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**Research Paper**

# Application on Sharp Weighted Trudinger–Moser–Adams Inequalities on Space and Their Existence Extremal functions

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***Abstract***
*Following L. Chen, G. Lu and C. Zhang [64] we show the existence of extremals for sharp weighted Trudinger–Moser–Adams type inequalities with the Dirichlet and Sobolev norms. We follow the method based on level-sets of functions and Fourier transform to generalize the weighted Trudinger–Moser–Adams type inequalities with the Dirichlet norm in $W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ and $W^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right)$ respectively. When the mentioned first order and its existence of extremal functions was obtained by using a quasi-conformal type transform, such a transform does not valid for the Adams inequality involving higher order derivatives. Through the compact embedding and scaling invariance of the subcritical Adams inequality, we estimated the best constants. We also follow the method developed the supremums of the critical and subcritical inequalities and show the existence of extremals for weighted Adams' inequalities with the Sobolev norm. Now using the Fourier rearrangement inequality, we reduce the problem to the radial case and then show the existence of the extremal functions for the non-weighted Adams inequalities. We derive new results on high-order critical Caffarelli–Kohn–Nirenberg interpolation inequalities for more parameters and show the relationship between the best constants of the weighted Trudinger–Moser–Adams type inequalities and the above inequalities as applications.*

***Keywords:*** *Trudinger–Moser inequality, Adams inequality, Dirichlet norm, Sobolev norm, Extermal functions, Caffarelli–Kohn–Nirenberg inequality.*

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## I. Introduction

We study and treat higher order on Sobolev spaces $W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ and $W^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right)$ the maximizer joint the singular Adams inequality. The known Sobolev embedding theorems established on bounded domain raised that $W_0^{1,1+\epsilon}(\Omega) \subset L^{1+\epsilon}(\Omega)$ for $\epsilon \geq 0$ which improve the exponent. When $\epsilon = 0$, this exponent tends to infinite so $W_0^{1,1+2\epsilon}(\Omega) \subset L^{1+\epsilon}(\Omega)$ for $0 \leq \epsilon < \infty$, but $W_0^{1,1+2\epsilon}(\Omega) \not\subseteq L^{\infty}(\Omega)$. (see for instances [59], [19], [55] and [54]). The growing process of the embedding. Hence named the Trudinger-Moser inequality and stated as.

**Theorem A** [54,59] Let $\Omega$ be a bounded domain in $\mathbb{R}^{2+\epsilon}, \epsilon \geq 0$. Then there exist a positive constant $C_{2+\epsilon}$ and a sharp constant $\alpha_{2+\epsilon} = (2+\epsilon)\omega_{1+\epsilon}^{\frac{1}{1+\epsilon}}$ such that

$$\frac{1}{|\Omega|}\int_{\Omega}\sum_j \exp\left(\alpha|u_j|^{\frac{2+\epsilon}{1+\epsilon}}\right)dx \leq C_{2+\epsilon},$$

for any $\alpha \leq \alpha_{2+\epsilon}$ and $u_j \in C_0^{\infty}(\Omega)$ with $\int_{\Omega}\sum_j |\nabla u_j|^{2+\epsilon}dx \leq 1$, where $\omega_{1+\epsilon}$ stand for the surface area of the unit ball.

The authors of [6] establish the existence of extremal functions of Trudinger-Moser inequality when $\Omega$ is a ball in $\mathbb{R}^{2+\epsilon}$. Extended results of [14] to bounded domains in $\mathbb{R}^2$ and by [41] in $\mathbb{R}^{2+\epsilon}$ for $\epsilon > 0$. [49] investigated the blow-up of a sequence of the Trudinger-Moser functionals on the planar disk.

Also for more extensions of Theorem **A**, see [5], [10] and [1], etc. We state from [1] the following:

**Theorem B [1]** For $\epsilon \geq 0$ and $0 < \alpha < \alpha_{2+\epsilon}$, there exists a positive constant $C_{2+\epsilon,\alpha}$ such that

$$\sup_{u_j \in W^{1,1+2\epsilon}, \int_{\mathbb{R}^{1+2\epsilon}} \sum_j |\nabla u_j|^{1+2\epsilon} dx \leq 1} \sum_j \frac{1}{\|u_j\|_{L^{1+2\epsilon}}^{1+2\epsilon}} \int_{\mathbb{R}^{1+2\epsilon}} \Psi_j(\alpha |u_j(x)|^{\frac{1+2\epsilon}{2\epsilon}})\, dx < \mathrm{C}_{1+2\epsilon,\alpha}, \qquad (1.1)$$

where $\Psi_j(1+\epsilon) := e^{1+\epsilon} - \sum_{i=0}^{\epsilon} \frac{(1+\epsilon)^i}{i!}$. Hence, the constant $\alpha_{2+\epsilon}$ is sharp if $\alpha \geq \alpha_{2+\epsilon}$, then the supremum will tend to infinite.

When approaching the singular Trudinger-Moser inequality in $\mathbb{R}^{2+\epsilon}$. The authors in [18] investigated the scaling invariant form of the singular Trudinger-Moser inequality for radially symmetric functions and showed the existence of a maximize. Then:

**Theorem C [18]** Assume $\epsilon \geq 0, -\infty < s \leq 1+\epsilon < 2+\epsilon$ and $0 < \alpha < \alpha_{2+\epsilon} := \frac{\omega_{1+\epsilon}^{\frac{1}{1+\epsilon}}}{2+\epsilon}$, then there exists a positive constant $C = C(2+\epsilon, s, 1+\epsilon, \alpha)$ such that the inequality

$$\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\Psi_j\left(\alpha\left(\frac{1}{2+\epsilon}\right)|u_j(x)|^{\frac{2+\epsilon}{1+\epsilon}}\right)}{|x|^{1+\epsilon}} dx \leq C \sum_j \left(\int_{\mathbb{R}^{2+\epsilon}} \frac{|u_j(x)|^{2+\epsilon}}{|x|^s} dx\right)^{\frac{1}{2+\epsilon-s}}, \qquad (1.2)$$

holds for all radially symmetric sequences of functions $u_j \in L^{2+\epsilon}(\mathbb{R}^{2+\epsilon}; |x|^{-s}dx) \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^{2+\epsilon})$ with $\| \nabla u_j \|_{2+\epsilon} \leq 1$, where $\dot{W}^{1,2+\epsilon}(\mathbb{R}^{2+\epsilon})$ denotes the class of functions $u_j$ which are locally integrable and $\| \nabla u_j \|_{2+\epsilon}$ are in $L^{2+\epsilon}(\mathbb{R}^{2+\epsilon})$. The constant $\alpha_{2+\epsilon,1+\epsilon}$ is sharp for the inequality.

They also showed that when $s = 0$, the constant $C$ has an infimum and could be attained by some functions $u_j \in W^{1,2+\epsilon}(\mathbb{R}^{2+\epsilon})$. However, when $s \neq 0$, they only verified inequality (1.2) and the existence of extremals on the class of radial functions. A natural problem is whether we can remove the radially symmetric condition for functions $u_j$ in inequality (1.2). [13] gave an answer.

**Theorem D [13]** Assume $\epsilon \geq 0, -\infty < s \leq 1+\epsilon < 1+2\epsilon$ and $0 < \alpha < \alpha_{2+\epsilon}$, then there exists a positive constant $C = C(2+\epsilon, s, 1+\epsilon, \alpha)$ such that the inequality

$$\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\Psi_j\left(\alpha\left(\frac{1}{2+\epsilon}\right)|u_j(x)|^{\frac{2+\epsilon}{1+\epsilon}}\right)}{|x|^{1+\epsilon}} dx \leq C \sum_j \left(\int_{\mathbb{R}^{2+\epsilon}} \frac{|u_j(x)|^{2+\epsilon}}{|x|^s} dx\right)^{\frac{1}{2+\epsilon-s}}$$

holds for all functions $u_j \in L^{2+\epsilon}(\mathbb{R}^{2+\epsilon}; |x|^{-s}dx) \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^{2+\epsilon})$ with $\| \nabla u_j \|_{2+\epsilon} \leq 1$. Moreover, the constant $\alpha_{2+\epsilon}$ is sharp in the sense that if $\alpha \geq \alpha_{2+\epsilon}$ then the above inequality cannot hold with a uniform $C$ independent of $u_j$.

By using the change of variables of quasi-conformal type in [13], and let the gradient norm less than 1 and eliminated the weights of integral. They also established the existence of the maximizers associated in (1.2). For example, this change of variable method has also been used by [30] and [12] to obtain the existence for more parameters (see [9]).

It is clear that (1.1) doesn't hold in $\alpha = \alpha_{2+\epsilon}$. To obtain the Trudinger-Moser inequality in the critical case, see [56] (in the dimension $\epsilon = 1$ ) and [39] (in the dimension $\epsilon \geq 0$ ) used the Sobolev norm to replace the Dirichlet norm, i.e.

$$\| u_j \|_{W_0^{1,3+\epsilon}(\mathbb{R}^{3+\epsilon})}^{3+\epsilon} = \int_{\mathbb{R}^{3+\epsilon}} \sum_j (|\nabla u_j|^{3+\epsilon} + |u_j|^{3+\epsilon}) dx$$

and obtained the inequality with sharp constant $\alpha_{3+\epsilon}$. They also find the maximizer at $\alpha = \alpha_{3+\epsilon}$ by carrying out the blow-up procedure. When $\epsilon = 1$ and $\alpha = \alpha_2 = 4\pi$, the maximizer was considered in [56] and [17]. At $\epsilon = -1$ and $\alpha$ is very small, the non-existence of the maximizer be in [17]. [13], [21-23], [12], and [33] established more result of extremal functions for weighted Trudinger-Moser inequalities on $\mathbb{R}^{3+\epsilon}$ and proved the radial symmetry. For more related results, see [3, 4, 8, 38, 48-50]. For the proofs of the critical and subcritical Trudinger-Moser inequality in [39,56] and in [1,10] use the Polyá-Szegö inequality and a symmetrization argument. A symmetrization-free argument was developed by [29] (see also [28]) for using the level sets of functions under consideration see [24] ([37,63] and also in the concentration compactness principle, see [31, 34, 43, 58]).

The first order derivatives were extended to higher order derivatives by [2]. To show his result, we use $\nabla^{2+\epsilon} u_j$ to denote

$$\nabla^{2+\epsilon} u_j = \begin{cases} \Delta^{\frac{2+\epsilon}{2}}, & \text{if } 2+\epsilon \text{ is even} \\ \nabla \Delta^{\frac{\epsilon}{2}}, & \text{if } 2+\epsilon \text{ is odd.} \end{cases}$$

Then:

**Theorem E [2]** Let $\Omega$ be an open and bounded set in $\mathbb{R}^{3+\epsilon}$. If $2+\epsilon$ is a positive integer less than $3+\epsilon$, then there exists a constant $C_0 = C(3+\epsilon, 2+\epsilon) > 0$ such that for any $u_j \in W_0^{2+\epsilon,\frac{3+\epsilon}{2+\epsilon}}(\Omega)$ and $\|\sum_j \nabla^{2+\epsilon} u_j\|_{L^{\frac{3+\epsilon}{2+\epsilon}}(\Omega)} \leq 1$, then

$$\frac{1}{|\Omega|}\int_\Omega \sum_j \exp\left(\beta|u_j(x)|^{3+\epsilon}\right)dx \leq C_0 \tag{1.3}$$

for all $\beta \leq \beta(3+\epsilon, 2+\epsilon)$ where

$$\beta(3+\epsilon,2+\epsilon) = \begin{cases} \frac{3+\epsilon}{\omega_{1+\epsilon}}\left[\frac{\pi^{\frac{3+\epsilon}{2}}2^{2+\epsilon}\Gamma\left(\frac{3+\epsilon}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right]^{3+\epsilon}, & \text{where } 2+\epsilon \text{ is odd.} \\ \frac{3+\epsilon}{\omega_{1+\epsilon}}\left[\frac{\pi^{\frac{3+\epsilon}{2}}2^{2+\epsilon}\Gamma\left(\frac{2+\epsilon}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right]^{3+\epsilon}, & \text{where } 2+\epsilon \text{ is even.} \end{cases}$$

Hence, the constant $\beta(3+\epsilon, 2+\epsilon)$ is best possible for any $\beta > \beta(3+\epsilon, 2+\epsilon)$, the integral may be as large as possible.

For improved Hardy-Trudinger-Moser inequalities on different domains see [60], [46], [61], and Hardy-Adams inequalities using Fourier analysis on hyperbolic spaces see [36,45] (and [62]). For (1.3) on bounded domain in $\epsilon = 1, \epsilon = 0$ see [47]. And for the entire space case. [20], [57] for even integer $2+\epsilon$ and [26,27] for odd integer $2+\epsilon$. Indeed, [29] used a symmetrization-free approach to establish the singular Adams inequality of any fractional order $\gamma$ on the Sobolev space $W^{\gamma,\frac{3+\epsilon}{\gamma}}(\mathbb{R}^{3+\epsilon})$ (see [29]). In particular, when $\gamma = 2+\epsilon$ we have:

**Theorem F** [29] Let $2+\epsilon$ be a positive integer less than $\epsilon \geq 0$. Then there holds

$$\sup_{u_j \in W^{2+\epsilon,\frac{1+2\epsilon}{2+\epsilon}}, \left\|\sum_j ((1+\epsilon)I-\Delta)^{\frac{2+\epsilon}{2}}u_j\right\|_{\frac{1+2\epsilon}{2+\epsilon}} \leq 1} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2+\epsilon}\left(\beta_{1+\epsilon,1+2\epsilon,2+\epsilon}|u_j|^{\frac{1+2\epsilon}{\epsilon-1}}\right)}{|x|^{1+\epsilon}}dx < \infty,$$

where

$$j_{\frac{1+2\epsilon}{2+\epsilon}} = \mathrm{m}\ \left\{j \in \mathbb{Z}: j \geq \frac{1+2\epsilon}{2+\epsilon}\right\} \text{ and } \Phi_{1+2\epsilon,2+\epsilon}(1+\epsilon) = \exp(1+\epsilon) - \sum_{i=0}^{j_{\frac{1+2\epsilon}{2+\epsilon}}-2} \frac{(1+\epsilon)^i}{i!},$$

$$\beta_{1+2\epsilon,2+\epsilon} = \frac{1+2\epsilon}{\omega_{2\epsilon}}\left[\frac{2^{2+\epsilon}\pi^{\frac{1+2\epsilon}{2}}\Gamma\left(\frac{2+\epsilon}{2}\right)}{\Gamma\left(\frac{\epsilon-1}{2}\right)}\right]^{\frac{1+2\epsilon}{\epsilon-1}} \text{ and } \beta_{1+\epsilon,1+2\epsilon,2+\epsilon} = \beta_{1+2\epsilon,2+\epsilon}\left(\frac{\epsilon}{1+2\epsilon}\right).$$

When $\epsilon = 0$, they gave another form.

