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Research Paper

On Fourier Step Multipliers and Multiplications Actingon Quasi-Banach Modulation Spaces

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Abstract
For the clearity, consistency and simplicity we restate and follow the pioneer of the paper [38] to show the boundedness of a general class of multipliers and Fourier multipliers, in particular of the Hilbert transform, on quasi-Banach modulation spaces. And deduce boundedness for multiplications and convolutions for elements in such spaces with slightly small changes in the sequel.

Keywords: *Hilbert transform, convolutions, multiplications*

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I. Introduction

We deduce mapping properties of step multipliers and Fourier step multipliers when acting on quasi-Banach modulation spaces. Some parts of our investigations are based on certain continuity properties for multiplications and convolutions for elements in such spaces, deduced in Section 3, and which might be of independent interests. So the Hilbert transform, i.e. multiplication by the signum function on the Fourier transform side, is frequently used in mathematics, science and technology. In physics it can be used to secure causality. For example, in optics, the refractive index of a material is the frequency response of a causal system whose real part gives the phase shift of the penetrating light and the imaginary part gives the attenuation. The relationship between the two are given by the Hilbert transform. Consequently, knowledge of one is sufficient to retrieve the other. Hence the inconveniently property with the Hilbert transform concerns lack of continuity when acting on commonly used spaces. For example, it is well-known that the Hilbert transform is continuous on L^2, but fails to be continuous on $L^{1+\epsilon}$ for $\epsilon \neq 1$ as well as on $\mathcal{S}$. (See [22]). A pioneering contribution which drastically improve the situation concerns [23], who showed that the Hilbert transform is continuous on the modulation space $M^{1+\epsilon,1+\epsilon}$ when $0 \le \epsilon \le \infty$. The result is surprising because $M^{1+\epsilon,1+\epsilon}$ is rather close to $L^{1+\epsilon}$ when $(1+\epsilon)$ stays between $(1+\epsilon)$ and $\left(\frac{1+\epsilon}{\epsilon}\right)$ (see e. g. [8, 29]).

The result in [23] was extended in [3], where Bényi, Grafakos, Gröchenig and Okoudjou show that Fourier step multipliers, i.e. Fourier multipliers of the form

$$f_\rho \mapsto \mathcal{F}^{-1}\left(\sum_{j\in b\mathbf{Z}}\sum_{\rho} a_0(j)\chi_{j+[0,b)}\hat{f}_\rho\right), a_0 \in \ell^\infty(b\mathbf{Z}) \tag{0.1}$$

are continuous on the modulation space $M^{1+\epsilon,1+\epsilon}(\mathbf{R}^d)$, when $0<\epsilon<\infty$ and $0 \le \epsilon \le \infty$. (See [3, Theorem 1].)

Note that modulation spaces is a family of function and distribution spaces introduced by [8] and further developed by [10-13, 17]. In particular, the modulation spaces $M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ and $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ are the set of tempered (or Gelfand-Shilov) distributions whose short-time Fourier transforms belong to the weighted and mixed Lebesgue spaces $L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^{2d})$ respectively $L^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\mathbf{R}^{2d})$. Here ω_ρ is a weight function on phase (or time-frequency shift) space and $0<\epsilon\le\infty$. Note that $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ is also an example on Wiener-amalgam spaces (cf. [10]).

There are several convenient characterizations of modulation spaces. For example, in [9, 14, 17, 18, it is shown that modulation spaces admit reconstructible sequence space representations using Gabor frames.
[38] extend [3, Theorem 1] in several ways (see Theorems 2.1 and (2.3).

(1) The condition $0 \le \epsilon \le \infty$ is relaxed into $0 < \epsilon \le \infty$.

(2) We allow weighted modulation spaces $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$, where the weight ω_ρ only depends on the momentum or frequency variable ξ, i.e. $\omega_\rho(x,\xi) = \omega_\rho(\xi)$. These weights are allowed to grow or decay at infinity, faster than polynomial growth.

(3) Our analysis also include continuity properties for the modulation spaces $W^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$.

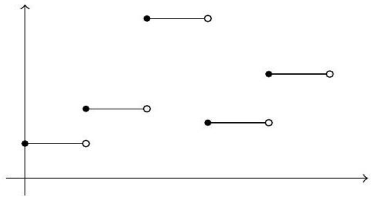
As in [3], we use Gabor analysis for modulation spaces to show these properties. In [3] the continuity for Fourier step multipliers are obtained by a convenient choice of Gabor atoms in terms of Fourier transforms of second order B-splines. This essentially transfer the critical continuity questions to a finite set of discrete convolution operators acting on $\ell^{1+\epsilon}$, with dominating operator being the discrete Hilbert transform. The choice of Gabor atoms then admit precise estimates of the appeared convolution operators.

In our situation the B-splines above are insufficient, because B-splines lack in regularity, and when $(1+\epsilon)$ approaches 0 , unbounded regularity on the Fourier transform of the Gabor atoms are required. In fact, in order to obtain continuity for weighted modulation spaces with general moderate weights in the momentum variables, it is required that the Fourier transform of Gabor atoms obey even stronger regularities of Gevrey types.

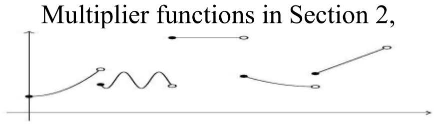
In Section 4 we obtain some further extensions and deduce precise estimates of the Fourier multipliers in (0.1), where more restrictive a_0 should belong to $\ell^{1+\epsilon}(b\mathbf{Z})$ for some $0 \le \epsilon \le \infty$. In the end we are able to prove that the Fourier multiplier in (0.1) is continuous from $M^{1+\epsilon,1+\epsilon}$ to $M^{1+\epsilon,1+2\epsilon}$ when $0 < \epsilon < \infty$ and $0 \le \epsilon \le \infty$ satisfy

$$\frac{1}{1+2\epsilon} - \frac{1}{1+\epsilon} \le \frac{1}{1+\epsilon}$$

More generally, in Section 4 we generalize the continuity properties for the step and Fourier step multiplier results in Section 2 with more general slope step multiplier and Fourier slope step multipliers.



Multiplier functions in Section 2,



Multiplier functions in Section 4 (see [38]).

An important ingredient for the proofs of the latter extension is multiplication and convolution properties for $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}$ and $W^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}$ spaces, given in Section 3.

Proposition 0.1 [38]. Let $0 \le \epsilon \le \infty$, then

$$\theta_1 = \max\left(1, \frac{1}{1+\epsilon}, \frac{1}{1+4\epsilon}, \frac{1}{1+5\epsilon}\right) \text{ and } \theta_2 = \max\left(1, \frac{1}{1+\epsilon}, \frac{1}{1+5\epsilon}\right)$$

Then

$$M^{1+3\epsilon,1+4\epsilon} \cdot M^{1+5\epsilon,1+6\epsilon} \subseteq M^{1+\epsilon,1+2\epsilon}, \quad \frac{1}{1+3\epsilon} + \frac{1}{1+5\epsilon} = \frac{1}{1+\epsilon}, \quad \frac{1}{1+4\epsilon} + \frac{1}{1+5\epsilon} = \theta_1 + \frac{1}{1+2\epsilon},$$

$$M^{1+3\epsilon,1+4\epsilon} * M^{1+5\epsilon,1+6\epsilon} \subseteq M^{1+\epsilon,1+2\epsilon}, \quad \frac{1}{1+3\epsilon} + \frac{1}{1+5\epsilon} = \theta_2 + \frac{1}{1+\epsilon}, \quad \frac{1}{1+4\epsilon} + \frac{1}{1+6\epsilon} = \frac{1}{1+2\epsilon}.$$

Similar result holds for $W^{1+\epsilon,1+\epsilon}$ spaces. The general multiplication and convolution properties in Section 3 also overlap with results by [1], [2], and [21].

The multiplication relation in Proposition 0.1 for $\epsilon \ge 0$ was obtained already in [8]. It is also obvious that the convolution relation was well-known since then (though a first formal proof of this relation seems to be given first in [30]). In general, these convolution and multiplication properties follow the rules

$$\ell^{1+3\epsilon} * \ell^{1+5\epsilon} \subseteq \ell^{1+\epsilon}, \ell^{1+4\epsilon} \cdot \ell^{1+6\epsilon} \subseteq \ell^{1+2\epsilon} \Rightarrow M^{1+3\epsilon,1+4\epsilon} * M^{1+5\epsilon,1+6\epsilon} \subseteq M^{1+\epsilon,1+2\epsilon}$$

and

$$\ell^{1+3\epsilon} \cdot \ell^{1+5\epsilon} \subseteq \ell^{1+\epsilon}, \ell^{1+4\epsilon} * \ell^{1+6\epsilon} \subseteq \ell^{1+2\epsilon} \Rightarrow M^{1+3\epsilon,1+6\epsilon} \cdot M^{1+5\epsilon,1+6\epsilon} \subseteq M^{1+\epsilon,1+2\epsilon}$$

which goes back to [8] in the Banach space case and to [14] in the quasi-Banach case. See also [11] and [26] for extensions of these relations to more general Banach function spaces and quasi-Banach function spaces, respectively.

In Section 3 we extend the multiplication and convolution results in [1, 2, 21] to allow more general weights as well as finding multi-linear versions. We stress that the results in Section 3 hold true for general moderate weights, while corresponding results in [21] are formulated only for polynomially moderate weights which also should be split, i.e. of the form $\omega_\rho(x,\xi) = (\omega_\rho)_1(x)(\omega_\rho)_2(\xi)$. In Section 3 we also carry out questions on uniqueness for extensions of multiplications and convolutions from the Gelfand-Shilov space $\Sigma_1(\mathbf{R}^d)$, to the involved modulation spaces. Note that $\Sigma_1(\mathbf{R}^d)$ is dense in $\mathcal{S}(\mathbf{R}^d)$ and is contained in all modulation spaces with moderate weights(see e. g. [31]). On the other hand, in contrast to [21], we do not deduce any sharpness for our results.

The analysis to show Proposition 0.1 is more complex compared to the restricted case when $\epsilon \geq 0$, because of absence of local-convexity of involved spaces when some of the Lebesgue exponents are smaller than one. In fact, the desired estimates when $\epsilon \geq 0$ can be achieved by straightforward applications of Hölder's and Young's inequalities. For corresponding estimates in Proposition 0.1, some additional arguments are needed. In our situation we discretize the situations in similar ways as in [1] by using Gabor analysis for modulation spaces, and then apply some further arguments, valid in non-convex analysis. This approach is slightly different compared with [21] which follows the discretization technique introduced in [36], and which has some traces of Gabor analysis.

A non-trivial question concerns wether the multiplications and convolutions in Propositions 0.1 and 0.1 are uniquely defined or not. If $\epsilon < \infty$, $j = 1,2$, then the uniqueness is evident because the Schwartz space is dense in M^{p_j,q_j}. In the case $\epsilon < \infty$, the uniqueness in Proposition 0.1 follows from the first case, duality and embedding properties for quasi-Banach modulation spaces into Banach modulation spaces. The uniqueness in 0.1 then follows from the uniqueness in Proposition 0.1 and the fact that $M^{1+\epsilon,1+\epsilon}$ increases with $(1+\epsilon)$.

