of distributions on Ω. The normed linear space $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\Omega)$ is complete. The set $C^\infty(\Omega)$ is dense in this space.

Theorem 1 [15]. Let $\epsilon^2 + \epsilon + 1 \geq 0$. Then the mapping (3) extends uniquely (by continuity) to a bounded linear operator

$$\Lambda: B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\Omega) \oplus \bigoplus_{1+\epsilon=1}^{\lambda} B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)+(1+2\epsilon)_{1+\epsilon}-\frac{1}{1+\epsilon}}(\Gamma)$$

$$\to B_{1+\epsilon,1+2\epsilon}^{-1+1/1+\epsilon}(\Omega) \oplus \bigoplus_{j=1}^{(1+\epsilon)+(1+2\epsilon)} B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)-m_j-\frac{1}{1+\epsilon}}(\Gamma). \qquad (8)$$

This operator is Fredholm with kernel N and index α. Its range consists of all vectors (f^L, g^L)that belong to the target space in (8) and satisfy (7).

A version of this theorem for Triebel–Lizorkin spaces is formulated as follows:

Theorem 2 [15]. Let $\epsilon^2 + \epsilon + 1 \geq 0$. Then the mapping (3) extends uniquely (by continuity) to a bounded linear operator

$$\Lambda:\ F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\Omega)\oplus\bigoplus_{1+\epsilon=1}^{1+2\epsilon}B_{1+\epsilon,1+\epsilon}^{(1+\epsilon)+(1+2\epsilon)_{1+\epsilon}-\frac{1}{1+\epsilon}}(\Gamma)$$

$$\to F_{1+\epsilon,1+2\epsilon}^{-1+1/1+\epsilon}(\Omega)\oplus\bigoplus_{j=1}^{(1+\epsilon)+(1+2\epsilon)}B_{1+\epsilon,1+\epsilon}^{(1+\epsilon)-m_j-\frac{1}{1+\epsilon}}(\Gamma). \qquad (9)$$

This operator is Fredholm with kernel N and index α. Its range consists of all vectors (f^L,g^L)that belong to the target space in (9) and satisfy (7).

We put

$$E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\rho,\Omega):=\left\{u^L\in E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega):Au^L\in\rho E_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)\right\},$$

$$\left\|u^L,E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\rho,\Omega)\right\|:=\left\|u^L,E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\right\|+\left\|Au^L,\rho E_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)\right\|,$$

with $0<\rho\in C^\infty(\Omega)$. The normed linear space $E_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\rho,\Omega)$ is complete. Let ∂_{ν^L}denote the differentiation operator along the inner normal to the boundary of Ω. As above, $\epsilon=1$and $\epsilon=1/2$.

Theorem 3 [15]. Let $\epsilon^2+\epsilon+1>0$. Suppose that a positive function $\rho\in C^\infty(\Omega)$ is a multiplier on the space $B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)}(\Omega)$ and satisfies the condition

$$\partial_{\nu^L}^j\rho\ =\ 0\ \text{on}\Gamma\text{whenever}\ \ j\in\mathbf{Z}\text{and}\ \ 0\le j<2(1+\epsilon)-(1+\epsilon)-1+1/(1+\epsilon). \quad(10)$$

Then the mapping (3), where $A(\sum_L\ \ u^L)\in\rho B_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)$ in addition, extends uniquely (by continuity) to a bounded linear operator

$$\Lambda:\ B_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\rho,\Omega)\oplus\bigoplus_{1+\epsilon=1}^{1+2\epsilon}B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)+(1+2\epsilon)_{1+\epsilon}-\frac{1}{1+\epsilon}}(\Gamma)$$

$$\to\rho B_{1+\epsilon,1+2\epsilon}^{-1+1/1+\epsilon}(\Omega)\oplus\bigoplus_{j=1}^{(1+\epsilon)+(1+2\epsilon)}B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)-m_j-\frac{1}{1+\epsilon}}(\Gamma). \qquad(11)$$

This operator is Fredholm with kernel N and index α. Its range consists of all vectors (f^L,g^L)that pertain to the target space in (11) and satisfy (7).