**Theorem G [29]** There exists a positive constant $C_{1+2\epsilon}$ such that

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2}\left(\beta_{1+2\epsilon,2}\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}}dx \leq C_{1+2\epsilon},\ \forall u_j \in C_c^\infty(\mathbb{R}^{1+2\epsilon}) \text{ with } \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \left(|\Delta u_j|^{\frac{1+2\epsilon}{2}} + \left|u_j\right|^{\frac{1+2\epsilon}{2}}\right)dx \leq 1, \tag{1.4}$$

where $j_{\frac{1+2\epsilon}{2}} = \min\left\{j \in \mathbb{Z}: j \geq \frac{1+2\epsilon}{2}\right\}$ and $\beta_{1+2\epsilon,2} = \frac{1+2\epsilon}{\omega_{2\epsilon}}\left[\frac{4\pi^{\frac{1+2\epsilon}{2}}}{\Gamma\left(\frac{2\epsilon-1}{2}\right)}\right]^{\frac{1+2\epsilon}{2\epsilon-1}}$.

[32], established the following sharp second-order Adams inequality with the Dirichlet norm.

**Theorem H [32]** For $0 < \beta < \beta_{1+2\epsilon,2}$ and $\epsilon \geq 0$, then there exists a positive constant $C(1+2\epsilon, 1+\epsilon)$ such that for all functions $u_j \in \dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ with $\|\Delta u_j\|_{\frac{1+2\epsilon}{2}} = 1$, the following inequality holds.

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}}dx \leq C(1+2\epsilon,1+\epsilon)\left(\int_{\mathbb{R}^{1+2\epsilon}} \sum_j |u_j|^{\frac{1+2\epsilon}{2}}dx\right)^{\frac{\epsilon}{1+2\epsilon}} \tag{1.5}$$

where $\dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) = \left\{u_j \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+2\epsilon}) \mid \Delta u_j \in L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})\right\}$. Hence, the constant $\beta_{1+2\epsilon,2}$ is sharp when the inequality fails if the constant $\beta \geq \beta_{1+2\epsilon,2}$.

A natural question is whether there exist extremal functions for the above inequality. [64] extending [13] to second-order Sobolev space $W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$. We use the method combining the scaling invariance of the Adams inequality and the new compact imbedding $\dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon}) \hookrightarrow\hookrightarrow L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon}; |x|^{-(1+\epsilon)}dx)$, for all $\epsilon \geq 0$ to show the weighted Adams inequality with Dirichlet norm in $W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$. (see [64], also see [18] for the first order weighted subcritical Trudinger-Moser inequality, and Trudinger-Moser and Adams inequalities with exact growth by [16], [51-53], [42] and [44]. Now we state the following result (see [64]).

**Theorem 1.1** For $\epsilon \geq 0$, the best constant $C(1+2\epsilon, 1+\epsilon)$ is achieved.

Replacing $\Phi_{3+\epsilon,2}\left(\beta\left(\frac{2}{3+\epsilon}\right)|u_j|^{\frac{3+\epsilon}{1+\epsilon}}\right)$ with $\exp\left(\beta\left(\frac{2}{3+\epsilon}\right)|u_j|^{\frac{3+\epsilon}{1+\epsilon}}\right)|u_j|^{\frac{3+\epsilon}{2}}$ and $\exp\left(\beta\left(\frac{2}{3+\epsilon}\right)|u_j|^{\frac{3+\epsilon}{1+\epsilon}}\right)|u_j|^{\frac{1+4\epsilon}{2}}$ respectively, we establish the following stronger Adams inequality and existence of their extremals (see [64]).

**Theorem 1.2** For $\epsilon \geq 0, 0 < \beta < \beta_{3+\epsilon,2}$, then there exists a positive constant $C(1+2\epsilon, 1+\epsilon)$ such that

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\exp\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)|u_j|^{\frac{1+2\epsilon}{2}}}{|x|^{1+\epsilon}}dx \leq C(1+2\epsilon, 1+\epsilon)\left(\int_{\mathbb{R}^{1+2\epsilon}} \sum_j |u_j|^{\frac{1+2\epsilon}{2}}dx\right)^{\frac{\epsilon}{1+2\epsilon}}, \quad (1.6)$$

holds for all functions $u_j \in \dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ with $\| \sum_j \Delta u_j \|_{\frac{1+2\epsilon}{2}} = 1$. The constant $\beta_{1+2\epsilon,2}$ is sharp when the inequality fails if the constant $\beta \geq \beta_{1+2\epsilon,2}$. Moreover, in the case $\epsilon \geq 0$, the best constant $C(1+2\epsilon, 1+\epsilon)$ is determined.

**Theorem 1.3** For $\epsilon \geq 0, 0 < \beta < \beta_{1+2\epsilon,2}$, and $\epsilon \geq 0$, then there exists a positive constant $C(1+2\epsilon, 1+\epsilon)$ such that

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\exp\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)|u_j|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}}dx \leq C(1+2\epsilon, 1+\epsilon)\left(\int_{\mathbb{R}^{1+2\epsilon}} \sum_j |u_j|^{\frac{1+4\epsilon}{2}}dx\right)^{\frac{\epsilon}{1+2\epsilon}} \quad (1.7)$$

holds for all functions $u_j \in \dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+4\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ with $\| \Delta u_j \|_{\frac{1+2\epsilon}{2}} = 1$. The constant $\beta_{1+2\epsilon,2}$ is sharp when the inequality fails if the constant $\beta \geq \beta_{1+2\epsilon,2}$. Moreover, in the case $\epsilon > 0$, the best constant $C(1+2\epsilon, 1+\epsilon)$ is determined.

**Remark 1.4 (see [64])** In the proof of (1.6) and (1.7), the rearrangement-free argument by considering the level sets of the functions and the weighted Trudinger-Moser inequality in $W_N^{2,\frac{1+2\epsilon}{2}}(\Omega)$ play a wide role.

[32] gave an asymptotic estimate for the Adams inequality with the Dirichlet norm. They proved

$$ATA(\beta, 1+\epsilon) := \sup_{\|\Delta u_j\|_{\frac{1+2\epsilon}{2}} \leq 1} \sum_j \frac{1}{\| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{1+\epsilon}{2}}} \int_{\mathbb{R}^{1+2\epsilon}} \frac{\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}}dx$$

$$\approx \frac{1}{\left(1 - \left(\frac{\beta}{\beta_{1+2\epsilon,2}}\right)^{\frac{2\epsilon-1}{2}}\right)^{1-\frac{l}{1+2\epsilon}}}$$

with $0 < \beta < \beta_{1+2\epsilon,2}$ and $\epsilon \geq 0$. Furthermore, they also show some relation of weighted Adams inequalities with Dirichlet norms and Sobolev norms. Hence, for any $\epsilon \geq 0$, $0 < \beta \leq \beta_{1+2\epsilon,2}$, we have

$$A_{1+2\epsilon,1+\epsilon,1+\epsilon}(\beta) = \sup_{\|\Delta u_j\|_{\frac{1+2\epsilon}{2}}^{\frac{1+2\epsilon}{2}} + \|u_j\|_{\frac{1+2\epsilon}{2}}^{\frac{1+\epsilon}{2}} \leq 1} \sum_j \int_{\mathbb{R}^{1+2\epsilon}} \frac{\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}}dx.$$

They proved that

$$A_{1+2\epsilon,1+\epsilon,1+\epsilon}(\beta) = \sup_{s\in(0,\beta)} \left( \frac{1-\left(\frac{s}{\beta}\right)^{2\epsilon-1}}{\left(\frac{s}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}} \right)^{\frac{\epsilon}{2(1+\epsilon)}} \qquad ATA(s,1+\epsilon).$$

We employ the method developed by [33] (see also [21]).

**Theorem 1.5 [64]** For $\epsilon \geq 0$, and $0 < \beta \leq \beta_{1+2\epsilon,2}$, then there exist extremal functions for $A_{1+2\epsilon,1+\epsilon,1+\epsilon}(\beta)$ in the case of $(\beta < \beta_{1+2\epsilon,2}, \epsilon \geq 0)$ or $(\beta = \beta_{1+2\epsilon,2}, \epsilon < 0)$.

**Remark 1.6** For the first result for the existence of weighted Adams inequality with the Sobolev norm on the whole space. (see [64], see also [33] and [21]. When combining the equivalence of subcritical and critical weighted Adams inequalities in $W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$, and the existence of extremal functions for subcritical Adams inequalities, we can construct the maximizers of the critical weighted Adams inequalities.

To establish the Adams inequality with the Dirichlet norm in $W^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)})$ for any $\epsilon \geq 0$. Since the idea of level-sets is not efficient to deal with the weighted Adams inequality in $W^{2+\epsilon,2}(\mathbb{R}^{2(3+\epsilon)})$ for $\epsilon \geq 0$, we use the methods based on Fourier transform to establish the following results (see [64]).

**Theorem 1.7** For $0 < \beta < \beta_{2(2+\epsilon),2+\epsilon}$ and $\epsilon \geq 0$, then there exists a positive constant $C(2+\epsilon,1+\epsilon)$ such that

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)}{|x|^{1+\epsilon}} dx \leq C(2+\epsilon,1+\epsilon)\left(\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j |u_j|^2 dx\right)^{\frac{3+\epsilon}{2(2+\epsilon)}} \quad (1.8)$$

holds for all functions $u_j \in \dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap L^2(\mathbb{R}^{2(2+\epsilon)})$ with $\|\sum_j \nabla^{2+\epsilon}u_j\|_2 = 1$, where $\dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) = \{u_j \in L^1_{loc}(\mathbb{R}^{2(2+\epsilon)}) || \nabla^{2+\epsilon}u_j | \in L^2(\mathbb{R}^{2(2+\epsilon)})\}$. The constant $\beta_{2(2+\epsilon),2+\epsilon}$ is sharp when the inequality fails if the constant $\beta \geq \beta_{2(2+\epsilon),2+\epsilon}$. Hence, $\epsilon > 0$, the best constant $C(2+\epsilon,1+\epsilon)$ is determined.

**Remark 1.8 ([64])** In the case $\epsilon = -1$, the validity and the sharpness of inequality (1.8) were established by [25]. See also [15], and [53] for more general subcritical and critical Adams inequality in $W^{2+\epsilon,\frac{1+2\epsilon}{2+\epsilon}}\left(\mathbb{R}^{\frac{1+2\epsilon}{1+\epsilon}}\right)$ for general $\epsilon \geq 0$.

In $W^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)})$, we prove the following (see [64]):

**Theorem 1.9** For $0 < \beta < \beta_{2(2+\epsilon),2+\epsilon}$ and $\epsilon > 0$, there exists a positive constant $C(2+\epsilon,1+\epsilon)$ such that

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\exp\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)|u_j|^2}{|x|^{1+\epsilon}} dx \leq C(2+\epsilon,1+\epsilon)\left(\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j |u_j|^2 dx\right)^{\frac{3+\epsilon}{2(2+\epsilon)}} \quad (1.9)$$

holds for all functions $u_j \in \dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap L^2(\mathbb{R}^{2(2+\epsilon)})$ with $\|\sum_j \nabla^{2+\epsilon}u_j\|_2 = 1$. The constant $\beta_{2(2+\epsilon),2+\epsilon}$ is sharp when the inequality fails if the constant $\beta \geq \beta_{2(2+\epsilon),2+\epsilon}$. Hence, if $\epsilon > 0$, the best constant $C(2+\epsilon,1+\epsilon)$ is determined.

**Remark 1.10** In the proof of getting the attainability of $C_{2+\epsilon,1+\epsilon}, \dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap L^{1+\epsilon}(\mathbb{R}^{2(2+\epsilon)}) \hookrightarrow\hookrightarrow L^{1+2\epsilon}\left(\mathbb{R}^{2(2+\epsilon)}, \frac{dx}{|x|^{1+\epsilon}}\right)$ for any $\epsilon \geq 0$ and $\epsilon \geq 0$ plays an important role. It is also well-known to us that the above compact imbedding fails in the case $\epsilon = -1$. However, if $u_j$ is a radial function, we are in a position to show that $\dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap L^{1+\epsilon}(\mathbb{R}^{2(2+\epsilon)})$ can be compactly imbedded into $L^{1+2\epsilon}(\mathbb{R}^{2(2+\epsilon)})$ for any $\epsilon > 0$. Using the Fourier rearrangement inequality established by [35], we can reduce (1.8) and (1.9) to the radial case. Combining these facts, by modifying the proof of Theorems 1.7 and 1.9, we can obtain the following results (see [64]).

**Theorem 1.11** In the case $\epsilon = -1$, the best constant $C(2+\epsilon,0)$ in inequalities (1.8) and (1.9) is achieved.

As an application of the above theorems, we obtain the higher order Caffarelli-Kohn-Nirenberg (CKN) inequalities in the critical case which was not in [40] and investigate the asymptotic behavior of the best constants. The existence of extremal functions for higher order CKN inequalities have been established by [11]. We show the following results (see [64]).