A critical situation appear when $2 + 7\epsilon = 2 + 4\epsilon = \infty$. Then $\mathcal{S}$ is neither dense in $M^{1+3\epsilon,1+4\epsilon}$ nor in $M^{1+5\epsilon,1+6\epsilon}$. For the multiplications in Propositions 0.1, the uniqueness can be obtained by suitable approaches based on the so-called narrow convergence, which is a weaker form of convergence compared to norm convergence (see [28, 29, 31]). However, for the convolution in Propositions 0.1, we are not able to show any uniqueness of these extensions in this critical situation.

[38] present well-known properties of Gelfand-Shilov spaces, modulation spaces, multipliers and Fourier multipliers. They deduce continuity properties for step and Fourier step multipliers when acting on (quasi-Banach) modulation spaces. Then establish convolution and continuity properties for quasi-Banach modulation spaces. They show how the multiplication and convolution results can be used to generalize the continuity results, to more general slope step multiplier and Fourier slope step multipliers. Finallythey present a proof of a multi-linear convolution result in Appendix A.

1. Preliminaries

We present some facts on Gelfand-Shilov spaces, modulation spaces, discrete convolutions, step and Fourier step multipliers. After explaining some properties of the Gelfand-Shilov spaces and their distribution spaces, we consider a suitable twisted convolution and recall some facts on weight functions and mixed norm spaces. Thereafter we consider classical modulation spaces, which are more general compared in [8] in the sense of more general weights as well as we permit the Lebesgue exponents to belong to the full interval $(0,\infty]$ instead of $[1,\infty]$. Here we also recall some facts on Gabor expansions for modulation spaces. Then we collect some facts on discrete convolution estimates on weighted $\ell^{1+\epsilon}$ spaces with the exponents in the full interval $(0,\infty]$. We finish the section by giving the definition of step and Fourier step multipliers (see [38]).

1.1. Gelfand-Shilov spaces and their distribution spaces. For any $\epsilon \geq 0$ and belong to$\mathbf{R}, \mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ consists of all $f_\rho \in C^\infty(\mathbf{R}^d)$ such that

$$\|f_\rho\|_{\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}} \equiv \sup \sum_\rho \frac{|x^\beta \partial^\alpha f_\rho(x)|}{(1+\epsilon)^{|\alpha+\beta|}\alpha!^{1+3\epsilon}\beta!^{1+2\epsilon}} \tag{1.1}$$

is finite. Then $\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ is a Banach space with norm $\|\cdot\|_{\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}}$. The GelfandShilov spaces $\mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ and $\Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$, of Roumieu and Beurling types respectively, are the inductive and projective limits of $\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ with respect to $\epsilon \geq 0$ (see e.g. [15]). It follows that

$$\mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) = \bigcup_{\epsilon\geq 0} \mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \text{ and } \Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) = \bigcap_{\epsilon\geq 0} \mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \tag{1.2}$$

We remark that $\Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \neq \{0\}$, if and only if $2+5\epsilon > 1$, and $\mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \neq \{0\}$, if and only if $2+5\epsilon \geq 1$, and that

$$\mathcal{S}_{s_1}^{\sigma_1}(\mathbf{R}^d) \subseteq \Sigma_{s_2}^{\sigma_2}(\mathbf{R}^d) \subseteq \mathcal{S}_{s_2}^{\sigma_2}(\mathbf{R}^d) \subseteq \mathcal{S}(\mathbf{R}^d), s_1 < s_2, \sigma_1 < \sigma_2$$

The Gelfand-Shilov distribution spaces $(\mathcal{S}_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d)$ and $(\Sigma_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d)$, of Roumieu and Beurling types respectively, are the (strong) duals of $\mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ and $\Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$, respectively. It follows that if $\left(\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}\right)'(\mathbf{R}^d)$ is the L^2-dual of $\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ and $2+5\epsilon \geq 1(2+5\epsilon > 1)$, then $(\mathcal{S}_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d)\left((\Sigma_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d)\right)$ can be identified with the projective limit (inductive limit) of $\left(\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}\right)'(\mathbf{R}^d)$ with respect to $\epsilon \geq 0$. It follows that

$$(\mathcal{S}_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d) = \bigcap_{\epsilon\geq 0} \left(\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}\right)'(\mathbf{R}^d) \text{ and } \Sigma_{1+2\epsilon}'(\mathbf{R}^d) = \bigcup_{\epsilon\geq 0} \left(\mathcal{S}_{1+2\epsilon,1+\epsilon}^{1+3\epsilon}\right)'(\mathbf{R}^d) \quad (1.3)$$

for such choices of $(1+2\epsilon)$ and $(1+3\epsilon)$. (See 24.) We remark that

$$\mathcal{S}'(\mathbf{R}^d) \subseteq \left(\mathcal{S}_{s_2}^{\sigma_2}\right)'(\mathbf{R}^d) \subseteq \left(\Sigma_{s_2}^{\sigma_2}\right)'(\mathbf{R}^d) \subseteq \left(\mathcal{S}_{s_1}^{\sigma_1}\right)'(\mathbf{R}^d),$$

when

$$s_1 < s_2, \sigma_1 < \sigma_2 \text{ and } s_1 + \sigma_1 \geq 1$$

For convenience we set $\mathcal{S}_{1+2\epsilon} = \mathcal{S}_{1+2\epsilon}^{1+2\epsilon}$ and $\Sigma_{1+2\epsilon} = \Sigma_{1+2\epsilon}^{1+2\epsilon}$.

The Gelfand-Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations and under (partial) Fourier transformations. In fact, let $\mathcal{F}$ be the Fourier transform which takes the form

$$(\mathcal{F}f_\rho)(\xi) = \hat{f}_\rho(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \sum_\rho f_\rho(x) e^{-i\langle x,\xi\rangle} dx$$

when $f_\rho \in L^1(\mathbf{R}^d)$. Here $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $\mathbf{R}^d$. The map $\mathcal{F}$ extends uniquely to homeomorphisms on $\mathcal{S}'(\mathbf{R}^d)$, from $(\mathcal{S}_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d)$ to $(\mathcal{S}_{1+3\epsilon}^{1+2\epsilon})'(\mathbf{R}^d)$ and from $(\Sigma_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d)$ to $(\Sigma_{1+3\epsilon}^{1+2\epsilon})'(\mathbf{R}^d)$. Then the map $\mathcal{F}$ restricts to homeomorphisms on $\mathcal{S}(\mathbf{R}^d)$, from $\mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ to $\mathcal{S}_{1+3\epsilon}^{1+2\epsilon}(\mathbf{R}^d)$, from $\Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)$ to $\Sigma_{1+3\epsilon}^{1+2\epsilon}(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

There are several characterizations of Gelfand-Shilov spaces and their distribution spaces (cf. [6,7,33] and the references therein). For example, it follows from [6,7] that the following is true. Here $g_\rho(\theta) \lesssim h(\theta), \theta \in \Omega$, means that there is a constant $c > 0$ such that $g_\rho(\theta) \leq ch(\theta)$ for all $\theta \in \Omega$.

Proposition 1.1 [38]. Let $f_\rho \in \mathcal{S}'(\mathbf{R}^d)$ and $\epsilon \geq 0$. Then the following conditions are equivalent:

(1) $f_\rho \in \mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \left(f_\rho \in \Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)\right)$;

(2) $|f_\rho(x)| \lesssim e^{-(1+\epsilon)|x|^{\frac{1}{1+2\epsilon}}}$ and $|\hat{f}_\rho(\xi)| \lesssim e^{-(1+\epsilon)|\xi|^{\frac{1}{1+3\epsilon}}}$ for some $\epsilon \geq 0$ (for every $\epsilon \geq 0$);

(3) $f_\rho \in C^\infty(\mathbf{R}^d)$ and $\left|\left(\partial^\alpha f_\rho\right)(x)\right| \lesssim (1+\epsilon)^{|\alpha|}\alpha!^{1+3\epsilon}\, e^{-(1+2\epsilon)|x|^{\frac{1}{1+2\epsilon}}}$ for some $\epsilon \geq 0$ (for every $\epsilon \geq 0$).

Gelfand-Shilov spaces and their distribution spaces can also be characterized by estimates on their short-time Fourier transforms Let $\phi_\rho \in \mathcal{S}_{1+2\epsilon}(\mathbf{R}^d) \left(\phi_\rho \in \Sigma_{1+2\epsilon}(\mathbf{R}^d)\right)$ be fixed. Then the short-time Fourier transform of $f_\rho \in \mathcal{S}_{1+2\epsilon}'(\mathbf{R}^d)$ (of $f_\rho \in \Sigma_{1+2\epsilon}'(\mathbf{R}^d)$) with respect to ϕ_ρ is defined by

$$\left(V_{\phi_\rho} f_\rho\right)(x,\xi) \equiv (2\pi)^{-\frac{d}{2}} \left(f_\rho, \phi_\rho(\cdot - x) e^{i\langle\cdot,\xi\rangle}\right)_{L^2} \quad (1.4)$$

We observe that

$$\left(V_{\phi_\rho} f_\rho\right)(x,\xi) = \mathcal{F}\left(f_\rho \cdot \overline{\phi_\rho(\cdot - x)}\right)(\xi) \quad (1.4)'$$

(cf. 34). If in addition $f_\rho \in L^{1+\epsilon}(\mathbf{R}^d)$ for some $0 \leq \epsilon \leq \infty$, then

$$\left(V_{\phi_\rho} f_\rho\right)(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \sum_\rho f_\rho(y) \overline{\phi_\rho(y-x)} e^{-i\langle y,\xi\rangle} dy \quad (1.4)''$$

In the next lemma we present characterizations of Gelfand-Shilov spaces and their distribution spaces in terms of estimates on the short-time Fourier transforms of the involved elements. The proof is omitted, since the first part follows from [20, and the second part from [31, 33].

Lemma 1.2 [38]. Let $0 \leq \epsilon \leq \infty, f_\rho \in \mathcal{S}_{\frac{1}{2}}'(\mathbf{R}^d), \epsilon \geq 0, \phi_\rho \in \mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \setminus 0 \left(\phi_\rho \in \Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \setminus 0\right)$ and

$$v_{1+2\epsilon}(x,\xi) = e^{(1+2\epsilon)\left(|x|^{\frac{1}{1+2\epsilon}} + |\xi|^{\frac{1}{1+3\epsilon}}\right)}, \epsilon \geq 0$$

Then the following is true:

(1) $f_\rho \in \mathcal{S}_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d) \left(f_\rho \in \Sigma_{1+2\epsilon}^{1+3\epsilon}(\mathbf{R}^d)\right)$, if and only if

$$\left\|\sum_\rho V_{\phi_\rho} f_\rho \cdot v_{1+\epsilon}\right\|_{L^{1+\epsilon}} < \infty \quad (1.5)$$

for some $\epsilon \geq 0$ (for every $\epsilon \geq 0$);

(2) $f_\rho \in (\mathcal{S}_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d) \left(f_\rho \in (\Sigma_{1+2\epsilon}^{1+3\epsilon})'(\mathbf{R}^d)\right)$, if and only if

$$\left\|\sum_\rho V_{\phi_\rho} f_\rho / v_{1+\epsilon}\right\|_{L^{1+\epsilon}} < \infty \quad (1.6)$$

for every $\epsilon \geq 0$ (for some $\epsilon \geq 0$).