As for Triebel–Lizorkin spaces, the result is formulated as follows:

Theorem 4 [15]. Let $0>\epsilon^2+\epsilon+1$. Suppose that a positive function $\rho\in C^\infty(\Omega)$ is a multiplier on the space $F_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)}(\Omega)$ and satisfies condition (10). Then the mapping (3), where $A(\sum_L\ \ u^L)\in\rho F_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)$ in addition, extends uniquely (by continuity) to a bounded linear operator

$$\Lambda:\ F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(A,\rho,\Omega)\oplus\bigoplus_{1+\epsilon=1}^{1+2\epsilon}B_{1+\epsilon,1+\epsilon}^{(1+\epsilon)+(1+2\epsilon)_{1+\epsilon}-\frac{1}{1+\epsilon}}(\Gamma)$$

$$\to\rho F_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)\oplus\bigoplus_{j=1}^{(1+\epsilon)+(1+2\epsilon)}B_{1+\epsilon,1+\epsilon}^{(1+\epsilon)-m_j-\frac{1}{1+\epsilon}}(\Gamma). \qquad(12)$$

This operator is Fredholm with kernel N and index α. Its range consists of all vectors (f^L,g^L)that pertain to the target space in (12) and satisfy (7).

Recall that a function $\rho\in C^\infty(\Omega)$ is called a multiplier on a normed linear space $Q\subset D'(\Omega)$ if the operator of multiplication by ρ is bounded on Q.

Remark 1. Condition (10) makes sense in Theorems 3 and 4. Indeed, if a function $\rho\in C^\infty(\Omega)$ is a multiplier on one of the spaces $B_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)}(\Omega)$ and $F_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)}(\Omega)$ where $0<\epsilon^2+\epsilon+1$,then ρ belongs to the same space. Hence, the trace $\partial_{\nu^L}^j\rho$ on $\Gamma$is well defined by [2, Theorem 3.3.3] whenever $(1+\epsilon)>j+1/(1+\epsilon)$. The latter inequality is equivalent to the condition $j<(1+\epsilon)$ used in (10).

Remark 2. Under the hypothesis of Theorem 3 or Theorem 4, we have the continuous embedding $\rho E_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)\subset E_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)$ where $E=B$ in the case of Theorem 3 or $E=F$in the case of Theorem 4. Moreover, the set $C^\infty(\overline\Omega)\cap\rho E_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)$ is dense in $\rho E_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)$.

The following result gives an important example of the above weight function ρ (see [15]).

Theorem 5. Let $0>\epsilon^2+\epsilon+1$, and let a positive function $\rho_1\in C^\infty(\Omega)$equal the distance to Γ in a neighbourhood of Γ. Assume that $\delta\ge\epsilon+1/(1+\epsilon)\in\mathbf{Z}$or that $\delta>\epsilon+1/(1+\epsilon)\notin\mathbf{Z}$. Then the function $\rho:=\rho_1^\delta$ satisfies the hypotheses of Theorems 3and 4.

In the case where $\epsilon=1,\epsilon=1/2$ and $\epsilon=-1/2$, these theorems were established in [5, Section 2] for regular elliptic problems (see also monograph [6, Theorems 4.27, 4.29, and 4.30]). Putting $\rho:=\rho_1^{(1+\epsilon)}$ in

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Theorem 4 in this case and assuming that $\epsilon + 1/2 \notin \mathbf{Z}$, we arrive at the classical Lions–Magenes theorems [4, Theorem 6.7 and 7.4]. They concern regular elliptic problems in inner product Sobolev spaces.