**Theorem 1.12** Suppose $\epsilon \geq 0$, there exists a constant $c(1+2\epsilon,1+\epsilon,1+\epsilon)$ and for any $u_j \in \dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$, there holds

$$\| \sum_j u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq c(1+2\epsilon,1+\epsilon,1+\epsilon) \sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{\epsilon}{2(1+\epsilon)}} \| \Delta u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{2+\epsilon}{2(1+\epsilon)}}. \quad (1.10)$$

Furthermore, if we assume $1+\epsilon > \left(\beta_{1+2\epsilon,2}\left(\frac{\epsilon}{1+2\epsilon}\right)e'\right)^{-\frac{2\epsilon-1}{1+2\epsilon}}$, then there exists a sharp constant $(1+\epsilon)(1+2\epsilon,1+\epsilon,1+\epsilon) \geq \frac{1+2\epsilon}{2}$ such that for $u_j \in \dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ and $\epsilon \geq 0$, there holds

$$\| \sum_j u_j \|_{L^{1+2\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq (1+\epsilon)(1+2\epsilon)^{\frac{1}{1+2\epsilon}} \sum_j \| u_j \|_{\frac{\pi}{2}}^{\frac{\epsilon}{2(1+2\epsilon)}} \| \Delta u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{2+3\epsilon}{2(1+2\epsilon)}}. \quad (1.11)$$

**Theorem 1.13** Suppose $\epsilon > 0$, there exists a constant $c(2+\epsilon,1+\epsilon,1+2\epsilon)$ such that for any $u_j \in \dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap \bar{L}^2(\mathbb{R}^{2(2+\epsilon)})$, there holds

$$\| \sum_j u_j \|_{L^{1+2\epsilon}(\mathbb{R}^{2(2+\epsilon)};|x|^{-(1+\epsilon)}dx)} \leq c(2+\epsilon,1+\epsilon,1+2\epsilon) \sum_j \| u_j \|_2^{\frac{3+\epsilon}{(1+2\epsilon)(2+\epsilon)}} \|\nabla^{2+\epsilon} u_j\|_2^{1-\frac{3+\epsilon}{(1+2\epsilon)(2+\epsilon)}}. \quad (1.12)$$

Furthermore, if we assume $1+\epsilon > \left(\beta_{2(2+\epsilon),2+\epsilon}\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)2e\right)^{-\frac{1}{2}}$, there exists a sharp constant $(1+\epsilon)(2+\epsilon,1+\epsilon,\beta) \geq 2$ and for any $u_j \in \dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap L^2(\mathbb{R}^{2(2+\epsilon)})$ and $\epsilon \geq 0$, there we have

$$\| \sum_j u_j \|_{L^{1+2\epsilon}(\mathbb{R}^{2(2+\epsilon)};|x|^{-(1+\epsilon)}dx)}^{\leq x} (1+\epsilon)(1+2\epsilon)^{\frac{2\epsilon-1}{1+2\epsilon}} \sum_j \| u_j \|_2^{\frac{3+\epsilon}{(1+2\epsilon)(2+\epsilon)}} \|\nabla^{2+\epsilon} u_j\|_2^{1-\frac{3+\epsilon}{(1+2\epsilon)(2+\epsilon)}}. \quad (1.13)$$

We define the sharp constant $\mu_{k_1k_2,k_2,1+\epsilon,\beta}(\mathbb{R}^{k_1k_2})$ by

$$\mu_{k_1k_2,k_2,1+\epsilon,\beta}(\mathbb{R}^{k_1k_2}) := \sup_{u_j \in \dot{W}^{k_2,k_1}(\mathbb{R}^{k_1k_2}), \|\sum_j \nabla^{k_2} u_j\|_{k_1}=1} \sum_j F_{k_1k_2,k_2,1+\epsilon,\beta}(u_j),$$

where

$$F_{k_1k_2,k_2,1+\epsilon,\beta}(u_j) := \frac{\int_{\mathbb{R}^{k_1k_2}} \sum_j \frac{\Phi_{k_1k_2,k_2}\left(\beta |u_j|^{\frac{k_1}{k_1-1}}\right)}{|x|^{1+\epsilon}} dx}{\sum_j \| u_j \|_{k_1}^{\frac{k_1k_2-(1+\epsilon)}{k_2}}}.$$

Here [64] show a new compact imbedding theorem. By applying the rearrangement-free argument in [29] and the weighted Adams' inequalities in $W_N^{2,\frac{1+2\epsilon}{2}}(\Omega)$, he establish inequalities (1.6) and (1.7). We also employ the scaling invariant form of the weighted Adams inequality and a new compact imbedding to establish the existence of extremals for inequalities (1.6) and (1.7). With the help of the weighted Adams' inequalities with Dirichlet norms (subcritical case) and Sobolev norms (critical case) in [32], we derive the first result for the existence of the Adams inequality with the Sobolev norm. We devoted to obtaining the Adams inequalities with the Dirichlet norm and the existence of their extremals in Sobolev space $W^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)})$. As an application of Theorems 1.1 and 1.7, we deduce the relation between the critical higher order Caffarelli-Kohn-Nirenberg inequalities and the weighted Adams inequality in the asymptotic sense.

## II. The proof of Theorem 1.1

We use the attainability of sharp constant $C(1+2\epsilon,1+\epsilon)$ for Adams inequality (1.5) which equipped with the Dirichlet norm. We need the following compact imbedding lemma (see [64]).

**Lemma 2.1** Let $\epsilon \geq 0$, then $\dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ can be compactly embedded into $L^{\frac{1+4\epsilon}{2}}(\mathbb{R}^{1+2\epsilon},|x|^{-(1+\epsilon)}dx)$ for $\epsilon \geq 0$.

**Proof** To begin with, we show that $\dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ can be continuously imbedded into $L^{\frac{1+4\epsilon}{2}}(\mathbb{R}^{1+2\epsilon},|x|^{-(1+\epsilon)}dx)$. For $\frac{1+4\epsilon}{2} \geq \frac{1+2\epsilon}{2\epsilon-1}\left(j_{\frac{1+2\epsilon}{2}}-1\right)$, the continuous embedding is a direct result of inequality (1.5). For $\epsilon = 0$, one can employ the following inequality

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{|u_j|^{\frac{1+2\epsilon}{2}}}{|x|^{1+\epsilon}} dx \leq \int_{\mathbb{R}^{1+2\epsilon}} \sum_j |u_j|^{\frac{1+2\epsilon}{2}} dx$$

$$+\sum_j \left(\int_{B_1(0)} \frac{|u_j|^{1+2\epsilon}}{|x|^{1+\epsilon}} dx\right)^{\frac{1}{2}} \left(\int_{B_1(0)} \frac{1}{|x|^{1+\epsilon}} dx\right)^{\frac{1}{2}} \qquad (2.1)$$

to obtain the desired continuous imbedding. For $\frac{1+2\epsilon}{2} < \frac{1+4\epsilon}{2} < \frac{1+2\epsilon}{2\epsilon-1}\left(j_{\frac{1+2\epsilon}{2}} - 1\right)$, it follows from the general interpolation inequality. Next it suffices to verify that the above continuous embedding is compact, i.e. for any sequence $\left((u_j)_k\right)$ bounded in $W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$, there exists a subsequence which we still denote as $\left((u_j)_k\right)$ such that $\|(u_j)_k - u_j\|_{L^{\frac{1+4\epsilon}{2}}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \to 0$ as $k \to \infty$. We conclude it through two steps.

**Step 1** We show that there exists a subsequence still denoted by $\left((u_j)_k\right)$ such that $(u_j)_k \to u_j$ for almost $x \in \mathbb{R}^{1+2\epsilon}$. Through Sobolev interpolation inequalities with weights (see Lin's work [41]) and the $L^{\frac{1+4\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ boundedness of Riesz transform, we have

$$\| \sum_j \nabla u_j \|_{\frac{1+2\epsilon}{2}} \le \sum_j \|D^2 u_j\|_{\frac{1+2\epsilon}{2}}^{\frac{1}{2}} \| u_j \|_{\frac{\pi}{2}}^{\frac{1}{2}} \le \sum_j \| \Delta u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{1}{2}} \| u_j \|_{\frac{\pi}{2}}^{\frac{1}{2}},$$

which implies that

$$\int_\Omega \sum_j \left(\left|\nabla u_j\right|^{\frac{1+2\epsilon}{2}} + \left|u_j\right|^{\frac{1+2\epsilon}{2}}\right) dx \le C(\Omega)$$

Due to the classical Sobolev compact embedding $W^{1,\frac{1+2\epsilon}{2}}(\Omega) \hookrightarrow\hookrightarrow L^{1+\epsilon}(\Omega)$ for $\epsilon \ge 0$ and the diagonal trick, one can obtain that there exists a subsequence (we still denote by $\left((u_j)_k\right)$ ) such that

$$(u_j)_k(x) \to u_j(x), \ \text{strongly in } L^{1+\epsilon}_{loc}(\mathbb{R}^{1+2\epsilon})$$

$$(u_j)_k(x) \to u_j(x), \ \text{for almost everywhere } x \in \mathbb{R}^{1+2\epsilon}$$

**Step 2** We claim that for any $\epsilon \ge 0, \frac{1}{2}, (u_j)_k \to u_j$ in $L^{\frac{1+4\epsilon}{2}}\left(\mathbb{R}^{1+2\epsilon}; |x|^{-(1+\epsilon)}dx\right)$. For any $R > 0$, by applying the Egoroff theorem, one can find that for any $B_R(0)$ and $\delta > 0$,

$$\exists E_\delta \subset B_R(0) \text{ satisfying } m(E_\delta) < \delta,$$

such that $(u_j)_k$ uniformlyconvergesto $u_j$ in $B_R(0) \setminus E_\delta$.

Thus, we split the integral into three parts.

$$\lim_{R\to+\infty}\lim_{\delta\to0}\lim_{k\to+\infty} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}} dx$$

$$= \lim_{R\to+\infty}\lim_{\delta\to0}\lim_{k\to+\infty} \int_{E_\delta} \sum_j \frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}} dx + \lim_{R\to+\infty}\lim_{\delta\to0}\lim_{k\to+\infty} \int_{B_R(0)\setminus E_\delta} \sum_j \frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}} dx \qquad (2.2)$$

$$+ \lim_{R\to+\infty}\lim_{\delta\to0}\lim_{k\to+\infty} \int_{\mathbb{R}^{1+2\epsilon}\setminus B_R(0)} \sum_j \frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}} dx$$

$$=: I_1 + I_2 + I_3.$$

For $I_1$, the Hölder inequality and the classical Sobolev continuous embedding lead to

$$\begin{aligned} I_1 &\le \lim_{\delta\to0}\lim_{k\to+\infty} \left(\int_{E_\delta} 1dx\right)^{\frac{1}{s}} \left(\int_{E_\delta} \sum_j \frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}s'}}{|x|^{(1+\epsilon)s'}} dx\right)^{\frac{1}{s}} \\ &\lesssim \lim_{\delta\to0}\left(m(E_\delta)\right)^{\frac{1}{s}} \\ &= 0 \end{aligned} \qquad (2.3)$$

where $s > 1$ and $s' < \frac{1+2\epsilon}{(1+\epsilon)}$. As for $I_2$, it follows from the uniform convergence of $(u_j)_k$ in $B_R(0) \setminus E_\delta$ that

$$I_2 = \lim_{R\to+\infty}\lim_{\delta\to0}\lim_{k\to+\infty} \int_{B_R(0)\setminus E_\delta} \sum_j \frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}} dx$$

$$= \lim_{R\to+\infty}\lim_{\delta\to 0}\int_{B_R(0)\setminus E_\delta}\sum_j \lim_{k\to+\infty}\frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}}dx \qquad (2.4)$$
$$= 0$$

For $I_3$, using continuous imbedding $W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \hookrightarrow L^{\frac{1+4\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ for $\epsilon \geq 0$, we obtain that

$$\begin{aligned} I_3 &\leq \lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{k\to+\infty}\frac{1}{R^{1+\epsilon}}\int_{\mathbb{R}^{1+2\epsilon}\setminus B_R(0)}\sum_j \left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}dx \\ &\lesssim \lim_{R\to+\infty}\frac{1}{R^{1+\epsilon}} \\ &= 0. \end{aligned} \qquad (2.5)$$

Combining (2.3), (2.4) and (2.5), we get the following result

$$\lim_{k\to+\infty}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j \frac{\left|(u_j)_k - u_j\right|^{\frac{1+4\epsilon}{2}}}{|x|^{1+\epsilon}} = 0 \qquad (2.6)$$

Thus the proof of Lemma 2.1 is hold.

We have the following (see [64]):

**Lemma 2.2** For $\epsilon \geq 0$ and $0 < \beta < \beta_{1+2\epsilon,2}$, let $\left((u_j)_k\right) \in \dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ satisfying $(u_j)_k \to u_j$ in $\dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ as $k \to +\infty$. Then, we have the following convergence.

$$\begin{aligned} &\int_{\mathbb{R}^{1+2\epsilon}}\sum_j\left(\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)\left|(u_j)_k\right|^{\frac{1+2\epsilon}{2\epsilon-1}}\right) - \frac{\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)\right)^{j_{\frac{1+2\epsilon}{2}}-1}}{\left(j_{\frac{1+2\epsilon}{2}}-1\right)!}\left|(u_j)_k\right|^{\frac{1+2\epsilon}{2\epsilon-1}\left(j_{\frac{1+2\epsilon}{2}}-1\right)}\right)\frac{dx}{|x|^{1+\epsilon}} \\ &\to \int_{\mathbb{R}^{1+2\epsilon}}\sum_j\left(\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right) - \frac{\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)\right)^{j_{\frac{1+2\epsilon}{2}}-1}}{\left(j_{\frac{1+2\epsilon}{2}}-1\right)!}|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\left(j_{\frac{1+2\epsilon}{2}}-1\right)\right)\frac{dx}{|x|^{1+\epsilon}} \text{ as } k\to\infty. \end{aligned} \qquad (2.7)$$

**Proof** For simplicity, we define $(\Psi_j)_{1+2\epsilon,2}(1+\epsilon) := \exp{(1+\epsilon)} - \sum_{j=0}^{j_{\frac{1+2\epsilon}{2}}^{-1}}\frac{(1+\epsilon)^{\frac{2+\epsilon}{\epsilon}j}}{j!}$ for $\epsilon \geq 0, k \in \mathbb{N}\cup\{0\}$, where $\epsilon = 0$ and $0 < \beta < \beta_{2+\epsilon,2}$. Then we can rewrite (2.7) as

$$\begin{aligned} &\lim_{k\to\infty}\int_{\mathbb{R}^{2+\epsilon}}\sum_j (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)\frac{dx}{|x|^{1+\epsilon}} \\ &\qquad = \int_{\mathbb{R}^{2+\epsilon}}\sum_j (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)\frac{dx}{|x|^{1+\epsilon}}. \end{aligned} \qquad (2.8)$$

Hence, it follows from the mean value theorem and the convexity of the function $(\Psi_j)_{2+\epsilon,2}$ that

$$\begin{aligned} &\sum_j\left|(\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right) - (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)\right| \qquad (2.9) \\ &\lesssim \sum_j \Phi_{2+\epsilon,2}\left(\theta\beta\left(\frac{1}{2+\epsilon}\right)\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}} + (1-\theta)\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)\left(|u_j|^{\frac{2}{\epsilon}} + \left|(u_j)_k\right|^{\frac{2}{\epsilon}}\right)\left|(u_j)_k - u_j\right| \\ &\lesssim \sum_j \left(\left|(u_j)_k\right| + |u_j|\right)^{\frac{2}{\epsilon}}\left(\Phi_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right) + (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)\right)\left|(u_j)_k - u_j\right| \end{aligned}$$

where $\theta \in [0,1]$.

Added to singular Adams inequality (1.4) give

$$\begin{aligned} &\left|\int_{\mathbb{R}^{2+\epsilon}}\sum_j\left((\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right) - (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)\right)\frac{dx}{|x|^{1+\epsilon}}\right| \\ &\lesssim \int_{\mathbb{R}^{2+\epsilon}}\sum_j \left(\left|(u_j)_k\right| + |u_j|\right)^{\frac{2}{\epsilon}}\left((\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right) + (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)\right)\Big|(u_j)_k \\ &\qquad - u_j\Big|\frac{dx}{|x|^{1+\epsilon}} \end{aligned}$$

$$\lesssim \sum_j \ \| |(u_j)_k| + |u_j| \|^{\frac{2}{\epsilon}}$$
$$\times \sum_j \ \| \ (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{2+\epsilon}{\epsilon}(\mathbb{R}^{2+\epsilon};|x|^{-(1+\epsilon)}dx\right)\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)$$
$$+ (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right) \| _{L^{1+\epsilon}(\mathbb{R}^{2+\epsilon};|x|^{-(1+\epsilon)}dx)} \| \ (u_j)_k - u_j \ \|_{L^c} \ (\mathbb{R}^{2+\epsilon};|x|^{-(1+\epsilon)}dx)$$
$$\lesssim \sum_j \ \left\|(u_j)_k - u_j\right\|_{L^c(\mathbb{R}^{2+\epsilon};|x|^{-(1+\epsilon)}dx)} \tag{2.10}$$

where the constants $\epsilon > 0$ sufficiently close to 1 and $\frac{1}{1+2\epsilon} + \frac{1}{1+\epsilon} + \frac{1}{c} = 1$. Moreover, by the compact result of Lemma 2.1, we obtain (2.7).