We also need the following. Here the first part is a straight-forward consequence of the definitions, and the second part follows from the first part and duality.

Proposition 1.3 [38]. Let $\phi_\rho \in \Sigma_{1+2\epsilon}(\mathbf{R}^d) \setminus 0$. Then the following is true:

(1) V_{ϕ_ρ} is continuous from $\Sigma_{1+2\epsilon}(\mathbf{R}^d)$ to $\Sigma_{1+2\epsilon}(\mathbf{R}^{2d})$ and from $\Sigma'_{1+2\epsilon}(\mathbf{R}^d)$ to $\Sigma'_{1+2\epsilon}(\mathbf{R}^{2d})$;

(2) $V^*_{\phi_\rho}$ is continuous from $\Sigma_{1+2\epsilon}(\mathbf{R}^{2d})$ to $\Sigma_{1+2\epsilon}(\mathbf{R}^d)$ and from $\Sigma'_{1+2\epsilon}(\mathbf{R}^{2d})$ to $\Sigma'_{1+2\epsilon}(\mathbf{R}^d)$.

The same holds true with $\mathcal{S}_{1+2\epsilon}$ or $\mathcal{S}$ in place of $\Sigma_{1+2\epsilon}$ at each occurrence.

1.2. A suitable twisted convolution. Let f_ρ be a distribution on $\mathbf{R}^d$, $\phi_\rho, (\phi_\rho)_j, j = 1,2,3$, be suitable test functions on $\mathbf{R}^d$, and let F_ρ and G be a pair of suitable distribution/test function on $\mathbf{R}^{2d}$. Then the twisted convolution $F_\rho *_V G$ of F_ρ and G is defined by

$$\left(F_\rho *_V G\right)(x,\xi) = (2\pi)^{-\frac{d}{2}} \iint_{\mathbf{R}^{2d}} \sum_\rho F_\rho(x-y,\xi-\eta)G(y,\eta)e^{-i\langle y,\xi-\eta\rangle}dyd\eta$$

$$= (2\pi)^{-\frac{d}{2}} \iint_{\mathbf{R}^{2d}} \sum_\rho F_\rho(y,\eta)G(x-y,\xi-\eta)e^{-i\langle x-y,\eta\rangle}dyd\eta \qquad (1.7)$$

The convolution above should be interpreted as

$$\left(F_\rho *_V G\right)(X) = (2\pi)^{-\frac{d}{2}}\left\langle F_\rho(X-\cdot)e^{-i\Phi_\rho(X,\cdot)}, G\right\rangle$$

$$= (2\pi)^{-\frac{d}{2}}\left\langle F_\rho, G(X-\cdot)e^{-i\Phi_\rho(X,X-\cdot)}\right\rangle \qquad (1.7)'$$

where$\Phi_\rho(X,Y) = \langle y,\xi-\eta\rangle, X=(x,\xi)\in\mathbf{R}^{2d}, Y=(y,\eta)\in\mathbf{R}^{2d}$.

when F_ρ belongs to a distribution space on $\mathbf{R}^{2d}$ and G belongs to the corresponding test function space. By straight-forward computations it follows that

$$\left(F_\rho *_V G\right) *_V H_\rho = F_\rho *_V \left(G *_V H_\rho\right) \qquad (1.8)$$

when F_ρ, H_ρ are distributions and G is a test function, or F_ρ, H_ρ are test functions and G is a distribution.

Remark 1.4 [38]. Let $\epsilon \geq 0$. An important property of $*_V$ above is that if $f_\rho \in \Sigma'_{1+\epsilon}(\mathbf{R}^d)$ and $(\phi_\rho)_j \in \Sigma_{1+\epsilon}(\mathbf{R}^d)$ and $\phi_\rho \in \Sigma_{1+\epsilon}(\mathbf{R}^d)\setminus 0, j=1,2,3$, then it follows by straight-forward applications of Parseval's formula that

$$\left(\left(V_{(\phi_\rho)_2}(\phi_\rho)_3\right) *_V \left(V_{(\phi_\rho)_1}f_\rho\right)\right)(x,\xi) = \left((\phi_\rho)_3,(\phi_\rho)_1\right)_{L^2}\cdot\left(V_{(\phi_\rho)_2}f_\rho\right)(x,\xi). \qquad (1.9)$$

and that if

$$P_{\phi_\rho} \equiv \|\phi_\rho\|_{L^2}^{-2}\cdot V_{\phi_\rho}\circ V^*_{\phi_\rho} \qquad (1.10)$$

then

$$P_{\phi_\rho}F_\rho = \|\phi_\rho\|_{L^2}^{-2}\cdot V_{\phi_\rho}\phi_\rho *_V F_\rho \qquad (1.11)$$

when $F_\rho \in \Sigma'_{1+\epsilon}(\mathbf{R}^{2d})$. We observe that

$$P^*_{\phi_\rho} = P_{\phi_\rho} \text{ and } P^2_{\phi_\rho} = P_{\phi_\rho}. \qquad (1.12)$$

(See e.g. Chapters 11 and 12 in [17].)

We also remark that if $F_\rho \in \Sigma'_{1+\epsilon}(\mathbf{R}^{2d})$, then $F_\rho = V_{\phi_\rho}f_\rho$ for some $f_\rho \in \Sigma'_{1+\epsilon}(\mathbf{R}^d)$, if and only if

$$F_\rho = P_{\phi_\rho}F_\rho \qquad (1.13)$$

Furthermore, if (1.13) holds, then $F_\rho = V_{\phi_\rho}f_\rho$ with

$$f_\rho = \|\phi_\rho\|_{L^2}^{-2}V^*_{\phi_\rho}F_\rho \qquad (1.14)$$

In fact, suppose that $f_\rho \in \Sigma'_{1+\epsilon}(\mathbf{R}^d)$ and let $F_\rho = V_{\phi_\rho}f_\rho$. Then (1.13) follows from (1.9).

On the other hand, suppose that (1.13) holds and let f_ρ be given by (1.14). Then

$$V_{\phi_\rho}f_\rho = P_{\phi_\rho}F_\rho = F_\rho$$

and the asserted equivalence follows.

We notice that the same holds true with $\mathcal{S}_{1+\epsilon}$ or $\mathcal{S}$ in place of $\Sigma_{1+\epsilon}$ at each occurrence.

1.3. Mixed norm space of Lebesgue types. A weight on $\mathbf{R}^d$ is a function $(\omega_\rho)_0 \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ such that $1/(\omega_\rho)_0 \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$. The weight $(\omega_\rho)_0$ on $\mathbf{R}^d$ is called moderate, if there is an other weight v on $\mathbf{R}^d$ such that

$$\omega_\rho(x+y) \lesssim \omega_\rho(x)v(y), x,y\in\mathbf{R}^d \qquad (1.15)$$

The set of moderate weights on $\mathbf{R}^d$ is denoted by $\mathcal{P}_E(\mathbf{R}^d)$, and if $\epsilon \geq 0$, then $\mathcal{P}_{E,1+\epsilon}(\mathbf{R}^d)$ is the set of all moderate weights $(\omega_\rho)_0$ on $\mathbf{R}^d$ such that (1.15) holds for $v(y) = e^{(1+\epsilon)|y|^{\frac{1}{1+\epsilon}}}$ for some $\epsilon \geq 0$. We also let $\mathcal{P}^{1+3\epsilon}_{E,1+\epsilon}(\mathbf{R}^{2d})$ be the set of all weights ω_ρ such that

$$\omega_\rho(x+y,\xi+\eta) \lesssim \omega_\rho(x,y)e^{(1+\epsilon)\left(|y|^{\frac{1}{1+\epsilon}}+|\eta|^{\frac{1}{1+3\epsilon}}\right)}$$

for some $\epsilon \geq 0$. We recall that if $\omega_\rho \in \mathcal{P}_E(\mathbf{R}^d)$, then there is a constant $\epsilon \geq -1$ such that

$$\omega_\rho(x+y) \lesssim \omega_\rho(x) e^{(1+\epsilon)|y|}, x, y \in \mathbf{R}^d$$

In particular, $\mathcal{P}_{E,1+\epsilon}(\mathbf{R}^d) = \mathcal{P}_E(\mathbf{R}^d)$ when $\epsilon \leq 0$ (see [19]).

For any weight ω_ρ on $\mathbf{R}^{2d}$ and for every $0 < \epsilon \leq \infty$, we set

$\|F_\rho\|_{L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^{2d})} \equiv \left\| G_{F_\rho,\omega_\rho,1+\epsilon} \right\|_{L^{1+2\epsilon}(\mathbf{R}^d)}$, where $G_{F_\rho,\omega_\rho,1+\epsilon}(\xi) = \|F_\rho(\cdot,\xi)\omega_\rho(\cdot,\xi)\|_{L^{1+\epsilon}(\mathbf{R}^d)}$

and

$\|F_\rho\|_{L^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\mathbf{R}^{2d})} \equiv \left\| (H_\rho)_{F_\rho,\omega_\rho,1+2\epsilon} \right\|_{L^{1+\epsilon}(\mathbf{R}^d)}$, where $(H_\rho)_{F_\rho,\omega_\rho,1+2\epsilon}(x) = \|F_\rho(x,\cdot)\omega_\rho(x,\cdot)\|_{L^{1+2\epsilon}(\mathbf{R}^d)}$, when F_ρ is (complex-valued) measurable function on $\mathbf{R}^{2d}$. Then $L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^{2d}) \left(L^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\mathbf{R}^{2d}) \right)$ consists of all measurable functions F_ρ such that $\|F_\rho\|_{L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} < \infty \left(\|F_\rho\|_{L^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}} < \infty \right)$.

In similar ways, let Ω_1, Ω_2 be discrete sets and $\ell'_0(\Omega_1 \times \Omega_2)$ consists of all formal (complex-valued) sequences $c = \{c(j,k)\}_{j\in\Omega_1, k\in\Omega_2}$. Then the discrete Lebesgue spaces

$$\ell^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\Omega_1 \times \Omega_2) \text{ and } \ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Omega_1 \times \Omega_2)$$

of mixed (quasi-)norm types consists of all $c \in \ell'_0(\Omega_1 \times \Omega_2)$ such that $\|c\|^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\Omega_1 \times \Omega_2) < \infty$ respectively $\|c\|_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Omega_1\times\Omega_2)} < \infty$. Here

$\|c\|^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\Omega_1 \times \Omega_2) \equiv \left\| G_{F_\rho,\omega_\rho,1+\epsilon} \right\|_{\ell^{1+2\epsilon}(\Omega_2)}$, where $G_{c,\omega_\rho,1+\epsilon}(k) = \|F_\rho(\cdot,k)\omega_\rho(\cdot,k)\|_{\ell^{1+\epsilon}(\Omega_1)}$

and $\|c\|_{L^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Omega_1\times\Omega_2)} \equiv \left\| (H_\rho)_{c,\omega_\rho,1+2\epsilon} \right\|_{\ell^{1+\epsilon}(\Omega_1)}$, where $(H_\rho)_{c,\omega_\rho,1+2\epsilon}(j) = \|c(j,\cdot)\omega_\rho(j,\cdot)\|_{\ell^{1+2\epsilon}(\Omega_2)}$, when $c \in \ell'_0(\Omega_1 \times \Omega_2)$.