It is worthwhile to compare Theorems 1–4 with the newest result by F. Hummel [14,Theorem 6.3] concerning some constant-coefficient parameter-elliptic problems in the half-space. He has found sufficient conditions under which the unique solution to such a problem belong to certain anisotropic distribution spaces built on the base of Sobolev, Besov, and Triebel–Lizorkin spaces of an arbitrary real order. Specifically, the right-hand side of the elliptic equation should have a nonnegative integer-valued Sobolev regularity in the normal direction with respect to the boundary of the half-space. This result does not allow us to derive the maximal regularity of solutions from these conditions, as is noted in [14, p. 1949]. Theorems 1–4 provide the maximal regularity in terms of isotropic spaces (see [15]).

V.   Applications

In Theorems 2 and 4, the spaces over $\Gamma$are independent of the parameter $(1 + 2\epsilon)$ in contrast to the spaces over Ω. This suggests that the set of all $u^L \in F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$such that Au^L satisfies a relevant condition does not depend on $(1 + 2\epsilon)$. The following two theorems give such conditions (see [15]). Let $0 < \epsilon < \infty$.

Theorem 6 [15]. Let $\epsilon^2 + \epsilon + 1 \geq 0$. Assume that a Banach space Q is continuously embedded in $F_{1+\epsilon,\min\{1+2\epsilon,1+2\epsilon\}}^{-1+\frac{1}{1+\epsilon}}(\Omega)$. Then

$$\left\{u^L \in F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega): A\left(\sum_L u^L\right) \in Q\right\} = \left\{u^L \in F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega): A\left(\sum_L u^L\right) \in Q\right\}, \qquad (13)$$

$$\left\|u^L, F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\right\| + \left\|A\left(\sum_L u^L\right), Q\right\| \asymp \left\|u^L, F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)\right\| + \left\|A\left(\sum_L u^L\right), Q\right\|. \qquad (14)$$

Recall that the symbol $\asymp$means equivalence of norms.

Theorem 7 [15]. Let $\epsilon^2 + \epsilon + 1 > 0$. Suppose that a positive function $\rho \in C^\infty(\Omega)$ is a multiplier on the space $F_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)}(\Omega)$ and satisfies condition (10). Suppose also that a positive function $\mu \in C^\infty(\Omega)$ is a multiplier on the space $F_{1+\epsilon,1+2\epsilon}^{(1+\epsilon)}(\Omega)$and satisfies condition (10) in which ρ is replaced with μ. Let a Banach space Q be continuously embedded in both of the spaces $\rho F_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)$ and $\mu F_{1+\epsilon,1+2\epsilon}^{-(1+\epsilon)}(\Omega)$. Then relations (13) and (14) hold true.

Theorems 6 and 7 complement the following property of Triebel–Lizorkin spaces of high enough regularity: if $\epsilon^2 + \epsilon + 1 < 0$ and if a Banach space Q is continuously embedded in$F_{1+\epsilon,\min\{1+2\epsilon,1+2\epsilon\}}^{-(1+\epsilon)}(\Omega)$, then (13) and (14) are valid. This property follows from the result stated at the beginning of Section 3. Specifically, the space

$$\left\{u^L \in F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega): A\left(\sum_L u^L\right) = 0 \ in \ \Omega\right\} \qquad (15)$$

does not depend on $(1 + 2\epsilon)$ whatever $1 + \epsilon \in \mathbf{R}$. Such a property is proved in [15, Theorem 1.6] under the additional assumption that A is a constant-coefficient homogeneous PDO but in the more general case where the bounded domain Ω is Lipschitz and when $0 < \epsilon < \infty$. Note that the space (15) will not change if we replace $F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ with the space (4) used in [15]. The latter space is broader than $F_{1+\epsilon,1+2\epsilon}^{1+\epsilon}(\Omega)$ if an only if $1 + \epsilon \notin \{-(1 + \epsilon) + 1/1 + \epsilon : 1 + \epsilon \in \mathbf{N}\}$.

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