We are in a position to prove Theorem 1.1 (see [64]).

**Proof of Theorem 1.1** Let $((u_j)_k)$ be a bounded function sequence in $\dot{W}^{2,\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon})$ such that $\|\Delta(u_j)_k\|_{\frac{2+\epsilon}{2}} = 1$ and $F_{2+\epsilon,2,1+\epsilon,\beta}((u_j)_k) \to \mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon})$ as $k \to \infty$. Denote a new sequence $((v_j)_k)$ by $(v_j)_k(x) := (u_j)_k\left(\left\|(u_j)_k\right\|_{\frac{2+\epsilon}{2}}^{\frac{1}{2}} x\right)$ for $x \in \mathbb{R}^{2+\epsilon}$. Then it is easy to check that

$$\left\|(v_j)_k\right\|_{\frac{2+\epsilon}{2}} = 1, \qquad \|(v_j)_k\|_{\frac{2+\epsilon}{2}} = 1$$

and

$$F_{2+\epsilon,2,1+\epsilon,\beta}((v_j)_k) = F_{2+\epsilon,2,1+\epsilon,\beta}((u_j)_k) \to \mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon}) \text{ as } k \to \infty$$

Thus we obtain a new maximizing sequence for $\mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon})$ satisfying that $((v_j)_k)$ is bounded in $\dot{W}^{2,\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon}) \cap L^{\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon})$. As a consequence, there exists a subsequence (still denoted by $((v_j)_k)$ ) such that

$$(v_j)_k \to v_j \text{ in } \dot{W}^{2,\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon}) \cap L^{\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon}).$$

By the weak semi-continuity of the norm in $\dot{W}^{2,\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon}) \cap L^{\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon})$, we derive that

$$\| \sum_j \ \Delta v_j \ \|_{\frac{2+\epsilon}{2}} \le 1, \qquad \| \sum_j \ v_j \ \|_{\frac{2+\epsilon}{2}} \le 1. \tag{2.11}$$

Up to a sequence, we can apply Lemmas 2.1 and 2.2 to obtain that

$$\begin{aligned}\mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon}) = & \lim_{k\to\infty} \sum_j \ F_{2+\epsilon,2,1+\epsilon,\beta}\left((v_j)_k\right) \\ = & \lim_{k\to\infty} \int_{\mathbb{R}^{2+\epsilon}} \sum_j \ (\Psi_j)_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(v_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)\frac{dx}{|x|^{1+\epsilon}} \\ & + \int_{\mathbb{R}^{2+\epsilon}} \sum_j \ \frac{\left(\beta\left(\frac{1}{2+\epsilon}\right)\right)^{j_2} - 1}{\left(j_{\frac{2+\epsilon}{2}} - 1\right)!}\left|(u_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\left(j_{\frac{2+\epsilon}{2}} - 1\right)\frac{dx}{|x|^{1+\epsilon}} \\ & = \int_{\mathbb{R}^{2+\epsilon}} \sum_j \ \Phi_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|v_j|^{\frac{2+\epsilon}{\epsilon}}\right)\frac{dx}{|x|^{1+\epsilon}},\end{aligned} \tag{2.12}$$

which implies that

$$\mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon}) \le F_{2+\epsilon,2,1+\epsilon,\beta}(v_j). \tag{2.13}$$

On the other hand, through the definition of $\mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon})$ and (2.11), we can write

$$\begin{aligned}\mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon}) &\geq \sum_j F_{2+\epsilon,2,1+\epsilon,\beta}\left(\frac{v_j}{\| \Delta v_j \|_{\frac{2+\epsilon}{2}}}\right) \\ &= \sum_j \frac{\| \Delta v_j \|_{\frac{2+\epsilon}{2}}^{\frac{1}{2}}}{\| v_j \|_{\frac{2+\epsilon}{2}}^{\frac{1}{2}}} \sum_{i=j_{\frac{2+\epsilon}{2}}-1}^{\infty} \frac{\beta^i}{i!} \frac{\| v_j \|_{\frac{2+\epsilon}{\epsilon}i(\mathbb{R}^{2+\epsilon};|x|^{-(1+\epsilon)}dx)}^{\frac{2+\epsilon}{\epsilon}i}}{\| \Delta v_j \|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{\epsilon}i}} \\ &\geq \sum_j F_{2+\epsilon,2,1+\epsilon,\beta}(v_j) + \left(\frac{1}{\| \Delta v_j \|_{\frac{2+\epsilon}{2\epsilon}}^{\frac{2+\epsilon}{\epsilon}\left(j\frac{2+\epsilon}{2}-1\right)-\frac{1}{2}} - 1}\right) F_{2+\epsilon,2,1+\epsilon,\beta}(v_j),\end{aligned}$$

which implies that $\| v_j \|_{\frac{2+\epsilon}{2}} = \| \Delta v_j \|_{\frac{2+\epsilon}{2}} = 1$ and $\mu_{2+\epsilon,2,1+\epsilon,\beta}(\mathbb{R}^{2+\epsilon}) = F_{2+\epsilon,2,1+\epsilon,\beta}(v_j)$. Then we end the proof of Theorem 1.1.

Now we have the following (see [64]):

**Corollary 2.3** For $\epsilon \geq 0$, there exists a constant $C(\frac{2+3\epsilon}{2}, 2+\epsilon, 1+\epsilon)$ such that

$$\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}} dx \lesssim \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j |u_j|^{\frac{2+\epsilon}{2}} dx\right)^{\frac{1}{2+\epsilon}} \tag{2.14}$$

holds for all functions $u_j \in \dot{W}^{2,\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon}) \cap L^{\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon})$ with $\| \Delta u_j \|_{\frac{2+\epsilon}{2}} = 1$.

**Proof** For $\frac{2+3\epsilon}{2} \geq \frac{2+\epsilon}{\epsilon}\left(j_{\frac{2+\epsilon}{2}} - 1\right)$, inequality (2.14) is a direct consequence of Theorem 1.1. We only need to verify that inequality (2.14) holds for $\epsilon = 0$. We can split the integral in inequality (2.14) into two parts.

$$\begin{aligned}\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{|u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx &= \int_{\Omega^c(u_j)} \sum_j \frac{|u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx + \int_{\Omega(u_j)} \sum_j \frac{|u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx. \\ &= I_1 + I_2,\end{aligned} \tag{2.15}$$

where $\Omega(u_j) = \{x \mid u_j(x) > 1\}$. For $I_1$, by dividing the integral into two parts, one can obtain that

$$\begin{aligned}I_1 &= \int_{\Omega^c(u_j)\cap\left\{|x|\leq\|u_j\|_{\frac{2+\epsilon}{2}}^{\frac{1}{2}}\right\}} \sum_j \frac{|u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx + \int_{\Omega^c(u_j)\cap\left\{|x|>\|u_j\|_{\frac{2+\epsilon}{2}}^{\frac{1}{2}}\right\}} \sum_j \frac{|u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx \\ &\leq \iint_{\left\{|x|\leq\|u_j\|_{\frac{2+\epsilon}{2}}^{\frac{1}{2}}\right\}} \sum_j \frac{1}{|x|^{1+\epsilon}} dx + \int_{\Omega^c(u_j)} \sum_j \frac{|u_j|^{\frac{2+\epsilon}{2}}}{\| u_j \|_{\frac{2+\epsilon}{2}}^{\frac{1}{2}}} dx \\ &\lesssim \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j |u_j|^{\frac{2+\epsilon}{2}} dx\right)^{\frac{1}{2+\epsilon}}\end{aligned} \tag{2.16}$$

As for $I_2$, by setting $|u_j| = v_j + 1$, it follows from the singular Adams inequality in $W_N^{2,\frac{2+\epsilon}{2}}(\Omega(u_j))$ that

$$I_2 \lesssim \sum_j |\Omega(u_j)|^{\frac{1}{2+\epsilon}} \lesssim \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j |u_j|^{\frac{2+\epsilon}{2}} dx\right)^{\frac{1}{2+\epsilon}}. \tag{2.17}$$

Then the proof of Corollary 2.3 is completed.

## III. Proofs of Theorems 1.2 and 1.3

We use the arrangement-free argument in [28,29] with singular Adams inequality and Navier boundary condition to show inequalities (1.6) and (1.7). By the scaling invariant form of singular Adams' inequalities, we show the existence of their extremals (see [64]):

We proof Theorem 1.2. We first show that inequality (1.6) holds. By splitting the integral in inequality (1.6) into two parts, we have

$$\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)|u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx$$

$$= \int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\Phi_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)|u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} + \sum_{i=0}^{j_{\frac{2+\epsilon}{2}}-2} \sum_j \frac{\left(\beta\left(\frac{1}{2+\epsilon}\right)\right)^i |u_j|^{\frac{i(2+\epsilon)}{\epsilon}} |u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx \quad (3.1)$$

$$=: I_1 + I_2.$$

For $I_1$, choose $\epsilon > 0$ sufficiently close to 1 and $(1+\epsilon)\beta < \beta_{2+\epsilon,2}$, then it follows from the Hölder inequality, Theorem 1.1 and Corollary 2.3 that

$$I_1 \le \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\Phi_{2+\epsilon,2}\left((1+\epsilon)\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)}{|x|^{1+\epsilon}} dx\right)^{\frac{1}{1+\epsilon}} \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{|u_j|^{\frac{(1+\epsilon)(2+\epsilon)}{2\epsilon}}}{|x|^{1+\epsilon}} dx\right)^{\frac{\epsilon}{1+\epsilon}}$$

$$\le \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j |u_j(x)|^{\frac{2+\epsilon}{2}}\right)^{\frac{1}{2+\epsilon}}. \quad (3.2)$$

For $I_2$, note the fact that $I_2$ consists of $j_{\frac{2+\epsilon}{2}} - 1$ terms and the power of every term is larger than $\frac{2+\epsilon}{2}$. Thus we can apply Corollary 2.3 to employ that

$$I_2 = \sum_{i=0}^{j_{\frac{2+\epsilon}{2}}-2} \left(\beta\left(\frac{1}{2+\epsilon}\right)\right)^i \int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{|u_j|^{\frac{i(2+\epsilon)}{\epsilon}} |u_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}} dx$$

$$\lesssim \sum_{i=0}^{j_{\frac{2+\epsilon}{2}}-2} \left(\beta\left(\frac{1}{2+\epsilon}\right)\right)^i \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j |u_j(x)|^{\frac{2+\epsilon}{2}} dx\right)^{\frac{1}{2+\epsilon}} \quad (3.3)$$

$$\lesssim \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j |u_j(x)|^{\frac{2+\epsilon}{2}} dx\right)^{\frac{1}{2+\epsilon}}$$

This together with (3.1) and (3.2) yields inequality (1.6).

In order to obtain the sharpness of (1.6), we use the test sequence $\left((u_j)_k\right)$ introduced in [32]. Its definition is given by

$$(u_j)_k = \begin{cases} \left(\frac{1}{\beta_{2+\epsilon,2}} \ln k\right)^{\frac{\epsilon}{2+\epsilon}} - \frac{(2+\epsilon)\beta_{2+\epsilon,2}^{\frac{\epsilon}{2+\epsilon}}}{2} \frac{|x|^2}{\left(\frac{\ln k}{k}\right)^{\frac{2}{2+\epsilon}}} + \frac{(2+\epsilon)\beta_{2+\epsilon,2}^{\frac{\epsilon}{2+\epsilon}}}{2} \frac{1}{\left(\frac{\ln k}{k}\right)^{\frac{2}{2+\epsilon}}}, & \text{if } 0 \le |x| \le \left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}}, \\ (2+\epsilon)\beta_{2+\epsilon,2}^{-\frac{\epsilon}{2+\epsilon}} (\ln k)^{-\frac{2}{2+\epsilon}} \ln \frac{1}{|x|}, & \text{if } \left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}} \le |x| \le 1, \\ \varsigma_k, & \text{if } |x| > 1, \end{cases}$$

where $\varsigma_k$ is a smooth function satisfying supp $(\varsigma_k) \subset \{|x| < 2\}$,

$$\varsigma_k|_{|x|=1} = 0, \left.\frac{\partial \zeta_k}{\partial v_j}\right|_{|x|=1} = (2+\epsilon)\beta_{2+\epsilon,2}^{-\frac{\epsilon}{2+\epsilon}} (\ln_k)^{-\frac{2}{2+\epsilon}}, \varsigma_k = O\left((\ln_k)^{\frac{2}{2+\epsilon}}\right), \Delta\zeta_k = O\left((\ln_k)^{\frac{2}{2+\epsilon}}\right).$$

Directly computations yield that

$$1 \le \sum_j \|\Delta(u_j)_k\|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{2}} \le 1 + O\left(\frac{1}{\ln k}\right).$$

Let $(\tilde{v}_j)_k = \frac{(u_j)_k}{\|\Delta(u_j)_k\|_{\frac{2+\epsilon}{2}}}$, we derive that

$$\|\Delta(\tilde{v}_j)_k\|_{\frac{2+\epsilon}{2}} = 1 \text{ and } \left\|\sum_j (\tilde{v}_j)_k\right\|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{2}} \le \sum_j \|\Delta(u_j)_k\|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{2}} \le A(\ln k)^{-1} + B(\ln k)^{\frac{\epsilon}{2}} \frac{1}{k}$$

Then we calculate as follows:

$$\sum_j \left\|(\tilde{v}_j)_k\right\|_{\frac{2+\epsilon}{2}}^{-\frac{1}{2}} \int_{\mathbb{R}^{2+\epsilon}} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$\geq \sum_j \left\|(\tilde{v}_j)_k\right\|_{\frac{2+\epsilon}{2}}^{-\frac{1}{2}} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}}} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$\geq \left(\frac{1}{\beta_{2+\epsilon,2}}\ln k\right)^{\frac{\epsilon}{2}} \sum_j \left\|(\tilde{v}_j)_k\right\|_{\frac{2+\epsilon}{2}}^{-\frac{1}{2}} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}}} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)}{|x|^{1+\epsilon}}dx$$

$$\geq \left(\frac{1}{\beta_{2+\epsilon,2}}\ln k\right)^{\frac{\epsilon}{2}} \sum_j \left\|(\tilde{v}_j)_k\right\|_{\frac{2+\epsilon}{2}}^{-\frac{1}{2}} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}}} \frac{\Phi_{2+\epsilon,2}\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)}{|x|^{1+\epsilon}}dx$$

$$\geq \left(\frac{1}{\beta_{2+\epsilon,2}}\ln k\right)^{\frac{\epsilon}{2}} \left(\frac{1}{1-\frac{\beta}{\beta_{2+\epsilon,2}}}\right)^{\frac{1}{2+\epsilon}} \tag{3.4}$$

$\to \infty$ as $\beta \to \beta_{2+\epsilon,2}$,

which completes the proof of the sharpness of (1.6).