1.4. Modulation spaces and other Wiener type spaces. The (classical) modulation spaces, essentially introduced in [8] are given in the following. (See e.g. [10] for definition of more general modulation spaces.)

Definition 1.5 [38]. Let $0 < \epsilon \leq \infty, \omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\phi_\rho \in \Sigma_1(\mathbf{R}^d) \setminus 0$.

(1) The modulation space $M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ consists of all $f_\rho \in \Sigma'_1(\mathbf{R}^d)$ such that

$$\|f_\rho\|_{M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} \equiv \left\| V_{\phi_\rho} f_\rho \right\|_{L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}}$$

is finite. The topology of $M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}}$;

(2) The modulation space (of Wiener amalgam type) $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ consists of all $f_\rho \in \Sigma'_1(\mathbf{R}^d)$ such that

$$\|f_\rho\|_{W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} \equiv \left\| V_{\phi_\rho} f_\rho \right\|_{L^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}}$$

is finite. The topology of $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}}$.

Remark 1.6 [38]. Modulation spaces possess several convenient properties. In fact, let $0 < \epsilon \leq \infty, \omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\phi_\rho \in \Sigma_1(\mathbf{R}^d) \setminus 0$. Then the following is true (see [8,10,12,14,17] and their analyses for verifications):

- the definitions of $M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ and $W^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ are independent of the choices of $\phi_\rho \in \Sigma_1(\mathbf{R}^d) \setminus 0$, and different choices give rise to equivalent quasi-norms;
- the spaces $M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ and $W^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ are quasi-Banach spaces which increase with $(1+2\epsilon)$ and $(1+\epsilon)$, and decrease with ω_ρ. If in addition $\epsilon \geq 0$, then they are Banach spaces.
- $\Sigma_1(\mathbf{R}^d) \subseteq M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d), W^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d) \subseteq \Sigma'_1(\mathbf{R}^d)$;
- If in addition $\epsilon \geq 0$, then the $L^2(\mathbf{R}^d)$ scalar product, $(\cdot,\cdot)_{L^2(\mathbf{R}^d)}$, on $\Sigma_1(\mathbf{R}^d) \times \Sigma(\mathbf{R}^d)$ is uniquely extendable to dualities between $M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ and $M^{\frac{1+2\epsilon}{2\epsilon},\frac{1+\epsilon}{\epsilon}}_{(1/\omega_\rho)}(\mathbf{R}^d)$, and between $W^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ and $W^{\frac{1+2\epsilon}{2\epsilon},\frac{1+\epsilon}{\epsilon}}_{(1/\omega_\rho)}(\mathbf{R}^d)$.

If in addition $\epsilon < \infty$, then the duals of $M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ and $W^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ can be identified with $M^{\frac{1+2\epsilon}{2\epsilon},\frac{1+\epsilon}{\epsilon}}_{(1/\omega_\rho)}(\mathbf{R}^d)$ respectively $W^{\frac{1+2\epsilon}{2\epsilon},\frac{1+\epsilon}{\epsilon}}_{(1/\omega_\rho)}(\mathbf{R}^d)$, through the form $(\cdot,\cdot)_{L^2(\mathbf{R}^d)}$;

- Let $(\omega_\rho)_0(x,\xi) = \omega_\rho(-\xi, x)$. Then $\mathcal{F}$ on $\Sigma'_1(\mathbf{R}^d)$ restricts to a homeomorphism from $M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ to $W^{1+\epsilon,1+2\epsilon}_{((\omega_\rho)_0)}(\mathbf{R}^d)$.

1.5. Gabor expansions for modulation spaces. A fundamental property for modulation spaces is that they can be discretized in convenient ways by Gabor expansions. For fundamental contributions, see e.g. [5,9,11,14, 16, 17, 20] and the references therein. Here we present a straight way to obtain such expansions in the case when we may find compactly supported Gabor atoms.

Let $\epsilon \geq 0$. Then $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ is the set of all compactly supported elements in $\mathcal{S}_{1+2\epsilon}^{1+\epsilon}(\mathbf{R}^d)$. That is, $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ consists of all $\phi_\rho \in C_0^\infty(\mathbf{R}^d)$ such that

$$\left\|\partial^\alpha \phi_\rho\right\|_{L^\infty} \lesssim (1+\epsilon)^{|\alpha|}\alpha!^{1+\epsilon}$$

holds true for some $\epsilon \geq 0$. We recall that if $\epsilon \leq 0$, then $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ is trivial (i. e. $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d) = \{0\}$). If instead $\epsilon > 0$, then $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ is dense in $C_0^\infty(\mathbf{R}^d)$.

From now on we suppose that $\epsilon > 0$, giving that $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ is non-trivial. In view of Sections 1.3 and 1.4 in [22], we may find $\phi_\rho, \psi_\rho \in \mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ with values in [0,1] such that

$$\operatorname{supp}\phi_\rho \subseteq \left[-\frac{3}{4},\frac{3}{4}\right]^d, \quad \phi_\rho(x) = 1 \quad \text{when } x \in \left[-\frac{1}{4},\frac{1}{4}\right]^d$$

and

$$\sum_{j\in\mathbf{Z}^d}\sum_{\rho} \phi_\rho(\cdot - j) = 1 \tag{1.18}$$

Let $f_\rho \in (\mathcal{S}_{1+2\epsilon}^{1+\epsilon})'(\mathbf{R}^d)$. Then $x \mapsto f_\rho(x)\phi_\rho(x-j)$ belongs to $(\mathcal{S}_{1+2\epsilon}^{1+\epsilon})'(\mathbf{R}^d)$ and is supported in $j + \left[-\frac{3}{4},\frac{3}{4}\right]^d$. Hence, by periodization it follows from Fourier analysis that

$$f_\rho(x)\phi_\rho(x-j) = \sum_{\iota\in\pi\mathbf{Z}^d} c(j,\iota)e^{i\langle x,\iota\rangle}, x \in j + [-1,1]^d \tag{1.19}$$

where

$$c(j,\iota) = 2^{-d}\left(f_\rho, \phi_\rho(\cdot - j)e^{i\langle\cdot,\iota\rangle}\right) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} V_{\phi_\rho}f_\rho(j,\iota), j \in \mathbf{Z}^d, \iota \in \pi\mathbf{Z}^d$$

Since $\psi_\rho = 1$ on the support of ϕ_ρ, (1.19) gives

$$f_\rho(x)\phi_\rho(x-j) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{\iota\in\pi\mathbf{Z}^d}\sum_{\rho} V_{\phi_\rho}f_\rho(j,\iota)\psi_\rho(x-j)e^{i\langle x,\iota\rangle}, x \in \mathbf{R}^d, \tag{1.19$'$}$$

By (1.18) it now follows that

$$f_\rho(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda}\sum_{\rho} V_{\phi_\rho}f_\rho(j,\iota)\psi_\rho(x-j)e^{i\langle x,\iota\rangle}, x \in \mathbf{R}^d \tag{1.20}$$

where

$$\Lambda = \mathbf{Z}^d \times (\pi\mathbf{Z}^d) \tag{1.21}$$

which is the Gabor expansion of f_ρ with respect to the Gabor pair (ϕ_ρ, ψ_ρ) and lattice Λ, i.e. with respect to the Gabor atom ϕ_ρ and the dual Gabor atom ψ_ρ. Here the series converges in $(\mathcal{S}_{1+2\epsilon}^{1+\epsilon})'(\mathbf{R}^d)$. By duality and the fact that $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ is dense in $(\mathcal{S}_{1+2\epsilon}^{1+\epsilon})'(\mathbf{R}^d)$ we also have

$$f_\rho(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda}\sum_{\rho} V_{\psi_\rho}f_\rho(j,\iota)\phi_\rho(x-j)e^{i\langle x,\iota\rangle}, x \in \mathbf{R}^d \tag{1.22}$$

with convergence in $(\mathcal{S}_{1+2\epsilon}^{1+\epsilon})'(\mathbf{R}^d)$.

Let T be a linear continuous operator from $\mathcal{S}_{1+2\epsilon}^{1+\epsilon}(\mathbf{R}^d)$ to $(\mathcal{S}_{1+2\epsilon}^{1+\epsilon})'(\mathbf{R}^d)$ and let $f_\rho \in \mathcal{S}_{1+2\epsilon}^{1+\epsilon}(\mathbf{R}^d)$. Then it follows from (1.20) that

$$(Tf_\rho)(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda}\sum_{\rho} V_{\phi_\rho}f_\rho(j,\iota)T\left(\psi_\rho(\cdot - j)e^{i\langle\cdot,\iota\rangle}\right)(x)$$

and

$$T\left(\psi_\rho(\cdot - j)e^{i\langle\cdot,\iota\rangle}\right)(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(k,\kappa)\in\Lambda}\sum_{\rho} \left(V_{\phi_\rho}\left(T\left(\psi_\rho(\cdot - j)e^{i\langle\cdot,\iota\rangle}\right)\right)\right)(k,\kappa)\psi_\rho(x-k)e^{i\langle x,\kappa\rangle}.$$

A combination of these expansions show that

$$(Tf_\rho)(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} \left(A \cdot V_{\phi_\rho}f_\rho\right)(j,\iota)\psi_\rho(x-j)e^{i\langle x,\iota\rangle} \tag{1.23}$$

where $A = (a(\boldsymbol{j},K))_{\boldsymbol{j},\boldsymbol{k}\in\Lambda}$ is the $\Lambda \times \Lambda$-matrix, given by

$$a(\boldsymbol{j},\boldsymbol{k}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \left(T\left(\psi_\rho(\cdot - j)e^{i\langle\cdot,\iota\rangle}\right), \phi_\rho(\cdot - k)e^{i\langle\cdot,\kappa\rangle}\right)_{L^2(\mathbf{R}^d)} \tag{1.24}$$

when $\boldsymbol{j} = (j,\iota)$ and $\boldsymbol{k} = (k,\kappa)$.

By the Gabor analysis for modulation spaces we get the following. See[9,11,14,16,17,32] for details.

Proposition 1.7 [38]. Let $\epsilon > 0, 0 < \epsilon \leq \infty, \omega_\rho \in \mathcal{P}_{E,1+\epsilon}^{1+\epsilon}(\mathbf{R}^{2d}), \phi_\rho, \psi_\rho \in \mathcal{D}^{1+\epsilon}(\mathbf{R}^d; [0,1])$ be such that (1.16), (1.17) and (1.18) hold true, and let $f_\rho \in (\mathcal{D}^{1+\epsilon})'(\mathbf{R}^d)$. Then the following is true:

(1) $f_\rho \in M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d)$, if and only if $\left\|V_{\phi_\rho} f_\rho\right\|_{\ell_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^d \times \pi\mathbf{Z}^d)}$;

(2) $f_\rho \in M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d)$, if and only if $\left\|V_{\psi_\rho} f_\rho\right\|_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^d \times \pi\mathbf{Z}^d)$;

(3) the quasi-norms

$$f_\rho \mapsto \left\|V_{\phi_\rho} f_\rho\right\|_{\ell_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^d \times \pi\mathbf{Z}^d)} \text{ and } f_\rho \mapsto \left\|V_{\psi_\rho} f_\rho\right\|_{\ell_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^d \times \pi\mathbf{Z}^d)}$$

are equivalent to $\|\cdot\|_{M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}}$.

The same holds true with $W_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}$ and $\ell_{*,(\omega_\rho)}^{1+2\epsilon,1+\epsilon}$ in place of $M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}$ respectively $\ell_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}$ at each occurrence.

Remark 1.8 [38]. There are weights $\omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d})$ such that corresponding modulation spaces $M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d)$ and $W_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d)$ do not contain $\mathcal{D}^{1+\epsilon}(\mathbf{R}^d)$ for any choice of $\epsilon > 0$. In this situation, it is not possible to find compactly supported elements in Gabor pairs which can be used for expanding all elements in $M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d)$ and $W_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d)$.

For a general weight $\omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d})$ which is moderated by the submultiplicative weight $v \in \mathcal{P}_E(\mathbf{R}^{2d})$, we may always find a lattice $\Lambda \in \mathbf{R}^d$ and a Gabor pair (ϕ_ρ, ψ_ρ) such that

$$\phi_\rho, \psi_\rho \in \bigcap_{\epsilon \geq 0} M_{(v)}^{1+2\epsilon}(\mathbf{R}^d)$$

and

$$\begin{aligned} f_\rho(x) &= C \sum_{j,\iota \in \Lambda} V_{\phi_\rho} f_\rho(j,\iota)\psi_\rho(x-j)e^{i\langle x,\iota\rangle} \\ &= C \sum_{j,\iota \in \Lambda} V_{\psi_\rho} f_\rho(j,\iota)\phi_\rho(x-j)e^{i\langle x,\iota\rangle}, f_\rho \in M_{(\omega_\rho)}^{\infty}(\mathbf{R}^d), \end{aligned} \tag{1.25}$$

for some constant C, where the series convergence with respect to the weak* topology in $M_{(\omega_\rho)}^{\infty}(\mathbf{R}^d)$. (See [16, Theorem S] and some further comments in [32]. See also [11-13] for more facts.) In such approach we still have that if $0 < \epsilon \leq \infty$, then

$$\begin{aligned} f_\rho \in M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d) &\Leftrightarrow \left\{V_{\phi_\rho} f_\rho(j,\iota)\right\}_{j,\iota \in \Lambda} \in \ell_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\Lambda \times \Lambda) \\ &\Leftrightarrow \left\{V_{\psi_\rho} f_\rho(j,\iota)\right\}_{j,\iota \in \Lambda} \in \ell_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\Lambda \times \Lambda) \end{aligned} \tag{1.26}$$

$$\begin{aligned} f_\rho \in W_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\mathbf{R}^d) &\Leftrightarrow \left\{V_{\phi_\rho} f_\rho(j,\iota)\right\}_{j,\iota \in \Lambda} \in \ell_{*,(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\Lambda \times \Lambda) \\ &\Leftrightarrow \left\{V_{\psi_\rho} f_\rho(j,\iota)\right\}_{j,\iota \in \Lambda} \in \ell_{*,(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\Lambda \times \Lambda) \end{aligned} \tag{1.27}$$

$$\|f_\rho\|_{M_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}} \asymp \left\|V_{\phi_\rho} f_\rho\right\|_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\Lambda \times \Lambda) \left\|V_{\psi_\rho} f_\rho\right\|_{(\omega_\rho)}^{1+2\epsilon,1+\epsilon}(\Lambda \times \Lambda) \tag{1.28}$$

and

$$\|f_\rho\|_{W_{(\omega_\rho)}^{1+2\epsilon.1+\epsilon}} \asymp \left\|V_{\phi_\rho} f_\rho\right\|_{\ell_{*,(\omega_\rho)}^{1+2\epsilon.1+\epsilon}(\Lambda \times \Lambda)} \asymp \left\|V_{\psi_\rho} f_\rho\right\|_{\ell_{*,(\omega_\rho)}^{1+2\epsilon.1+\epsilon}(\Lambda \times \Lambda)} \tag{1.29}$$

Furthermore, if $f_\rho \in M_{(\omega_\rho)}^{1+2\epsilon.1+\epsilon}(\mathbf{R}^d)\left(f_\rho \in W_{(\omega_\rho)}^{1+2\epsilon.1+\epsilon}(\mathbf{R}^d)\right)$ and in addition $\epsilon < \infty$, then the series in (1.25) converges with respect to the $M_{(\omega_\rho)}^{1+2\epsilon.1+\epsilon}$ quasi-norm $\left(W_{(\omega_\rho)}^{1+2\epsilon.1+\epsilon}\right.$ quasi-norm).

Remark 1.9 [38]. Let $(\omega_\rho)_0 \in \mathcal{P}_E(\mathbf{R}^d), \omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d}), 0 \leq \epsilon \leq \infty, Q_d = [0,1]^d$ be the unit cube, and set for measurable f_ρ on $\mathbf{R}^d$,

$$\|f_\rho\|_{\mathrm{W}^{1+3\epsilon}(\omega_\rho,\ell^{1+\epsilon})} \equiv \|a_0\|_{\ell^{1+\epsilon}(\mathbf{Z}^d)} \tag{1.30}$$

when

$$a_0(j) \equiv \|f_\rho \cdot (\omega_\rho)_0\|_{L^{1+3\epsilon}(j+Q_d)}, j \in \mathbf{Z}^d$$

and measurable F_ρ on $\mathbf{R}^{2d}$,

$$\|F_\rho\|_{\mathrm{W}^{1+3\epsilon}(\omega_\rho,\ell^{1+\epsilon,1+2\epsilon})} \equiv \|a\|_{\ell^{1+\epsilon,1+2\epsilon}(\mathbf{Z}^{2d})} \text{ and } \|F_\rho\|_{\mathrm{W}(\omega_\rho,\ell_*^{1+\epsilon,1+2\epsilon})} \equiv \|a\|_{\ell_*^{1+\epsilon,1+2\epsilon}(\mathbf{Z}^{2d})} \tag{1.31}$$

w-hen

$$a(j,\iota) \equiv \|F_\rho \cdot \omega_\rho\|_{L^{1+3\epsilon}((j,\iota)+Q_{2d})}, j,\iota \in \mathbf{Z}^d$$

The Wiener space

$$\mathrm{W}^{1+3\epsilon}\big((\omega_\rho)_0, \ell^{1+\epsilon}\big) = \mathrm{W}^{1+3\epsilon}\left((\omega_\rho)_0, \ell^{1+\epsilon}(\mathbf{Z}^2)\right)$$

consists of all measurable $f_\rho \in L^{1+3\epsilon}_{\mathrm{loc}}(\mathbf{R}^d)$ such that $\|F_\rho\|_{\mathrm{W}^{1+3\epsilon}((\omega_\rho)_0,\ell^{1+\epsilon})}$ is finite, and the Wiener spaces $\mathrm{W}^{1+3\epsilon}\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big) = \mathrm{W}^{1+3\epsilon}\left(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}(\mathbf{Z}^{2d})\right)$ and $\mathrm{W}^{1+3\epsilon}\big(\omega_\rho, \ell_*^{1+\epsilon,1+2\epsilon}\big) = \mathrm{W}^{1+3\epsilon}\left(\omega_\rho, \ell_*^{1+\epsilon,1+2\epsilon}(\mathbf{Z}^{2d})\right)$ consist of all measurable $F_\rho \in L^{1+3\epsilon}_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that $\|F_\rho\|_{\mathrm{W}^{1+3\epsilon}(\omega_\rho,\ell^{1+\epsilon,1+2\epsilon})}$ respectively $\|F_\rho\|_{\mathrm{W}^{1+3\epsilon}(\omega_\rho,\ell_*^{1+\epsilon,1+2\epsilon})}$ are finite. The topologies are defined through their respectively quasi-norms in (1.30) and (1.31). For conveniency we set

$$\mathrm{W}\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big) = \mathrm{W}^\infty\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big) \text{ and } \mathrm{W}\big(\omega_\rho, \ell_*^{1+\epsilon,1+2\epsilon}\big) = \mathrm{W}^\infty\big(\omega_\rho, \ell_*^{1+\epsilon,1+2\epsilon}\big)$$

Obviously, $\mathrm{W}^{1+3\epsilon}\big((\omega_\rho)_0, \ell^{1+\epsilon}\big)$ and $\mathrm{W}^{1+3\epsilon}\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big)$ increase with $1+\epsilon, 1+2\epsilon$, decrease with $(1+3\epsilon)$, and

$$\mathrm{W}\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big) \hookrightarrow L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^{2d}) \cap \Sigma_1'(\mathbf{R}^{2d}) \hookrightarrow L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^{2d}) \hookrightarrow \mathrm{W}^{1+3\epsilon}\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big) \tag{1.32}$$

and

$$\|\cdot\|_{\mathrm{W}^{1+3\epsilon}(\omega_\rho,\ell^{1+\epsilon,1+2\epsilon})} \le \|\cdot\|_{L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} \le \|\cdot\|_{\mathrm{W}(\omega_\rho,\ell^{1+\epsilon,1+2\epsilon})}, 1+3\epsilon \le \min(1,1+\epsilon,1+2\epsilon). \tag{1.33}$$

On the other hand, for modulation spaces we have

$$f_\rho \in M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d) \iff V_{\phi_\rho} f_\rho \in L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^{2d}) \iff V_{\phi_\rho} f_\rho \in \mathrm{W}^{1+3\epsilon}\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big) \tag{1.34}$$

with

$$\|f_\rho\|_{M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} = \left\|V_{\phi_\rho} f_\rho\right\|_{L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} \asymp \left\|V_{\phi_\rho} f_\rho\right\|_{\mathrm{W}^{1+3\epsilon}(\omega_\rho,\ell^{1+\epsilon,1+2\epsilon})} \tag{1.35}$$

The same holds true with $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}, L^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}$ and $\mathrm{W}\big(\omega_\rho, \ell_*^{1+\epsilon,1+2\epsilon}\big)$ in place of $M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}, L^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}$ and $\mathrm{W}\big(\omega_\rho, \ell^{1+\epsilon,1+2\epsilon}\big)$, respectively, at each occurrence. (For $\epsilon = \infty$, see [17] when $0 \le \epsilon \le \infty$, [14,32] when $0 < \epsilon \le \infty$, and for $0 \le \epsilon \le \infty$, see [35].)