The proof of the attainability of the best constant $C(2+\epsilon, 1+\epsilon)$ for inequality (1.6) is similar to that of Theorem 1.1. In fact, by the scaling invariant form of the weighted Trudinger-Moser inequality, we can choose a maximizing sequence $\left((v_j)_k\right)$ for $C(2+\epsilon, 1+\epsilon)$ satisfying that $\left((v_j)_k\right)$ is bounded in $\dot{W}^{2,\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon}) \cap L^{\frac{2+\epsilon}{2}}(\mathbb{R}^{2+\epsilon})$. Following the same procedure as that of Lemma 2.2 and Theorem 1.1, we can obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\left(\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(v_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)-1\right)\left|(v_j)_k\right|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$= \int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\left(\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)|v_j|^{\frac{2+\epsilon}{\epsilon}}\right)-1\right)|v_j|^{\frac{2+\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

and $\|\sum_j \Delta v_j\|_{\frac{2+\epsilon}{2}} = \sum_j \|v_j\|_{\frac{2+\epsilon}{2}} = 1$, which implies the attainability of the best constant $C(2+\epsilon, 1+\epsilon)$.

We now prove Theorem 1.3. We first apply the arrangement-free argument introduced in [29] and the weighted Adams inequality in $W_N^{\frac{2+\epsilon}{2}}(\Omega)$ to obtain inequality (1.7). Indeed, by dividing the integral into two parts, we have

$$\begin{aligned}\int_{\mathbb{R}^{2+\epsilon}} \sum_j \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx &= \int_{|u_j|^{1+\epsilon}\leq 1} \sum_j \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx \\ &\quad + \int_{|u_j|>1} \sum_j \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx \\ &=: I_1 + I_2 \end{aligned} \tag{3.5}$$

For $I_2$, setting $|u_j| = v_j + 1$ and using an elementary inequality

$$|u_j|^{\frac{2+\epsilon}{\epsilon}} \leq (1+\varepsilon)v_j^{\frac{2+\epsilon}{\epsilon}} + C_\varepsilon, \forall\varepsilon > 0,$$

one can obtain

$$I_2 \quad \le \int_{|u_j|>1} \sum_j \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)(1+\varepsilon)v_j^{\frac{2+\epsilon}{\epsilon}}\right)\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)C_\varepsilon\right)|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$\lesssim \sum_j \left(\int_{|u_j|>1} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)(1+\epsilon)(1+\varepsilon)v_j^{\frac{2+\epsilon}{\epsilon}}\right)}{|x|^{1+\epsilon}}dx\right)^{\frac{1}{1+\epsilon}} \left(\int_{|u_j|>1} \frac{(v_j+1)^{\frac{(2+3\epsilon)(1+\epsilon)}{2\epsilon}}}{|x|^{1+\epsilon}}dx\right)^{\frac{\epsilon}{1+\epsilon}}$$

$$\lesssim \sum_j \ |\{|u_j|>1\}|^{\frac{1}{(2+\epsilon)(1+\epsilon)}}\left(\int_{|u_j|>1} \frac{|v_j|^{\frac{(2+3\epsilon)(1+\epsilon)}{2\epsilon}}}{|x|^{1+\epsilon}}+\frac{1}{|x|^{1+\epsilon}}dx\right)^{\frac{1}{1+\epsilon}}$$

$$\lesssim \sum_j \ |\{|u_j|>1\}|^{\frac{1}{(2+\epsilon)(1+\epsilon)}}|\{|u_j|>1\}|^{\frac{1}{(2+\epsilon)\frac{1+\epsilon}{\epsilon}}}+\sum_j \ |\{|u_j|>1\}|^{\frac{1}{(2+\epsilon)(1+\epsilon)}}\left(\int_{|u_j|>1} \frac{1}{|x|^{1+\epsilon}}dx\right)^{\frac{\epsilon}{1+\epsilon}}$$

$$\lesssim \left(\int_{\mathbb{R}^{2+\epsilon}} \sum_j \ |u_j|^{\frac{2+3\epsilon}{2}}dx\right)^{\frac{1}{2+\epsilon}}. \tag{3.6}$$

For $I_1$, direct computations show that

$$\int_{|u_j|\le 1} \sum_j \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)|u_j|^{\frac{2+\epsilon}{\epsilon}}\right)|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$\int_{\{|u_j|\le 1\}\cap\left\{|x|\le\|u_j\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2}\big/\frac{2+3\epsilon}{2}}\right\}} \sum_j \frac{|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx + \int_{\{|u_j|\le 1\}\cap\left\{|x|>\|u_j\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2}\big/2+\epsilon}\right\}} \sum_j \frac{|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx \tag{3.7}$$

$$=: I_{11}+I_{12}.$$

We can estimate $I_{11}$ as follows

$$\int_{\{|u_j|\le 1\}\cap\left\{|x|\le\|u_j\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2(2+\epsilon)}}\right\}} \sum_j \frac{|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dxdx$$

$$\le \int \sum_j \left\{|x|\le\|u_j\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2(2+\epsilon)}}\right\}^{|x|^{-(1+\epsilon)}} dx = \|u_j\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2(2+\epsilon)}}. \tag{3.8}$$

Similarly, we also derive that

$$\int_{\left\{|u_j|\le 1|\cap\left\{|x|\ge\|u_j\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2(2+\epsilon)}}\right\}\right\}} \sum_j \frac{|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx \le \sum_j \ \|u_j\|_{\frac{2+3\epsilon}{2}}^{-\frac{(2+3\epsilon)(1+\epsilon)}{2(2+\epsilon)}} \int_{\{|u_j|\le 1\}} |u_j|^{\frac{2+3\epsilon}{2}}dx = \|u_j\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2(2+\epsilon)}} \tag{3.9}$$

Combining inequalities (3.5), (3.6), (3.8) with (3.9), we obtain the required inequality (1.7). Next, we show the sharpness of inequality (1.7). Using the same test function sequence $\left((u_j)_k\right)_k$ as that of Theorem 1.2, one can easily calculate that

$$\|(\tilde{v}_j)_k\|_{\frac{2+3\epsilon}{2}}^{\frac{2+3\epsilon}{2}} \le A(\ln\ k)^{-\frac{2}{2+\epsilon}\frac{2+3\epsilon}{2}} + B\frac{(\ln\ k)^{\frac{\epsilon(2+3\epsilon)}{2(2+\epsilon)}}}{k} + C\frac{(\ln\ k)^{-\frac{2+3\epsilon}{2+\epsilon}}}{k}$$

Then, it follows that

$$\left\|(\tilde{v}_j)_k\right\|_{\frac{2+3\epsilon}{2}}^{-\frac{(2+3\epsilon)}{2(2+\epsilon)}} \int_{\mathbb{R}^{2+\epsilon}} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$\geq \sum_j \left\|(\tilde{v}_j)_k\right\|_{\frac{2+3\epsilon}{2}}^{-\frac{(2+3\epsilon)}{2(2+\epsilon)}} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}}} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$\geq \left(\frac{1}{\beta_{2+\epsilon,2}}\ln k\right)^{\frac{\epsilon(2+3\epsilon)}{2(2+\epsilon)}} \sum_j \left\|(\tilde{v}_j)_k\right\|_{\frac{2+\epsilon}{2}}^{-\frac{(2+3\epsilon)}{2(2+\epsilon)}} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}}} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)}{|x|^{1+\epsilon}}dx$$

$$\geq \left(\frac{1}{\beta_{2+\epsilon,2}}\ln k\right)^{\frac{\epsilon(2+3\epsilon)}{2(2+\epsilon)}} \sum_j \left\|(\tilde{v}_j)_k\right\|_{\frac{2+\epsilon}{2}}^{-\frac{1}{2}} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{2+\epsilon}}} \frac{\exp\left(\beta\left(\frac{1}{2+\epsilon}\right)\left|(\tilde{v}_j)_k\right|^{\frac{2+\epsilon}{\epsilon}}\right)}{|x|^{1+\epsilon}}dx \quad (3.10)$$

$\to \infty$ as $\beta \to \beta_{2+\epsilon,2}$

Then, we show the attainability of the sharp constant $C(2+\epsilon, 1+\epsilon)$ for inequality (1.7). We need the following compact imbedding (see [64]).

**Lemma 3.1** For $\epsilon \geq 0, \dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{\frac{2+3\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ can be embedded compactly into $L^{1+\epsilon}\left(\mathbb{R}^{1+2\epsilon}; |x|^{-(1+\epsilon)}dx\right)$.

For the continuity of the proof, we postpone the proof of Lemma 3.1. With the help of Lemma 3.1, applying the same method in Lemma 2.2, we can derive the following required convergence.

$$\lim_{k\to\infty} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\left(\exp\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)\left|(u_j)_k\right|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)-1\right)\left|(u_j)_k\right|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx$$

$$= \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\left(\exp\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)-1\right)|u_j|^{\frac{2+3\epsilon}{2}}}{|x|^{1+\epsilon}}dx. \quad (3.11)$$

Then, we can use the same procedure as Theorem 1.1 to obtain the attainability of the best constant. At last, we focus on the proof of Lemma 3.1.

The continuity of the embedding is a direct result of inequality (1.7) and the Hölder inequality. Then it is sufficient to show that for any bounded sequence $\left((u_j)_k\right)$ in $\dot{W}^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \cap L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})$, there exists a subsequence which we still denote as $\left((u_j)_k\right)$ such that

$$\lim_{k\to\infty} \sum_j \left\|(u_j)_k - u_j\right\|_{L^{1+2\epsilon}\left(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx\right)} = 0 \text{ as } k\to\infty \text{ for } \epsilon \geq 0.$$

Similar to the proof of Lemma 2.1, we carry out the process of proof by two steps.

**Step I** We first show that there exists a subsequence still denoted by $\left((u_j)_k\right)$ such that $(u_j)_k \to u_j$ for almost $x \in \mathbb{R}^{1+2\epsilon}$. In fact, through Sobolev interpolation inequalities with weights (see Lin's work [41]), we can obtain

$$\| \sum_j \nabla u_j \|_{2(1+\epsilon)} \leq \sum_j \| \Delta u_j \|_{\frac{\pi}{2}}^{\frac{1}{2}} \| u_j \|_{1+\epsilon}^{\frac{1}{2}}.$$

Then it follows from the Hölder inequality that

$$\int_\Omega \sum_j \left(|\nabla u_j|^{\frac{1+2\epsilon}{2}} + \left|u_j\right|^{\frac{1+2\epsilon}{2}}\right) dx \leq C(\Omega)$$

According to the classical Sobolev compact embedding $W^{1,\frac{\pi}{2}}(\Omega) \hookrightarrow\hookrightarrow L^{1+2\epsilon}(\Omega)$ for $\epsilon > 0$ and the diagonal trick, one can obtain that there exists a subsequence (we still denote by $\left((u_j)_k\right)$) such that $(u_j)_k(x) \to u_j(x)$, strongly in $L^{1+2\epsilon}_{loc}(\mathbb{R}^{1+2\epsilon})$, $(u_j)_k(x) \to u_j(x)$, for almost everywhere $x \in \mathbb{R}^{1+2\epsilon}$.

**Step 2** We claim that for any $\epsilon \geq 0, (u_j)_k \to u_j$ in $L^{1+2\epsilon}\left(\mathbb{R}^{1+2\epsilon}; |x|^{-(1+\epsilon)}dx\right)$. Since the process of the proof is similar to that of Lemma 2.1, we omit the details.

## IV. Proof of Theorem 1.5

We use the relationship between the supremums of the subcritical and critical inequalities in [33] to establish the existence of maximizers for the singular Adams inequality with the Sobolev norm. We need the following lemmas whose proofs can be found in [21], and [33].

**Lemma 4.1** For $\epsilon \geq 0$, then ATA(*,t) is continuous on $\left(0, \beta_{1+2\epsilon,2}\right)$.

**Lemma 4.2** If $\epsilon \geq 0$, then

$$\lim_{1+\epsilon\to 0}\left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} ATA(1+\epsilon,1+\epsilon)=0$$

**Lemma 4.3** For $\epsilon \geq 0$, if $\left(\beta < \beta_{1+2\epsilon,2}, \epsilon \geq 0\right)$ or $\left(\beta = \beta_{1+2\epsilon,2}, 0 < 1+\epsilon < \frac{1+2\epsilon}{2}\right)$, then

$$\lim_{1+\epsilon\to \beta}\left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} ATA(1+\epsilon,1+\epsilon)=0.$$

**Proof of Theorem 1.5 (see [64])** With the help of Theorem 1.1, Lemmas 4.1,4.2 and 4.3, we establish the existence of extremals for the singular Adams inequality with the Sobolev norm. We only need to prove that there exists an extremal function for $A_{1+\epsilon,1+\epsilon,1+\epsilon}(\beta)$ in the case of $\left(\beta < \beta_{1+2\epsilon,2}, \epsilon \geq 0\right)$ or $\left(\beta = \beta_{1+2\epsilon,2}, 1+\epsilon < \frac{1+2\epsilon}{2}\right)$. It is easy to check that

$$\lim_{1+\epsilon\to 0}\left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{1}{2(1+\epsilon)}} ATA(\beta,1+\epsilon)<A_{1+\epsilon,1+\epsilon,1+\epsilon}(\beta)$$

and

$$\lim_{1+\epsilon\to 1+\epsilon}\left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} ATA(\beta,1+\epsilon)<A_{1+\epsilon,1+\epsilon,1+\epsilon}(\beta)$$

On the other hand, we also have

$$A_{1+\epsilon,1+\epsilon,1+\epsilon}(\beta)=\sup_{1+\epsilon\in(0,\beta)}\left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} ATA(1+\epsilon,1+\epsilon).$$

This together with Lemma 4.2 and Lemma 4.3 yields that there exists $1+\epsilon \in (0,\beta)$ such that

$$\left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} ATA(1+\epsilon,1+\epsilon)=A_{1+\epsilon,1+\epsilon,1+\epsilon}(\beta)$$

Assume that $u_j \in W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ with $\| \sum_j \Delta u_j \|_{\frac{1+2\epsilon}{2}} \leq 1 = \sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}$ is the maximizer for $ATA(1+\epsilon,1+\epsilon)$. Define

$$v_j(x)=\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}}\sum_j u_j(\lambda x),$$

$$\lambda=\left(\frac{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{1}{2(1+\epsilon)}},$$

then it follows that

$$\| \sum_j \Delta v_j \|_{\frac{1+2\epsilon}{2}}^{1+\epsilon} = \left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)} \sum_j \| \Delta u_j \|_{\frac{\pi}{2}}^{1+\epsilon} \le \left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}$$

$$\| \sum_j v_j \|_{\frac{1+2\epsilon}{2}}^{1+\epsilon} = \left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)} \frac{1}{\lambda^{1+\epsilon}} \sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{1+\epsilon} = 1 - \left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}$$

which implies that $\| \Delta v_j \|_{\frac{1+2\epsilon}{2}}^{1+\epsilon} + \| v_j \|_{\frac{1+2\epsilon}{2}}^{1+\epsilon} \le 1$. Hence,

$$\begin{aligned} A_{1+\epsilon,1+\epsilon,1+\epsilon}(\beta) &= \left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2}\left((1+\epsilon)\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}} dx \\ &= \left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2}\left((1+\epsilon)\left(\frac{\epsilon}{1+2\epsilon}\right)|u_j(\lambda x)|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|\lambda x|^{1+\epsilon}} d(\lambda x) \\ &= \left(\frac{1-\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}{\left(\frac{1+\epsilon}{\beta}\right)^{\frac{2\epsilon-1}{1+2\epsilon}(1+\epsilon)}}\right)^{\frac{\epsilon}{2(1+\epsilon)}} \lambda^{\epsilon} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|v_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}} dx \qquad (4.1) \\ &= \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{\epsilon}{1+2\epsilon}\right)|v_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}} dx. \end{aligned}$$

This implies that $v_j$ is actually a maximizer for $A_{1+\epsilon,1+\epsilon,1+\epsilon}(\beta)$.

## V. Proofs of Theorems 1.7, 1. 9 and 1.11

We show inequalities (1.8) and (1.9) which equipped with the Dirichlet norm and the existence of their extremal functions. The arrangement-free argument in [29] is a useful tool in dealing with the Trudinger-Moser and the second-order Adams inequalities. This method may fail when we come to consider the higher order inequalities. We use the method based on Fourier transform to establish (1.8) and (1.9). We need the following (see [64]).

**Lemma 5.1** For any $\beta \in \left(0, \beta_{2(2+\epsilon),2+\epsilon}\right)$, there exists a positive constant $C_\beta$ such that

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)}{|x|^{1+\epsilon}} dx \le C_\beta, \qquad (5.1)$$

where $u_j \in W^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right), \left\|\sum_j \nabla^{2+\epsilon} u_j\right\|_2 \le 1$ and $\| u_j \|_2 = 1$.

**Proof** We first claim that for any fixed $\beta \in \left(0, \beta_{2(2+\epsilon),2+\epsilon}\right)$, there exists sufficient small $\epsilon \ge 0$ such that for all $u_j \in \dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right) \cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right)$ with $\left\|\nabla^{2+\epsilon} u_j\right\|_2 \le 1$ and $\| u_j \|_2 = 1$, there holds

$$\left\| ((1+\epsilon)I - \Delta)^{\frac{2+\epsilon}{2}} \sum_j u_j \right\|_2^2 \le \frac{\beta_{2(2+\epsilon),2+\epsilon}}{\beta} \qquad (5.2)$$

Indeed, by Fourier transform, we have

$$\left\| \sum_j ((1+\epsilon)I - \Delta)^{\frac{2+\epsilon}{2}} u_j \right\|_2^2 = \sum_{j_0=0}^{2+\epsilon} \sum_j C_{2+\epsilon}^{j_0} (1+\epsilon)^{2+\epsilon-j_0} \left\|\nabla^{j_0} u_j\right\|_2^2.$$

Thanks to the Sobolev interpolation inequalities, one can derive that for every $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$ such that

$$\left\| \sum_j ((1+\epsilon)I - \Delta)^{\frac{2+\epsilon}{2}} u_j \right\|_2^2 \le (1+\varepsilon) \sum_j \left\|\nabla^{2+\epsilon} u_j\right\|_2^2 + C_\varepsilon (1+\epsilon) \sum_j \| u_j \|_2^2,$$

which implies inequality (5.2). With the help of Theorem D in [29], we derive that

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)}{|x|^{1+\epsilon}} dx$$

$$= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left\|((1+\epsilon)I-\Delta)^{\frac{2+\epsilon}{2}}u_j\right\|_2^2 \left|\frac{u_j}{\left\|((1+\epsilon)I-\Delta)^{\frac{\pi}{2}}u_j\right\|_2}\right|^2\right)}{|x|^{1+\epsilon}} dx$$

$$\leq \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta_{2(2+\epsilon),2+\epsilon}\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|\frac{u_j}{\left\|((1+\epsilon)I-\Delta)^{\frac{4}{2}}u_j\right\|_2}\right|^2\right)}{|x|^{1+\epsilon}} dx \leq C_B, \quad (5.3)$$

which finishes the proof.

With the help of Lemma 5.1, we start the proof of inequality (1.8). In fact, for any $u_j \in W^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)})$ satisfying $\left\|\sum_j \nabla^{2+\epsilon}u_j\right\|_2 \leq 1$, we define $(u_j)_\lambda(x) = u_j(\lambda x)$ with $\lambda = \| u_j \|_2^{\frac{1}{(2+\epsilon)(1+\epsilon)}}$. Through direct calculations, we derive that

$$\left\|\sum_j (u_j)_\lambda\right\|_2^2 = \lambda^{-2(2+\epsilon)} \sum_j \| u_j \|_2^2 = 1$$

$$\left\|\sum_j \nabla^{2+\epsilon}(u_j)_\lambda\right\|_2 = \sum_j \left\|\nabla^{2+\epsilon}u_j\right\|_2 = 1$$

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|(u_j)_\lambda\right|^2\right)}{|x|^{1+\epsilon}} dx$$

$$= \lambda^{-2(3+\epsilon)} \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)}{|x|^{1+\epsilon}} dx.$$

Then it follows from inequality (5.1) that

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)}{|x|^{1+\epsilon}} dx = \lambda^{3+\epsilon} \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|(u_j)_\lambda\right|^2\right)}{|x|^{1+\epsilon}} dx$$

$$\leq \lambda^{3+\epsilon} C_\beta$$

$$= C_\beta \sum_j \| u_j \|_2^{2\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)}. \quad (5.4)$$

Next, we show the sharpness of inequality (1.8).

We will modify the idea of constructing test functions for the Adams inequality on domains of finite measure in Euclidean spaces [2]. Let $\phi \in C_0^\infty([0,1])$ such that

$$\phi(0) = \phi'(0) = \cdots = \phi^{1+\epsilon}(0) = 0,\ \phi(1) = \phi'(1) = 1$$

$$\phi''(1) = \cdots = \phi^{1+\epsilon}(1) = 0$$

For $0 < \varepsilon < \frac{1}{2}$, set

$$H(1+\epsilon) := \begin{cases} \varepsilon\phi\left(\frac{1+\epsilon}{\varepsilon}\right), & \text{if } 0 < 1+\epsilon \leq \varepsilon \\ 1+\epsilon, & \text{if } \varepsilon < 1+\epsilon \leq 1-\varepsilon \\ 1-\varepsilon\phi\left(\frac{-\epsilon}{\varepsilon}\right), & \text{if } 1-\varepsilon < 1+\epsilon \leq 1 \\ 1, & \text{if } \epsilon \geq 0. \end{cases}$$

For any fixed $\epsilon \geq 0$ sufficiently small, we define

$$(\psi_j)_{1+\epsilon}(|x|):=H_{\varepsilon(1+\epsilon)}\left(\frac{\log\frac{1}{|x|}}{\log\frac{1}{1+\epsilon}}\right)$$

where $\varepsilon(1+\epsilon)=\frac{1}{\log\frac{1}{1+\epsilon}}$. Obviously, $(\psi_j)_{1+\epsilon}\in W_0^{2+\epsilon,2}(B_1)$ and

$$(\psi_j)_{1+\epsilon}=1 \text{ on } B_{1+\epsilon}$$

It was proved in Adams [2] that

$$\left\|\sum_j \nabla^{2+\epsilon}(\psi_j)_{1+\epsilon}\right\|_2^2\le(2(2+\epsilon))^{-1}\beta_{2(2+\epsilon),2+\epsilon}\left(\log\frac{1}{1+\epsilon}\right)^{-1}A_{1+\epsilon}$$

where

$$A_{1+\epsilon}:=1+O\left(\frac{1}{\log\frac{1}{1+\epsilon}}\right)$$

Moreover, direct computations show

$$\|(\psi_j)_{1+\epsilon}\|_2\lesssim\frac{1}{\log\frac{1}{1+\epsilon}}$$

Define

$$(u_j)_{1+\epsilon}:=\sum_j\frac{(\psi_j)_{1+\epsilon}}{\left((2(2+\epsilon))^{-1}\beta_{2(2+\epsilon),2+\epsilon}\left(\log\frac{1}{1+\epsilon}\right)^{-1}A_{1+\epsilon}\right)^{\frac{1}{2}}}.$$

By direct calculations, one can obtain that

$$\left\|\sum_j \nabla^{2+\epsilon}(u_j)_{1+\epsilon}\right\|_2\le 1$$

and

$$\|(u_j)_{1+\epsilon}\|_2^2\sim\sum_j\frac{\|(\psi_j)_{1+\epsilon}\|_2^2}{\left(\log\frac{1}{1+\epsilon}\right)^{-1}A_{1+\epsilon}}\lesssim\frac{1}{\log\frac{1}{1+\epsilon}}.$$

Let $1+\epsilon\to 0$, it follows that

$$\begin{aligned}
&\lim_{1+\epsilon\to0}\sum_j\frac{\int_{\mathbb{R}^{2(2+\epsilon)}}\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta_{2(2+\epsilon),2+\epsilon}\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|(u_j)_{1+\epsilon}\right|^2\right)|x|^{-(1+\epsilon)}dx}{\left\|(u_j)_{1+\epsilon}\right\|_2^{2\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)}}\\
&\gtrsim\lim_{1+\epsilon\to0}\left(\log\frac{1}{1+\epsilon}\right)^{\frac{3+\epsilon}{2(2+\epsilon)}}\int_{B_{1+\epsilon}}\sum_j\exp\left(\beta_{2(2+\epsilon),2+\epsilon}\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|(u_j)_{1+\epsilon}\right|^2\right)|x|^{-(1+\epsilon)}dx\\
&=\lim_{1+\epsilon\to0}\left(\log\frac{1}{1+\epsilon}\right)^{\frac{3+\epsilon}{2(2+\epsilon)}}\int_{B_{1+\epsilon}}\exp\left((3+\epsilon)\log\frac{1}{1+\epsilon}A_{1+\epsilon}^{-1}\right)|x|^{-(1+\epsilon)}dx \qquad (5.5)\\
&\gtrsim\lim_{1+\epsilon\to0}\left(\log\frac{1}{1+\epsilon}\right)^{\frac{3+\epsilon}{2(2+\epsilon)}}(1+\epsilon)^{3+\epsilon}\exp\left((3+\epsilon)\log\frac{1}{1+\epsilon}A_{1+\epsilon}^{-1}\right)\\
&\gtrsim\lim_{1+\epsilon\to0}\left(\log\frac{1}{1+\epsilon}\right)^{\frac{3+\epsilon}{2(2+\epsilon)}}\exp\left(\left(2(2+\epsilon)-(1+\epsilon)\right)\log\frac{1}{1+\epsilon}(A_{1+\epsilon}^{-1}-1)\right)\\
&\gtrsim\lim_{1+\epsilon\to0}\left(\log\frac{1}{1+\epsilon}\right)^{\frac{3+\epsilon}{2(2+\epsilon)}}\to\infty,
\end{aligned}$$

which completes the proof of sharpness.

At last, we show the attainability of $\mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}$. Just as what we did in Theorem 1.1, we need the following compactness lemma (see [64]).

**Lemma 5.2** For $\epsilon\ge0$, then $\dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right)\cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right)$ can be compactly embedded into $L^{2+\epsilon}\left(\mathbb{R}^{2(2+\epsilon)},|x|^{-(1+\epsilon)}dx\right)$.

**Proof** The proof is similar to that of Lemma 2.1 once we prove the equivalence between the space $\dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right) \cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right)$ and the standard Sobolev space $W^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right)$. Indeed, it suffices to show that

$$\left\|\partial^{1+\epsilon}u_j\right\|_2^2 \leq \| u_j \|_2^2 + \left\|\nabla^{2+\epsilon}u_j\right\|_2^2,\ \forall |1+\epsilon| \leq 2+\epsilon. \tag{5.6}$$

We first prove the

$$\left\|\nabla^{1+\epsilon}u_j\right\|_2^2 \lesssim \| u_j \|_2^2 + \left\|\nabla^{2+\epsilon}u_j\right\|_2^2, \qquad \forall \epsilon \geq 0, 1+\epsilon \in \mathbb{N}. \tag{5.7}$$

In fact, by the Fourier transform, we have

$$\begin{aligned}
\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\nabla^{1+\epsilon}u_j\right|^2 dx &= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\xi\right|^{2(1+\epsilon)}\left|\hat{u}_j(\xi)\right|^2 d\xi \\
&\leq \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left(1+|\xi|^{2(2+\epsilon)}\right)|\hat{u}_j(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j |\hat{u}_j(\xi)|^2 d\xi + \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\xi\right|^{2(2+\epsilon)}\left|\hat{u}_j(\xi)\right|^2 d\xi \\
&= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j |u_j|^2 dx + \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\nabla^{2+\epsilon}u_j\right|^2 dx.
\end{aligned}$$

Combining this result, in order to obtain the equivalence result, we only need to show that

$$\left\|\sum_j \partial^\alpha u_j\right\|_2^2 \leq \sum_j \left\|\nabla^{|\alpha|}u_j\right\|_2^2 \tag{5.8}$$

One can derive it by induction. For $|\alpha| \geq 2, \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{1+2\epsilon})$, there exist $\alpha_j + \alpha_{1+\epsilon} \geq 2$ such that $\partial^\alpha = \frac{\partial^2}{\partial x_j\, \partial x_{1+\epsilon}}\partial^\beta$. Hence, it follows from the Fourier transform and the Riesz transform that

$$\begin{aligned}
\left\|\sum_j \partial^\alpha u_j\right\|_2^2 &= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\left(\frac{\partial^2}{\partial x_j\, \partial x_{1+\epsilon}}\partial^\beta u_j\right)^\wedge(\xi)\right|^2 d\xi \\
&= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left| 4\pi^2 \xi_j \xi_{1+\epsilon} \widehat{\partial^\beta u_j(\xi)}\right|^2 d\xi \\
&= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\left(-i\frac{\xi_j}{|\xi|}\right)\left(-i\frac{\xi_{1+\epsilon}}{|\xi|}\right)(4\pi^2|\xi|^2)\, \widehat{\partial^\beta u_j(\xi)}\right|^2 d\xi \\
&= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\left(R_j R_{1+\epsilon}\Delta\left(\partial^\beta u_j\right)\right)^\wedge(\xi)\right|^2 d\xi
\end{aligned}$$

Then, with the help of the induction and the definition of $\nabla^{2+\epsilon}$, one can get

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left|\left(R_j R_{1+\epsilon}\Delta\left(\partial^\beta u_j\right)\right)^\wedge(\xi)\right|^2 d\xi \leq \sum_j \left\|\partial^\beta(\Delta u_j)\right\|_2^2 \leq \sum_j \left\|\nabla^{|\beta|}\Delta u_j\right\|_2^2 = \sum_j \left\|\nabla^{|\alpha|} u_j\right\|_2^2,$$

which proves the required equivalence.