-- Next we discuss extended Hölder and Young relations for multiplications and convolutions on discrete Lebesgue spaces. Here the involved weights should satisfy

$$(\omega_\rho)_0(x) \le \sum_\rho \prod_{j=1}^N (\omega_\rho)_j(x) \tag{1.36}$$

or

$$(\omega_\rho)_0(x_1 + \cdots + x_N) \le \sum_\rho \prod_{j=1}^N (\omega_\rho)_j(x_j), \tag{1.37}$$

and it is convenient to make use of the functional

$$R_N(p_1, \dots, p_N) = \left(\sum_{j=1}^N \max\left(1, \frac{1}{p_j}\right)\right) - \min_{1\le j\le N}\left(\max\left(1, \frac{1}{p_j}\right)\right) \tag{1.38}$$

The Hölder and Young conditions on Lebesgue exponent are then

$$\frac{1}{1+2\epsilon} \le \sum_{j=1}^N \frac{1}{q_j} \tag{1.39}$$

respectively

$$\frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{p_j} - R_N(p_1, \dots, p_N) \tag{1.40}$$

Proposition 1.10 [38]. Let $0 \le \epsilon \le \infty$ be such that (1.38), (1.39) and (1.40) hold, $(\omega_\rho)_j \in \mathcal{P}_E(\mathbf{R}^d)$, and let $\Lambda \subseteq \mathbf{R}^d$ be a lattice containing origin. Then the following is true:

(1) if (1.36) holds true, then the map $(a_1, \dots, a_N) \mapsto a_1 \cdots a_N$ from $\ell_0(\Lambda) \times \cdots \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{q_1}_{((\omega_\rho)_1)}(\Lambda) \times \cdots \times \ell^{q_N}_{((\omega_\rho)_N)}(\Lambda)$ to $\ell^{q_0}_{((\omega_\rho)_0)}(\Lambda)$, and

$$\|a_1 \cdots a_N\|_{\ell^{1+2\epsilon}_{((\omega_\rho)_0)}} \le \sum_\rho \prod_{j=1^N} \|a_j\|_{\ell^{q_j}_{((\omega_\rho)_j)}}, a_j \in \ell^{q_j}_{((\omega_\rho)_j)}(\Lambda), j = 1, \dots, N \tag{1.41}$$

(2) if (1.37) holds true, then the map $(a_1, \dots, a_N) \mapsto a_1 * \cdots * a_N$ from $\ell_0(\Lambda) \times \cdots \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{p_1}_{((\omega_\rho)_1)}(\Lambda) \times \cdots \times \ell^{p_N}_{((\omega_\rho)_N)}(\Lambda)$ to $\ell^{p_0}_{((\omega_\rho)_0)}(\Lambda)$, and

$$\|a_1 * \cdots * a_N\|_{\ell^{1+\epsilon}_{((\omega_\rho)_0)}} \le \sum_\rho \prod_{j=1^N} \|a_j\|_{\ell^{p_j}_{((\omega_\rho)_j)}}, a_j \in \ell^{p_j}_{((\omega_\rho)_j)}(\Lambda), j = 1, \dots, N \tag{1.42}$$

The assertion (1) in Proposition 1.10 is the standard Hölder's inequality for discrete Lebesgue spaces. The assertion (2) in that proposition is the usual Young's inequality for Lebesgue spaces on lattices in the case when $p_1, \dots, p_N \in [1, \infty]$. In order to be self-contained we give a proof when $p_1, \dots, p_N$ are allowed to belong to the full interval $(0, \infty]$ in Appendix A.

1.7. Step and Fourier step multipliers. Let $b \in \mathbf{R}^d_+$ be fixed, Λ_b be the lattice given by

$$\Lambda_b = \{(b_1 n_1, \dots, b_d n_d) \in \mathbf{R}^d; (n_1, \dots, n_d) \in \mathbf{Z}^d\} \tag{1.43}$$

Q_b be the b-cube, given by

$$Q_b = \{(b_1 x_1, \dots, b_d x_d) \in \mathbf{R}^d; (x_1, \dots, x_d) \in [0,1]^d\} \tag{1.44}$$

and $a_0 \in \ell^\infty(\Lambda_b)$. Then we let the Fourier step multiplier $M_{\mathcal{F},b,a_0}$ (with respect to b and a_0) be defined by

$$M_{\mathcal{F},b,a_0} \equiv \mathcal{F}^{-1} \circ M_{b,a_0} \circ \mathcal{F}, \tag{1.45}$$

where M_{b,a_0} is the multiplier

$$M_{b,a_0}: f_\rho \mapsto \sum_{j \in \Lambda_b} a_0(j) \chi_{j+Q_b} f_\rho \tag{1.46}$$

Here χ_Ω is the characteristic function of Ω.

II. Step and Fourier step multipliers on modulation spaces

We deduce continuity properties for step and Fourier step multipliers on modulation spaces. In contrast to [3], the results presented here permit Lebesgue exponents to be smaller than one

We begin with step multipliers when acting on modulation spaces. Here involved Lebesgue exponents should fullfil

$$\frac{1}{1+2\epsilon} - \frac{1}{1+3\epsilon} \ge \max\left(\frac{1}{1+\epsilon} - 1, 0\right) \tag{2.1}$$

Theorem 2.1 (see [38]). Let $0 \le \epsilon \le \infty, 0 < \epsilon \le \infty, 1+\epsilon, 1+2\epsilon \in (\min(1, 1+\epsilon), \infty)$ be such that (2.1) holds, $b > 0, (\omega_\rho)_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and $\omega_\rho(x, \xi) = (\omega_\rho)_0(x), x, \xi \in \mathbf{R}^d$. Let $a_0 \in \ell^\infty(\Lambda_b)$. Then the following is true:

(1) M_{b,a_0} is continuous on $W^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$;

(2) M_{b,a_0} is continuous from $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ to $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$.

We observe that the conditions on $(1+\epsilon)$ in Theorem 2.1 implies that $\epsilon > 0$, since otherwise (2.1) should lead to $1+\epsilon \le \min(1+\epsilon, 1)$, which contradicts the assumptions on $(1+\epsilon)$.

We need the following lemma for the proof of Theorem 2.1.

Lemma 2.2 [38]. Let $0 < \epsilon < \infty$ and $\theta \in (0,1)$ be such that

$$\theta + \frac{1}{1+\epsilon} = 1 + \frac{1}{1+2\epsilon} \tag{2.2}$$

and suppose that $a = \{a(j)\}_{j \in \mathbf{Z}^d} \subseteq \mathbf{C}$ satisfies

$$|a(j)| \lesssim (\langle j_1 \rangle \cdots \langle j_d \rangle)^{-\theta}$$

Then the map $b \mapsto a * b$ from $\ell_0(\mathbf{Z}^d)$ to $\ell_0'(\mathbf{Z}^d)$ is uniquely extendable to a continuous mapping from $\ell^{1+\epsilon}(\mathbf{Z}^d)$ to $\ell^{1+2\epsilon}(\mathbf{Z}^d)$.

We observe that the conditions in Lemma 2.2 implies that $1+\epsilon < 1+2\epsilon$.

Lemma 2.2 is a straight-forward consequence of [22, Theorem 4.5.3]. In fact, by that theorem we have for

$$h(x) = (|x_1| \cdots |x_d|)^{-1} \text{ and } h_0(x) = (\langle x_1 \rangle \cdots \langle x_d \rangle)^{-1} \tag{2.3}$$

that

$$\left\| \sum_\rho f_\rho * h^\theta \right\|_{L^{1+2\epsilon}(\mathbf{R}^d)} \lesssim \sum_\rho \|f_\rho\|_{L^{1+\epsilon}(\mathbf{R}^d)}$$

when (2.2) holds. Since $0 < h_0(x) < h(x)$, we obtain

$$\left\|\sum_{\rho} f_\rho * h_0^\theta\right\|_{L^{1+2\epsilon}(\mathbf{R}^d)} \le \sum_{\rho} \||f_\rho| * h^\theta\|_{L^{1+2\epsilon}(\mathbf{R}^d)} \lesssim \sum_{\rho} \|f_\rho\|_{L^{1+\epsilon}(\mathbf{R}^d)} \tag{2.4}$$

which gives suitable boundedness properties for $f_\rho \mapsto f_\rho * h_0^\theta$.

We also have

$$\|(g_\rho)_a\|_{L^{1+\epsilon}(\mathbf{R}^d)} = \|a\|_{\ell^{1+\epsilon}(\mathbf{Z}^d)}, (g_\rho)_a(x) = \sum_{j\in\mathbf{Z}^d} a(j)\chi_{j+[0,1]^d}(x)$$

By a straight-forward combination of this estimate with (2.4) we obtain

$$\|b * h_0^\theta\|_{\ell^{1+2\epsilon}(\mathbf{Z}^d)} \lesssim \|b\|_{\ell^{1+\epsilon}(\mathbf{Z}^d)}$$

where now * denotes the discrete convolution. The continuity assertions in Lemma 2.2 now follows from

$$\|b * a\|_{\ell^{1+2\epsilon}(\mathbf{Z}^d)} \lesssim \||b| * h_0^\theta\|_{\ell^{1+2\epsilon}(\mathbf{Z}^d)} \lesssim \|b\|_{\ell^{1+\epsilon}(\mathbf{Z}^d)}$$

and the uniqueness assertions follows from the fact that $\ell_0(\mathbf{Z}^d)$ is dense in $\ell^{1+\epsilon}(\mathbf{Z}^d)$ when $\epsilon < \infty$.

Proof of Theorem 2.1. By straight-forward computations it follows that if $(\omega_\rho)_b(x,\xi) = \omega_\rho(bx, b^{-1}\xi)$, then $f_\rho \in W_{(\omega_\rho)}^{1+\epsilon,1+2\epsilon}(\mathbf{R}^d)$, if and only if $f_\rho(b\,\cdot) \in W_{((\omega_\rho)_b)}^{1+\epsilon,1+2\epsilon}(\mathbf{R}^d)$, and

$$\|f_\rho\|_{W_{(\omega_\rho)}^{1+\epsilon,1+2\epsilon}} \asymp \|f_\rho(b\,\cdot)\|_{W_{((\omega_\rho)_b)}^{1+\epsilon,1+2\epsilon}}$$

and similarly with $M^{1+\epsilon,1+2\epsilon}$ in place of $W^{1+\epsilon,1+2\epsilon}$ at each occurrence. This reduce ourself to the case when $b = 1$.

Let ϕ_ρ, ψ_ρ and Λ be the same as in (1.16)-(1.18) and (1.21). By (1.24) we have

$$M_{b,a_0} f_\rho(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} \sum_{\rho} \left(A \cdot V_{\phi_\rho} f_\rho\right)\Bigg)(j,\iota) e^{i\langle x,\iota\rangle} \psi_\rho(x-j), \tag{2.5}$$

where $A = (a(\boldsymbol{j}, K))_{\boldsymbol{j},\boldsymbol{k}\in\Lambda}$ is the matrix with elements

$$a(\boldsymbol{j},\boldsymbol{k}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \left(M_{b,a_0} e^{i\langle\cdot,\iota\rangle}\psi_\rho(\cdot - j), e^{i\langle\cdot,\kappa\rangle}\phi_\rho(\cdot - k)\right),$$

$$\text{when } \boldsymbol{j} = (j,\iota) \in \Lambda, \boldsymbol{k} = (k,\kappa) \in \Lambda.$$

Let $Q = [0,1]^d$ and

$$\Omega_m = \{j \in \mathbf{Z}^d; |j_n| \le m \text{ for every } n \in \{1,\dots,d\}\}, m \in \mathbf{Z}_+$$

By the support properties of ϕ_ρ and ψ_ρ we have

$$a(\boldsymbol{j},\boldsymbol{k}) = 0 \text{ when } j - k \notin \Omega_2,$$

and for $j - k \in \Omega_2$ we get

$$|a(\boldsymbol{j},\boldsymbol{k})| \asymp \left|\int_{\mathbf{R}^d} \sum_{\rho} M_{b,a_0}\left(e^{-i\langle\cdot,\kappa-\iota\rangle}\psi_\rho(\cdot - j)\right)(y)\phi_\rho(y-k)dy\right|$$

$$= \left|\int_{\mathbf{R}^d} \sum_{\rho} \left(\sum_{l\in\mathbf{Z}^d} a_0(l)\chi_Q(y-l)e^{-i\langle y,\kappa-\iota\rangle}\psi_\rho(y-j)\phi_\rho(y-k)dy\right)\right|$$

$$= \left|\int_{\mathbf{R}^d} \sum_{\rho} \left(\sum a_0(l)\chi_Q(y-(l-k))e^{-i\langle y,\kappa-\iota\rangle}\psi_\rho(y-(j-k))\phi_\rho(y)dy\right)\right|$$

$$\le \sum_{\rho} \sum |a_0(l)| \cdot \left|\int_{\mathbf{R}^d} \left(\chi_Q(y-(l-k))e^{-i\langle y,\kappa-\iota\rangle}\psi_\rho(y-(j-k))\phi_\rho(y)dy\right)\right|$$

$$\le \sum_{\rho} \|a_0\|_{\ell^\infty(\mathbf{Z}^d)} \sum \left|\left(\chi_Q(\cdot - (l-k)), e^{-i\langle\cdot,\kappa-\iota\rangle}\psi_\rho(\cdot - (j-k))\phi_\rho\right)_{L^2(\mathbf{R}^d)}\right| \tag{2.6}$$

where the last three sums are taken over all $l \in \mathbf{Z}^d$ such that $l - (j-k) \in \Omega_3$.

We have to estimate

$$\left|\left(\chi_Q(\cdot - (l-k)), e^{-i\langle\cdot,\kappa-\iota\rangle}\psi_\rho(\cdot - (j-k))\phi_\rho\right)_{L^2(\mathbf{R}^d)}\right|$$

when $j - k \in \Omega_2$ and $l - (j-k) \in \Omega_3$. By Parseval's formula we get

$$\sum_{\rho} |\left(\chi_Q(\cdot-(l-k)), e^{-i\langle\cdot,\kappa-\iota\rangle}\psi_\rho(\cdot-(j-k))\phi_\rho\right)_{L^2(\mathbf{R}^d)}|$$

$$=\sum_{\rho} \left|\left(e^{-i\langle l-k,\cdot\rangle}g_\rho, V_{\psi_\rho}\phi_\rho(j-k,\cdot-(\iota-\kappa))\right)_{L^2(\mathbf{R}^d)}\right|$$

where

$$g_\rho(\xi)=(2\pi)^{-\frac{d}{2}}e^{-\frac{i}{2}(\xi_1+\cdots+\xi_d)}\operatorname{sinc}(\xi_1/2)\cdots\operatorname{sinc}(\xi_d/2)$$

Here

$$\operatorname{sinc}t=\begin{cases}\dfrac{\sin\, t}{t}, & t\neq 0\\ 1, & t=0\end{cases}$$

is the sinc function.

Since

$$\operatorname{sinc}t\lesssim\langle t\rangle^{-1} \text{ and } \left|V_{\psi_\rho}\phi_\rho(j-k,\xi-(\iota-\kappa))\right|\lesssim e^{-(1+\epsilon)|\xi-(\iota-\kappa)|^{\frac{1}{1+\epsilon}}}$$

we obtain

$$|\left(\chi_Q(\cdot-(l-k)), e^{-i\langle\cdot,\kappa-\iota\rangle}\psi_\rho(\cdot-(j-k))\phi_\rho\right)_{L^2(\mathbf{R}^d)}|$$

$$\leq\int_{\mathbf{R}^d}\sum_{\rho}|g_\rho(\xi)|\cdot\left|V_{\psi_\rho}\phi_\rho(j-k,\xi-(\iota-\kappa))\right|d\xi$$

$$\lesssim\int_{\mathbf{R}^d}h_0(\xi)e^{-(1+\epsilon)|\xi-(\iota-\kappa)|^{\frac{1}{1+\epsilon}}}d\xi$$

$$=\int_{\mathbf{R}^d}h_0(\xi+\iota-\kappa)e^{-(1+\epsilon)|\xi|^{\frac{1}{1+\epsilon}}}d\xi$$

$$\leq h_0(\iota-\kappa)\int_{\mathbf{R}^d}\langle\xi_1\rangle\cdots\langle\xi_d\rangle e^{-(1+\epsilon)|\xi|^{\frac{1}{1+\epsilon}}}d\xi\asymp h_0(\iota-\kappa)$$

Here h_0 is given by (2.3), and we have used

$$h_0(\xi+\eta)=(\langle\xi_1+\eta_1\rangle\cdots\langle\xi_d+\eta_d\rangle)^{-1}\leq(\langle\eta_1\rangle\cdots\langle\eta_d\rangle)^{-1}\langle\xi_1\rangle\cdots\langle\xi_d\rangle.$$

By inserting this into (2.6) we get

$$|a(\boldsymbol{j},\boldsymbol{k})|\lesssim\sum_{l\in(j-k)+\Omega_3}h_0(\iota-\kappa)(\langle\iota_1-\kappa_1\rangle\cdots\langle\iota_d-\kappa_d\rangle)^{-1}=7^d h_0(\iota-\kappa)$$

Hence,

$$|a(\boldsymbol{j},\boldsymbol{k})|\lesssim\begin{cases}h_0(\iota-\kappa), & j-k\in\Omega_2\\ 0, & j-k\notin\Omega_2\end{cases}\tag{2.7}$$

If $c(j,\iota)=\left|V_{\phi_\rho}f_\rho(j,\iota)\right|$, then (2.7) gives

$$\left|\left(A\cdot V_{\phi_\rho}f_\rho\right)(j,\iota)\right| \lesssim\sum_{k\in j+\Omega_2}\left(\sum_{\kappa\in\pi\mathbf{Z}^d}h_0(\iota-\kappa)c(k,\kappa)\right)$$

$$=\sum_{k\in j+\Omega_2}\left(h_0*c(k,\cdot)\right)(\iota)=\sum_{k\in\Omega_2}\left(h_0*c(j+k,\cdot)\right)(\iota)\tag{2.8}$$

and Lemma 2.2 gives

$$\left\|\left(A\cdot V_{\phi_\rho}f_\rho\right)(j,\cdot)\right\|_{\ell^{1+2\epsilon}(\pi\mathbf{Z}^d)}\lesssim\sum_{k\in\Omega_2}\|h_0*c(j+k,\cdot)\|_{\ell^{1+2\epsilon}(\pi\mathbf{Z}^d)}$$

$$\lesssim\sum_{k\in\Omega_2}\|c(j+k,\cdot)\|_{\ell^{1+2\epsilon}(\pi\mathbf{Z}^d)}$$

By applying the $\ell^{1+\epsilon}_{((\omega_\rho)_0)}$ norm on the last inequality and raise it to the power $1+\epsilon=\min(1,1+\epsilon)$, we obtain

$$\left\|\sum_{\rho}\left(A\cdot V_{\phi_\rho}f_\rho\right)\right\|^{1+\epsilon}_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Lambda)}\lesssim\sum_{k\in\Omega_2}\sum_{\rho}\|c(\cdot+(k,0))\|^{1+\epsilon}_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Lambda)}$$

$$=\sum_{k\in\Omega_2}\sum_{\rho}\|c\|^{1+\epsilon}_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho(\cdot-(k,0)))}(\Lambda)}\asymp\sum_{k\in\Omega_2}\sum_{\rho}\|c\|^{1+\epsilon}_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Lambda)}=5^d\sum_{\rho}\|c\|^{1+\epsilon}_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Lambda)}.\tag{2.9}$$

Here we have used the fact that the number of elements in Ω_2 is equal to 5^d.

The asserted continuity in (1) now follows in the case when $\epsilon<\infty$ by combining (2.9) and the facts that

$$\left\|V_{\phi_\rho} f_\rho\right\|_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Lambda)} \asymp \|f_\rho\|_{W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} \text{ and } \left\|\left(A\cdot V_{\phi_\rho} f_\rho\right)\right\|_{\ell^{1+\epsilon,1+2\epsilon}_{*,(\omega_\rho)}(\Lambda)} \asymp \|M_{b,a_0} f_\rho\|_{W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}}.$$

The uniqueness of the map M_{b,a_0} on $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ follows from the fact that finite sequences in (1.22) are dense in $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ gives. The case when $\epsilon=\infty$ now follows from the case when $\epsilon=0$ and duality, and (1) follows.

In order to prove (2) we first consider the case when $\epsilon<\infty$. By applying the $\ell^{1+\epsilon}_{((\omega_\rho)_0)}$ norm with respect to the j variable in (2.8), we get

$$\left\|\sum_\rho \left(A\cdot V_{\phi_\rho} f_\rho\right)(\cdot,\iota)\right\|_{\ell^{1+\epsilon}_{((\omega_\rho)_0)}} \lesssim \left(\sum_{j\in\mathbf{Z}^d}\sum_\rho \left(\sum_{k\in\Omega_2}\left(h_0 * (c(j+k,\cdot))(\iota)(\omega_\rho)_0(j)\right)^{1+\epsilon}\right)\right)^{\frac{1}{1+\epsilon}}$$

$$\asymp \sum_{k\in\Omega_2}\sum_\rho \left(h_0^{1+\epsilon} * \left(\sum_{j\in\mathbf{Z}^d}\left(c(j+k,\cdot)(\omega_\rho)_0(j+k)\right)^{1+\epsilon}\right)^1\right)(\iota)$$

$$= \sum_{k\in\Omega_2}(h_0^{1+\epsilon} * c_0^{1+\epsilon})(\iota) \asymp (h_0^{1+\epsilon} * c_0^{1+\epsilon})(\iota)$$

where $c_0(\iota)=\|c(\cdot,\iota)\|_{\ell^{1+\epsilon}_{((\omega_\rho)_0)}}$.

Let $1+\epsilon=(1+\epsilon)^{-1}1+2\epsilon, 1+2\epsilon=(1+\epsilon)^{-1}(1+2\epsilon)$ and $u=(1+\epsilon)^{-1}$. Then (2.2) holds with $(1+\epsilon)$ and $(1+2\epsilon)$ in place of $(1+\epsilon)$ and $(1+2\epsilon)$, respectively. Hence by applying the $\ell^{1+2\epsilon}$ norm on the last estimates, Lemma 2.2 gives

$$\left\|A\cdot V_{\phi_\rho} f_\rho\right\|^{1+\epsilon}_{\ell^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} \lesssim \|h_0^\theta * c_0^{1+\epsilon}\|_{\ell^{1+2\epsilon}} \lesssim \|c_0^{1+\epsilon}\|_{\ell^{1+\epsilon}} = \|c_0\|^{1+\epsilon}_{\ell^{1+2\epsilon}}$$

The asserted continuity in (2) now follows in the case when $\epsilon<\infty$ by combining (2.9) and the facts that

$$\left\|V_{\phi_\rho} f_\rho\right\|_{\ell^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\Lambda)} \asymp \|f_\rho\|_{M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}} \text{ and } \left\|\left(A\cdot V_{\phi_\rho} f_\rho\right)\right\|_{\ell^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\Lambda)} \asymp \|M_{b,a_0} f_\rho\|_{M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}}$$

The uniqueness assertions as well as the continuity in the case $\epsilon=\infty$ follow by similar arguments as in the proof of (1).