Now we show that the best constant $\mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}$ could be attained by a function in $\dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right) \cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right)$. Assume that $\left((u_j)_{1+\epsilon}\right) \subset \dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right) \cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right)$ satisfying

$$\left\|\sum_j \nabla^{2+\epsilon}(u_j)_{1+\epsilon}\right\|_2 = 1 \text{ and } F_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left((u_j)_{1+\epsilon}\right) \to \mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\mathbb{R}^{2(2+\epsilon)}\right) \text{ as } 1+\epsilon \to \infty.$$

Constructing a new function sequence $\left((v_j)_{1+\epsilon}\right)$ defined by $(v_j)_{1+\epsilon}(x) := (u_j)_{1+\epsilon}\left(\left\|(u_j)_{1+\epsilon}\right\|_2^{\frac{1}{2+\epsilon}}x\right)$ for $x \in \mathbb{R}^{2(2+\epsilon)}$, one can easily verify that

$$\left\|\nabla^{2+\epsilon}(v_j)_{1+\epsilon}\right\|_2 = 1,\ \left\|(v_j)_{1+\epsilon}\right\|_2 = 1$$

and

$$F_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left((v_j)_{1+\epsilon}\right) = F_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left((u_j)_{1+\epsilon}\right) \to \mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\mathbb{R}^{2(2+\epsilon)}\right) \text{ as } 1+\epsilon \to \infty$$

Hence, $\left((v_j)_{1+\epsilon}\right)$ is also a maximizing sequence for $\mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\mathbb{R}^{2(2+\epsilon)}\right)$. Note that $\left((v_j)_{1+\epsilon}\right)$ is bounded in $\dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right) \cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right)$, thus up to a sequence, we may assume that

$$(v_j)_{1+\epsilon} \to v_j \text{ in } \dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right) \cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right).$$

It follows from weak semicontinuity of the norm in $\dot{W}^{2+\epsilon,2}\left(\mathbb{R}^{2(2+\epsilon)}\right) \cap L^2\left(\mathbb{R}^{2(2+\epsilon)}\right)$ that

$$\left\|\sum_j \nabla^{2+\epsilon} v_j\right\|_2 \leq 1, \qquad \left\|\sum_j v_j\right\|_2 \leq 1 \tag{5.9}$$

Then, implementing same procedures as we did in Lemma 2.2, we have

$$\begin{aligned}\lim_{1+\epsilon\to\infty}\int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j \left(\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|(u_j)_{1+\epsilon}\right|^2\right)-\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|(u_j)_{1+\epsilon}\right|^2\right)\frac{dx}{|x|^{1+\epsilon}} \\ = \int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j \left(\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)-\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)\frac{dx}{|x|^{1+\epsilon}}\end{aligned} \tag{5.10}$$

Combining (5.10) with Lemma 5.2, we derive that up to a sequence,

$$\begin{aligned}\mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\mathbb{R}^{2(2+\epsilon)}\right) &= \int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j \Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)\left|(v_j)_{1+\epsilon}\right|^2\right)\frac{dx}{|x|^{1+\epsilon}}+o(1) \\ &= \int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j \Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|v_j|^2\right)\frac{dx}{|x|^{1+\epsilon}},\end{aligned} \tag{5.11}$$

which implies $v_j \neq 0$. Then we can deduce from (5.9) and (5.11) that

$$\begin{aligned}\mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\mathbb{R}^{2(2+\epsilon)}\right) &\leq \sum_j \frac{\int_{\mathbb{R}^{2(2+\epsilon)}} \Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|v_j|^2\right)\frac{dx}{|x|^{1+\epsilon}}}{\| v_j \|_2^{2-\frac{1+\epsilon}{2+\epsilon}}} \\ &= F_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}(v_j).\end{aligned} \tag{5.12}$$

Therefore, it remains to show that $\|\nabla^{2+\epsilon}v_j\|_2 = 1$. By the definition of $\mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\mathbb{R}^{2(2+\epsilon)}\right)$ and (5.9), we see that

$$\begin{aligned}\mu_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\mathbb{R}^{2(2+\epsilon)}\right) &\geq \sum_j F_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}\left(\frac{v_j}{\|\nabla^{2+\epsilon}v_j\|_2}\right) \\ &= \sum_{i=1}^{\infty}\sum_j \frac{\beta^i}{i!}\| v_j \|^{2i}_{L^{2i}(\mathbb{R}^{2(2+\epsilon)};|x|^{-1+\epsilon}dx)}\| v_j \|_2^{-\frac{3+\epsilon}{2+\epsilon}}\|\nabla^{2+\epsilon}v_j\|_2^{\frac{3+\epsilon}{2+\epsilon}-2i} \\ &\geq \sum_j F_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}(v_j)+\sum_j\left(\|\nabla^{2+\epsilon}v_j\|_2^{-\frac{1+\epsilon}{2+\epsilon}}-1\right)F_{2(2+\epsilon),2+\epsilon,1+\epsilon,\beta}(v_j)\end{aligned} \tag{5.13}$$

This together with (5.9) and (5.12) implies that $\|\nabla^{2+\epsilon}v_j\|_2 = 1$. Then we complete the proof of Theorem 1.7.

**The Proof of Theorem 1.9 (see [64])** We first establish inequality (1.9). Just as what we did in Theorem 1.2, we divide the integral in inequality (1.9) into two parts.

$$\begin{aligned}&\int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j \frac{\exp\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)|u_j|^2}{|x|^{1+\epsilon}}dx \\ &= \int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)|u_j|^2}{|x|^{1+\epsilon}}dx+\int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j \frac{|u_j|^2}{|x|^{1+\epsilon}}dx \\ &= :I_1+I_2.\end{aligned} \tag{5.14}$$

By applying the Hölder inequality and inequality (1.8), one can estimate $I_1$ as follows

$$\begin{aligned}I_1 &\leq \sum_j\left(\int_{\mathbb{R}^{2(2+\epsilon)}}\frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta(2+\epsilon)\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)}{|x|^{1+\epsilon}}dx\right)^{\frac{1}{2+\epsilon}}\left(\int_{\mathbb{R}^{2(2+\epsilon)}}\frac{|u_j|^{2\left(\frac{1+\epsilon}{\epsilon}\right)}}{|x|^{1+\epsilon}}dx\right)^{\frac{\epsilon}{1+\epsilon}} \\ &\leq \sum_j\left(\int_{\mathbb{R}^{2(2+\epsilon)}}|u_j|^2dx\right)^{\frac{3+\epsilon}{2(2+\epsilon)^2}}\left(\int_{\mathbb{R}^{2(2+\epsilon)}}|u_j|^2dx\right)^{\frac{3+\epsilon}{2(2+\epsilon)^2}} \\ &= \left(\int_{\mathbb{R}^{2(2+\epsilon)}}\sum_j |u_j|^2dx\right)^{\frac{3+\epsilon}{2(2+\epsilon)}}\end{aligned} \tag{5.15}$$

where $\epsilon > 0$ and $\beta(1+\epsilon) < \beta_{2(2+\epsilon),2+\epsilon}$. As for $I_2$, it is an immediate result of inequality (1.8).

One can deduce the sharpness of inequality (1.9) from the sharpness of inequality (1.13).

In fact, one only needs to observe the following fact

$$\int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\exp\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)|u_j|^2}{|x|^{1+\epsilon}} dx \geq \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \frac{\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left(\frac{3+\epsilon}{2(2+\epsilon)}\right)|u_j|^2\right)}{|x|^{1+\epsilon}} dx$$

For the attainability of the best constant $C(2+\epsilon, 1+\epsilon)$ of inequality (1.9), one can manage the same steps as what we do in Theorem 1.7 to obtain the required results.

**The Proof of Theorem 1.11 (see [64])** We first employ the Fourier rearrangement tools to prove there exists radially maximizing sequence for $\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)})$. In fact, assume that ( $(u_j)_{1+\epsilon}$ ) is a maximizing sequence for $\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)})$, that is

$$\left\|\sum_j (-\Delta)^{\frac{2+\epsilon}{2}}(u_j)_{1+\epsilon}\right\|_2 = 1, \lim_{1+\epsilon\to\infty} F_{2(2+\epsilon),2+\epsilon,0,\beta}\left((u_j)_{1+\epsilon}\right) \to \mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)})$$

Define $(u_j)^{\sharp}_{1+\epsilon}$ by $(u_j)^{\sharp}_{1+\epsilon} = F^{-1}\left\{\left(F\left((u_j)_{1+\epsilon}\right)\right)^*\right\}$, where $F$ denotes the Fourier transform on $\mathbb{R}^{2(2+\epsilon)}$ (with its inverse $F^{-1}$ ) and $f^*$ stands for the Schwarz symmetrization of $f$. Using the property of the Fourier rearrangement from [35], one can derive that

$$\left\|\sum_j (-\Delta)^{\frac{2+\epsilon}{2}}(u_j)^{\sharp}_{1+\epsilon}\right\|_2 \leq \sum_j \left\|(-\Delta)^{\frac{2+\epsilon}{2}}(u_j)_{1+\epsilon}\right\|_2, \quad \left\|\sum_j (u_j)^{\sharp}_{1+\epsilon}\right\|_2 = \sum_j \left\|(u_j)_{1+\epsilon}\right\|_2,$$

$$\left\|\sum_j (u_j)^{\sharp}_{1+\epsilon}\right\|_{1+\epsilon} \geq \sum_j \left\|(u_j)_{1+\epsilon}\right\|_{1+\epsilon}.$$

Hence, $\lim_{1+\epsilon\to\infty} F_{2(2+\epsilon),2+\epsilon,0,\beta}\left((u_j)_{1+\epsilon}\right) \leq \lim_{1+\epsilon\to\infty} F_{2(2+\epsilon),2+\epsilon,0,\beta}\left((u_j)^{\sharp}_{1+\epsilon}\right)$, which implies that $\left((u_j)^{\sharp}_{1+\epsilon}\right)$ is also the maximizing sequence for $\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)})$. Constructing a new function sequence $\left((v_j)_{1+\epsilon}\right)$ defined by $(v_j)_{1+\epsilon}(x) := (u_j)_{1+\epsilon}\left(\|(u_j)_{1+\epsilon}\|_2^{\frac{1}{2+\epsilon}} x\right)$ for $x \in \mathbb{R}^{2(2+\epsilon)}$, one can easily verify that ( $(v_j)_{1+\epsilon}$ ) is also a maximizing sequence for $\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)})$ with $\|\nabla^{2+\epsilon}(v_j)_{1+\epsilon}\|_2 = 1$ and $\|(v_j)_{1+\epsilon}\|_2 = 1$. Note $\left((v_j)_{1+\epsilon}\right)$ is bounded in $W^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)})$, up to a sequence, we may assume that

$$(v_j)_{1+\epsilon} \to v_j \text{ in } \dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap L^2(\mathbb{R}^{2(2+\epsilon)}),$$

thus $v_j$ satisfies that $\| v_j \|_2 \leq 1$ and $\|\nabla^{2+\epsilon} v_j\|_2^2 \leq 1$. Since $W^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)})$ can be compactly imbedded into $L^{2+\epsilon}(\mathbb{R}^{2(2+\epsilon)})$ for any $\epsilon > 0$ (please refer to [7], Lemma 5.3), implementing same procedures as what we did in Lemma 2.2, one can deduce that

$$\lim_{1+\epsilon\to\infty} \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left(\Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left|(u_j)_{1+\epsilon}\right|^2\right) - \beta\left|(u_j)_{1+\epsilon}\right|^2\right)$$

$$= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left(\Phi_{2(2+\epsilon),2+\epsilon}(\beta|u_j|^2) - \beta|u_j|^2\right) \quad (5.16)$$

Then it follows that

$$\begin{aligned} \mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)}) &= \sum_j F_{2(2+\epsilon),2+\epsilon,0,\beta}\left((v_j)_{1+\epsilon}\right) + o(1) \\ &= \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \Phi_{2(2+\epsilon),2+\epsilon}\left(\beta\left|(v_j)_{1+\epsilon}\right|^2\right) dx + o(1) \\ &= \beta + \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \Phi_{2(2+\epsilon),2+\epsilon}\left(\left(\beta\left|(v_j)_{1+\epsilon}\right|^2\right) - \beta\left|(v_j)_{1+\epsilon}\right|^2\right) dx + o(1) \\ &= \beta + \int_{\mathbb{R}^{2(2+\epsilon)}} \sum_j \left(\Phi_{2(2+\epsilon),2+\epsilon}(\beta|v_j|^2) - \beta|v_j|^2\right) dx \end{aligned} \quad (5.17)$$

Next, we show $v_j \neq 0$. Indeed, one can pick $(u_j)_0$ in $\dot{W}^{2+\epsilon,2}(\mathbb{R}^{2(2+\epsilon)}) \cap L^2(\mathbb{R}^{2(2+\epsilon)})$ satisfying $\|\nabla^{2+\epsilon}(u_j)_0\|_2 = 1$ arbitrarily. Then, we have

$$\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)}) \geq F_{2(2+\epsilon),2+\epsilon,0,\beta}((u_j)_0) = \sum_j \frac{\int_{\mathbb{R}^{2(2+\epsilon)}} \Phi_{2(2+\epsilon),2+\epsilon}\left(\beta|(u_j)_0|^2\right)dx}{\|(u_j)_0\|_2^2}$$
$$= \sum_j \frac{\sum_{j_0=1}^{\infty} \frac{\beta^j}{j_0!}\|(u_j)_0\|_{2j_0}^{2j_0}}{\|(u_j)_0\|_2^2}$$
$$= \beta + \sum_j \frac{\sum_{j_0=2}^{\infty} \frac{\beta^{j_0}}{j_0!}\|(u_j)_0\|_{2j_0}^{2j_0}}{\|(u_j)_0\|_2^2} > \beta.$$

Hence,

$$\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)}) \leq \beta + \sum_j \frac{\int_{\mathbb{R}^{2(2+\epsilon)}} \Phi_{2(2+\epsilon),2+\epsilon}(\beta|v_j|^2) - \beta|v_j|^2 dx}{\| v_j \|_2^2}$$
$$= \sum_j \frac{\int_{\mathbb{R}^{2(2+\epsilon)}} \Phi_{2(2+\epsilon),2+\epsilon}(\beta|v_j|^2)dx}{\| v_j \|_2^2} = F_{2(2+\epsilon),2+\epsilon,0,\beta}(v_j).$$

Therefore, it remains to show $\|\nabla^{2+\epsilon}v_j\|_2^2 - 1$. Recall that $\|\nabla^{2+\epsilon}v_j\|_2^2 \leq 1$, it suffices to show that $\|\nabla^{2+\epsilon}v_j\|_2^2 \geq 1$. Through the definition of $\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)})$, one can obtain that

$$\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)}) \geq \sum_j F_{2(2+\epsilon),2+\epsilon,0,\beta}\left(\frac{v_j}{\|\nabla^{2+\epsilon}v_j\|_2}\right)$$
$$= \sum_{j_0=1}^{\infty}\sum_j \frac{\beta^{j_0}}{j_0!}\frac{\| v_j \|_{2j_0}^{2j_0}}{\| v_j \|_2^2}\|\nabla^{2+\epsilon}v_j\|_2^{2-2j_0}$$
$$\geq \beta + \frac{\beta^2}{2}\sum_j \frac{\| v_j \|_4^4}{\| v_j \|_2^2}\|\nabla^{2+\epsilon}v_j\|_2^{-2} + \sum_{j_0=2}^{\infty}\sum_j \frac{\beta^{j_0}}{j_0!}\frac{\| v_j \|_{2j_0}^{2j_0}}{\| v_j \|_2^2} \quad (5.18)$$
$$= \sum_j F_{2(2+\epsilon),2+\epsilon,0,\beta}(v_j) + \frac{\beta^2}{2}\sum_j \frac{\| v_j \|_4^4}{\| v_j \|_2^2}\left(\|\nabla^{2+\epsilon}v_j\|_2^{-2} - 1\right)$$
$$\geq \mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)}) + \frac{\beta^2}{2}\sum_j \frac{\| v_j \|_4^4}{\| v_j \|_2^2}\left(\|\nabla^{2+\epsilon}v_j\|_2^{-2} - 1\right)$$

which implies that $\|\nabla^{2+\epsilon}v_j\|_2^2 \geq 1$. Thus, $v_j$ is a maximizer for $\mu_{2(2+\epsilon),2+\epsilon,0,\beta}(\mathbb{R}^{2(2+\epsilon)})$ which completes the proof of Theorem 1.11.

## 6 Proofs of Theorems 1.12 and 1.13

We show some applications of Theorem 1.1 and Theorem 1.7. We first show the higher order critical Caffarelli-Kohn-Nirenberg inequalities which are not in [40]. We also investigate the relationship between the best constants of the singular Adams inequality and the Caffarelli-Kohn-Nirenberg inequality in the asymptotic sense.

**Proof of Theorem 1.12 (see [64])** We first give the proof of inequality (1.10). Denoting

$$\beta_0 := s\left\{\beta: \int_{\mathbb{R}^{1+2\epsilon}}\sum_j \frac{\Phi_{1+2\epsilon,2}\left(\beta|u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}}dx \right.$$
$$\left. \leq C(1+2\epsilon,1+\epsilon)\left(\int_{\mathbb{R}^{1+2\epsilon}}\sum_j |u_j(x)|^{\frac{1+2\epsilon}{2}}dx\right)^{\frac{\epsilon}{1+2\epsilon}}, \forall u_j \in W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon}) \text{ with } \| \Delta u_j \|_{\frac{1+2\epsilon}{2}} \leq 1\right\}$$

then for any $\beta < \beta_0$, there exists a constant $C(1+2\epsilon,1+\epsilon) > 0$ such that for $u_j \in W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ and $1+\epsilon \geq j_{\frac{1+2\epsilon}{2}} - 1$, there holds

$$C(1+2\epsilon,1+\epsilon)\sum_j\left(\frac{\| u_j \|_{\frac{1+2\epsilon}{2}}}{\| \Delta u_j \|_{\frac{1+2\epsilon}{2}}}\right)^{\frac{\epsilon}{2}} \geq \int_{\mathbb{R}^{1+2\epsilon}}\sum_j \frac{\Phi_{1+2\epsilon,2}\left(\beta\left(\frac{|u_j|}{\| \Delta u_j \|_{\frac{1+2\epsilon}{2}}}\right)^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}}dx$$

$$\geq \frac{\beta^{1+\epsilon}}{(1+\epsilon)!}\sum_j\left(\frac{\| u_j \|_{L^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)}}{\| \Delta u_j \|_{\frac{1+2\epsilon}{2}}}\right)^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)}, \tag{6.1}$$

which implies that for $u_j \in W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ and $1+\epsilon \geq j_{\frac{1+2\epsilon}{2}} - 1$,

$$\| \sum_j u_j \|_{L^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq \left(C(1+2\epsilon,1+\epsilon)\frac{(1+\epsilon)!}{\beta^{1+\epsilon}}\right)^{\frac{2\epsilon-1}{(1+2\epsilon)(1+\epsilon)}}\sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{\epsilon}{2\left(\frac{1+2\epsilon}{2\epsilon-1}\right)(1+\epsilon)}}$$
$$\| \Delta u_j \|_{\frac{1+2\epsilon}{2}}^{1-\frac{\epsilon}{2\left(\frac{1+2\epsilon}{2\epsilon-1}\right)(1+\epsilon)}} \tag{6.2}$$

For any $1+\epsilon \geq \frac{1+2\epsilon}{2\epsilon-1}\left(j_{\frac{1+2\epsilon}{2}} - 1\right)$, there exists $1+\epsilon \geq j_{\frac{1+2\epsilon}{2}} - 1$ satisfying $\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon) \leq 1+\epsilon < \frac{1+2\epsilon}{2\epsilon-1}(\epsilon)$ such that

$$\| \sum_j u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq \sum_j \| u_j \|^{\theta}_{L^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)}$$
$$\| u_j \|^{1-\theta}_{L^{\frac{1+2\epsilon}{2\epsilon-1}(2+\epsilon)}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)}{}^{\circ} \tag{6.3}$$

Combining (6.2) with (6.3) and the fact $\frac{1}{1+\epsilon} = \frac{\theta}{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)} + \frac{1-\theta}{\frac{1+2\epsilon}{2\epsilon-1}(2+\epsilon)}$, one can conclude that

$$\| \sum_j u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq \mathrm{C}(1+2\epsilon,1+\epsilon)^{\frac{1}{1+2\epsilon}}\beta^{-\frac{2\epsilon-1}{1+2\epsilon}}\left((k+1)!\right)^{\frac{1}{1+2\epsilon}}\sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{2\epsilon}{2(1+2\epsilon)}}$$
$$\| \Delta u_j \|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{2(1+2\epsilon)}} \tag{6.4}$$

since $\frac{(1+\epsilon)(2\epsilon-1)}{1+2\epsilon} \geq 1+\epsilon$, we have

$$((2+\epsilon)!)^{\frac{1}{1+\epsilon}} \leq \left(\Gamma\left(\frac{(1+\epsilon)(2\epsilon-1)}{1+2\epsilon}+2\right)\right)^{\frac{1}{1+\epsilon}} \tag{6.5}$$

By (6.4) and (6.5), one can derive inequality (1.10) with estimating $c(1+2\epsilon,1+\epsilon,1+\epsilon) \approx C(1+2\epsilon,1+\epsilon)^{\frac{1}{1+\epsilon}}\beta^{-\frac{2\epsilon-1}{1+2\epsilon}}\left(\Gamma\left(\frac{(1+\epsilon)(2\epsilon-1)}{1+2\epsilon}+2\right)\right)^{\frac{1}{1+\epsilon}}$.

Next, we claim that there exists $\epsilon \geq 0$ such that $c(1+2\epsilon,1+\epsilon,1+\epsilon)$ behaves like $c(1+2\epsilon,1+\epsilon,1+\epsilon) \simeq (1+\epsilon)(1+\epsilon)^{\frac{2\epsilon-1}{1+2\epsilon}}$ as $1+\epsilon \to +\infty$ which is equivalent to say by recalling Stirling's asymptotic formula, we see that as $1+\epsilon \to \infty$,

$$\left(\Gamma\left(\frac{(1+\epsilon)(2\epsilon-1)}{1+2\epsilon}+2\right)\right)^{\frac{1}{1+\epsilon}} = (1+o(1))\left(\frac{1+\epsilon}{e\,\frac{1+2\epsilon}{2\epsilon-1}}\right)^{\frac{2\epsilon-1}{1+2\epsilon}}$$

Therefore, we derive that

$$\| \sum_j u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq \left(1+\mathrm{o}(1)\right)\left(\frac{1+2\epsilon}{\beta e\left(\frac{2\epsilon-1}{1+2\epsilon}\right)}\right)^{\frac{2\epsilon-1}{1+2\epsilon}}\sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{2\epsilon}{2(1+2\epsilon)}} \| \Delta u_j \|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{2(1+2\epsilon)}} \tag{6.6}$$

which accomplishes the claim.

At last, we show the relationship between $\beta_0$ and $\alpha_{1+2\epsilon,1+\epsilon}$,

$$\alpha_{1+2\epsilon,1+\epsilon} := \inf\left\{\epsilon \geq 0: \exists 1+\epsilon \geq j_{\frac{1+2\epsilon}{2}}, \forall\epsilon \geq 0, \| \sum_j u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq (1+\epsilon)(1+2\epsilon)^{\frac{2\epsilon-1}{1+2\epsilon}} \sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{2\epsilon}{2(1+2\epsilon)}} \| \Delta u_j \|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{2(1+2\epsilon)}}\right\}$$

where according to the definition of $\alpha_{1+2\epsilon,1+\epsilon}$, combining inequality (6.6), one can derive that $\alpha_{1+2\epsilon,1+\epsilon} \leq \left(\frac{1}{\beta e \frac{1+2\epsilon}{2\epsilon-1}}\right)^{\frac{2\epsilon-1}{1+2\epsilon}}$. Then it follows from the definition of $\beta_0$ that

$$\alpha_{1+2\epsilon,1+\epsilon} \leq \left(\frac{1}{\beta_0 e \frac{1+2\epsilon}{2\epsilon-1}}\right)^{\frac{2\epsilon-1}{1+2\epsilon}} \tag{6.7}$$

Then it suffices to show that

$$\alpha_{1+2\epsilon,1+\epsilon} \geq \left(\frac{1}{\beta_0 e \frac{1+2\epsilon}{2\epsilon-1}}\right)^{\frac{2\epsilon-1}{1+2\epsilon}}$$

Pick any $\alpha > \alpha_{1+2\epsilon,1+\epsilon}$, through the definition of $\alpha_{1+2\epsilon,1+\epsilon}$, there exists $1+\epsilon \geq j_{\frac{1+2\epsilon}{2}}$ such that for any $\epsilon \geq 0$

$$\| \sum_j u_j \|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \leq (1+\epsilon)(1+2\epsilon)^{\frac{2\epsilon-1}{1+2\epsilon}} \sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{2\epsilon}{2(1+2\epsilon)}} \| \Delta u_j \|_{\frac{2+\epsilon}{2}}^{\frac{2+\epsilon}{2(1+2\epsilon)}} \tag{6.8}$$

Then for $u_j \in W^{2,\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon})$ and $\| \Delta u_j \|_{\frac{1+2\epsilon}{2}} \leq 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^{1+2\epsilon}} \sum_j \frac{\Phi_{1+2\epsilon,2}\left(\beta |u_j|^{\frac{1+2\epsilon}{2\epsilon-1}}\right)}{|x|^{1+\epsilon}} dx \\
&= \int_{\mathbb{R}^{1+2\epsilon}} \left( \sum_{j\frac{\pi}{2} \leq \frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon) < 1+\epsilon} \sum_j \frac{\beta^{1+\epsilon}}{(1+\epsilon)!} |u_j(x)|^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)} \right) \frac{dx}{|x|^{1+\epsilon}} \\
&+ \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \left( \sum_{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon) \geq 1+\epsilon} \frac{\beta^{1+\epsilon}}{(1+\epsilon)!} |u_j(x)|^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)} \right) \frac{dx}{|x|^{1+\epsilon}} \\
&=: J_1 + J_2. \qquad (6.9)
\end{aligned}$$

Since $J_1$ consists of finite weighted norms and $\frac{1+2\epsilon}{2} \leq \frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon) < 1+\epsilon$, one can get

$$\| \sum_j u_j \|_{L^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)}(\mathbb{R}^{1+2\epsilon};|x|^{-1+\epsilon}dx)} \leq C \sum_j \| u_j \|^{\theta}_{L^{\frac{1+2\epsilon}{2}}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \| u_j \|^{1-\theta}_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon};|x|^{-(1+\epsilon)}dx)} \tag{6.10}$$

through using the Hölder inequality. Taking (6.8) and (6.10) into consideration, we get that for all $\frac{1+2\epsilon}{2} \leq \frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon) < 1+\epsilon$,

$$\| \sum_j u_j \|_{L^{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)}(\mathbb{R}^{1+2\epsilon};|x|^{-1+\epsilon}dx)} \leq C \sum_j \| u_j \|_{\frac{\pi}{2}}^{\frac{\epsilon}{2\left(\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)\right)}}, \tag{6.11}$$

where we used the fact that $\| \Delta u_j \|_{\frac{1+2\epsilon}{2}} \leq 1$. Then it follows from (6.11) that

$$J_1 \leq C \left( \sum_{\frac{1+2\epsilon}{2} \leq \frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon) < 1+\epsilon} \frac{\beta^{1+\epsilon}}{(1+\epsilon)!} \right) \sum_j \| u_j \|_{\frac{1+2\epsilon}{2}}^{\frac{\epsilon}{2}} \tag{6.12}$$

For $J_2$, inequality (6.8) leads to

$$J_2 \leq \left( \sum_{\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon) \geq 1+\epsilon} \frac{(1+\epsilon)^{1+\epsilon}}{(1+\epsilon)!} \left( \beta \frac{1+2\epsilon}{2\epsilon-1} (1+\epsilon)^{\frac{1+2\epsilon}{2\epsilon-1}} \right)^{1+\epsilon} \right) \sum_j \| u_j \|_{\frac{1+2\epsilon}{\epsilon}}. \tag{6.13}$$

Then it follows from the Stirling's asymptotic formula that the power in (6.13) converges if $\beta\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)^{\frac{1+2\epsilon}{2\epsilon-1}}<\frac{1}{e}$, which implies that $\beta\in\left(0,\frac{1}{e\frac{1+2\epsilon}{2\epsilon-1}\alpha'}\right)$. Hence, the definition of $\beta_0$ leads to $\beta_0\geq\frac{1}{e\frac{1+2\epsilon}{2\epsilon-1}(1+\epsilon)^{\frac{1+2\epsilon}{2\epsilon-1}}}$. Moreover, through the definition of $\alpha_{1+2\epsilon,1+\epsilon}$, we get that

$$\beta_0\geq\frac{1}{e\dfrac{1+2\epsilon}{2\epsilon-1}\alpha_{1+2\epsilon,1+\epsilon}^{\frac{1+2\epsilon}{2\epsilon-1}}}$$

which is equivalent to

$$\alpha_{1+2\epsilon,1+\epsilon}\geq\left(\frac{1}{e\dfrac{1+2\epsilon}{2\epsilon-1}\beta_0}\right)^{\frac{2\epsilon-1}{1+2\epsilon}} \tag{6.14}$$

Combining (6.7) and (6.14), we complete the proof.

**Remark 6.1** The proof of Theorem 1.13 is similar to that of Theorem 1.12.

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