By the links between $M^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ and $W^{1+\epsilon,1+2\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ via the Fourier transform, explained in Remark 1.6, the following result follows from Theorem 2.1 and Fourier transformation.

Theorem 2.3 [38]. Let $0<\epsilon<\infty, 0<\epsilon\le\infty, 1+\epsilon, 1+\epsilon\in(\min(1,1+\epsilon),\infty)$ be such that

$$\frac{1}{1+\epsilon}-\frac{1}{1+\epsilon}\ge \max\left(\frac{1}{1+\epsilon}-1,0\right) \tag{2.10}$$

$b>0, a_0\in\ell^\infty(\Lambda_b), (\omega_\rho)_0\in\mathcal{P}_{E,1+\epsilon}(\mathbf{R}^d)$ and $\omega_\rho(x,\xi)=(\omega_\rho)_0(\xi), x,\xi\in\mathbf{R}^d$. Then the following is true:

(1) $M_{\mathcal{F},b,a_0}$ is continuous on $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$;

(2) $M_{\mathcal{F},b,a_0}$ is continuous from $W^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ to $W^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$.

We observe that Theorem 2.3 generalizes [3, Theorem 1] and [37, Theorem 4.16].

III. Multiplications and convolutions of quasi-BanachModulation Spaces

We extend the multiplication and convolution properties on modulation spaces in [8,30] to allow the Lebesgue exponents to belong to the full interval $(0,\infty]$ instead of $[1,\infty]$, and to allow general moderate weights. There are several approaches in the case when the involved Lebesgue exponents belong to $[1,\infty]$ (see [4,8,11,21,27,30]). There are also some results when such exponents belong to the full interval $(0,\infty]$ (see [1, 2, 14, 25, 26, 32]). Here we remark that our results in this section cover several of these earlier results. For example, we observe that Theorem 3.2 below extends [1 Proposition 3.1].

We recall that convolutions and multiplications on $\Sigma_1(\mathbf{R}^d)$ are commutative and associative. That is, for any $N\ge 1, (f_\rho)_1,\dots,(f_\rho)_N\in\Sigma_1(\mathbf{R}^d)$ and $j,k\in\{1,\dots,N\}$ one has

$$(f_\rho)_1\cdots(f_\rho)_N = \left((f_\rho)_1\cdots(f_\rho)_j\right)\cdot\left((f_\rho)_{j+1}\cdots(f_\rho)_N\right) \text{ and } (f_\rho)_1\cdots(f_\rho)_N = (g_\rho)_1\cdots(g_\rho)_N$$

when

$$(g_\rho)_m=(f_\rho)_m, (g_\rho)_j=(f_\rho)_k \text{ and } (g_\rho)_k=(f_\rho)_j, m\ne j,k$$

and similarly for convolutions in place of multiplications at each occurrence.

Because of possible lacks of density properties, we do not always reach the uniqueness when extending the convolutions and multiplications from the case when each $(f_\rho)_j$ belong to $\Sigma_1(\mathbf{R}^d)$ to the case when each $(f_\rho)_j$ belong to suitable modulation spaces. In some cases we manage the uniqueness by replacing the (quasi-)norm convergence by a weaker convergence, the socalled narrow convergence (see [28,29,31]). In the other situations we define multiplications and convolutions in terms of short-time Fourier transforms, in similar ways as in [30].

Let $(\phi_\rho)_0, \dots, (\phi_\rho)_N \in \Sigma_1(\mathbf{R}^d)$ be fixed such that

$$\left((\phi_\rho)_1 \cdots (\phi_\rho)_N, (\phi_\rho)_0\right)_{L^2} = (2\pi)^{-(N-1)\frac{d}{2}} \tag{3.1}$$

and let $(f_\rho)_1, \dots, (f_\rho)_N, g_\rho \in \Sigma_1(\mathbf{R}^d)$. Then the multiplication $(f_\rho)_1 \cdots (f_\rho)_N$ can be expressed by

$$\left((f_\rho)_1 \cdots (f_\rho)_N, \varphi_\rho\right)_{L^2(\mathbf{R}^d)} = \iint_{\mathbf{R}^d \times \mathbf{R}^{Nd}} \sum_\rho \left(\prod_{j=1}^N (F_\rho)_j(x, \xi_j)\right) \overline{\Phi_\rho(x, \xi_1 + \cdots + \xi_N)}\, dx\, d\xi$$

$$= \int \cdots \int_{\mathbf{R}^{(N+1)d}} \sum_\rho \left(\prod_{j=1}^N (F_\rho)_j(x, \xi_j)\right) \overline{\Phi_\rho(x, \xi_1 + \cdots + \xi_N)}\, dx\, d\xi_1 \cdots d\xi_N \tag{3.2}$$

for every $\varphi_\rho \in \Sigma_1(\mathbf{R}^d)$, where

$$(F_\rho)_j = V_{(\phi_\rho)_j}(f_\rho)_j \text{ and } \Phi_\rho = V_{(\phi_\rho)_0}\varphi_\rho. \tag{3.3}$$

We observe that (3.2) is the same as

$$(F_\rho)_0(x, \xi) = \left(\left(V_{(\phi_\rho)_1}(f_\rho)_1\right)(x, \cdot) * \cdots * \left(V_{(\phi_\rho)_N}(f_\rho)_N\right)(x, \cdot)\right)(\xi) \tag{3.2$'$}$$

where

$$(F_\rho)_0(x, \xi) = \left(\left\|(\phi_\rho)_0\right\|_{L^2}\right)^{-2} \cdot V_{(\phi_\rho)_0}\left((f_\rho)_1 \cdots (f_\rho)_N\right)(x, \xi) \tag{3.4}$$

and that we may extract $(f_\rho)_0 = (f_\rho)_1 \cdots (f_\rho)_N$ by the formula

$$(f_\rho)_0 = V^*_{(\phi_\rho)_0}(F_\rho)_0 \tag{3.5}$$

In the same way, let $(\phi_\rho)_0, \dots, (\phi_\rho)_N \in \Sigma_1(\mathbf{R}^d)$ be fixed such that

$$\left((\phi_\rho)_1 * \cdots * (\phi_\rho)_N, (\phi_\rho)_0\right)_{L^2} = 1 \tag{3.6}$$

and let $(f_\rho)_1, \dots, (f_\rho)_N, g_\rho \in \Sigma_1(\mathbf{R}^d)$. Then the convolution $(f_\rho)_1 * \cdots * (f_\rho)_N$ can be expressed by

$$\left((f_\rho)_1 * \cdots * (f_\rho)_N, \varphi_\rho\right)_{L^2(\mathbf{R}^d)} = \iint_{\mathbf{R}^{Nd} \times \mathbf{R}^d} \sum_\rho \left(\prod_{j=1}^N (F_\rho)_j(x_j, \xi)\right) \overline{\Phi_\rho(x_1 + \cdots + x_N, \xi)}\, dx\, d\xi$$

$$= \int \cdots \int_{\mathbf{R}^{(N+1)d}} \sum_\rho \left(\prod_{j=1}^N (F_\rho)_j(x_j, \xi)\right) \overline{\Phi_\rho(x_1 + \cdots + x_N, \xi)}\, dx_1 \cdots dx_N\, d\xi \tag{3.7}$$

for every $\varphi_\rho \in \Sigma_1(\mathbf{R}^d)$, where $(F_\rho)_j$ and Φ_ρ are given by (3.3). We observe that (3.7) is the same as

$$(F_\rho)_0(x, \xi) = \left(\left(V_{(\phi_\rho)_1}(f_\rho)_1\right)(\cdot, \xi) * \cdots * \left(V_{(\phi_\rho)_N}(f_\rho)_N\right)(\cdot, \xi)\right)(x) \tag{3.7$'$}$$

where

$$(F_\rho)_0 = \left(\left\|(\phi_\rho)_0\right\|_{L^2}\right)^{-2} V_{(\phi_\rho)_0}\left((f_\rho)_1 * \cdots * (f_\rho)_N\right) \tag{3.8}$$

and that we may extract $(f_\rho)_0 = (f_\rho)_1 * \cdots * (f_\rho)_N$ from (3.5).

Definition 3.1 [38]. Let $(f_\rho)_1, \dots, (f_\rho)_N \in \Sigma_1'(\mathbf{R}^d)$.

(1) Let $(\phi_\rho)_0, \dots, (\phi_\rho)_N \in \Sigma_1(\mathbf{R}^d)$ be fixed and such that (3.1) holds, and suppose that the integrand in (3.2) belongs to $L^1(\mathbf{R}^{(N+1)d})$ for every $\varphi_\rho \in \Sigma_1(\mathbf{R}^d)$, where $(F_\rho)_j = V_{(\phi_\rho)_j}(f_\rho)_j$ and $\Phi_\rho = V_{(\phi_\rho)_0}\varphi_\rho, j = 1, \dots, N$. Then $(f_\rho)_0 \equiv (f_\rho)_1 \cdots (f_\rho)_N \in \Sigma_1'(\mathbf{R}^d)$ is defined by (3.2);

(2) Let $(\phi_\rho)_0, \dots, (\phi_\rho)_N \in \Sigma_1(\mathbf{R}^d)$ be fixed and such that (3.6) holds, and suppose that the integrand in (3.7) belongs to $L^1(\mathbf{R}^{(N+1)d})$ for every $\varphi_\rho \in \Sigma_1(\mathbf{R}^d)$, where $(F_\rho)_j = V_{(\phi_\rho)_j}(f_\rho)_j$ and $\Phi_\rho = V_{(\phi_\rho)_0}\varphi_\rho, j = 1, \dots, N$. Then $(f_\rho)_0 \equiv (f_\rho)_1 * \cdots * (f_\rho)_N \in \Sigma_1'(\mathbf{R}^d)$ is defined by (3.7).

Next we discuss convolutions and multiplications for modulation spaces, and start with the following convolution result for modulation spaces. Here the conditions for the involved weight functions are given by

$$(\omega_\rho)_0(x, \xi_1 + \cdots + \xi_N) \lesssim \sum_\rho \prod_{j=1}^N (\omega_\rho)_j(x, \xi_j), x, \xi_1, \dots, \xi_N \in \mathbf{R}^d \tag{3.9}$$

or by

$$(\omega_\rho)_0(x_1 + \cdots + x_N, \xi) \lesssim \sum_\rho \prod_{j=1}^N (\omega_\rho)_j(x_j, \xi), x_1, \dots, x_N, \xi \in \mathbf{R}^d \tag{3.10}$$

For multiplications of elements in modulation spaces we need to swap the conditions for the involved Lebesgue exponents compared to (1.39) and (1.40). That is, these conditions become

$$\frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{p_j}, \frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{q_j} - R_{1+\epsilon,N}(q_1, \dots, q_N) \tag{3.11}$$

or

$$\frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{p_j}, \frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{q_j} - R_N(q_1, \dots, q_N) \tag{3.12}$$

where

$$R_{1+\epsilon,N}(q_1, \dots, q_N) = \left(\sum_{j=1}^N \frac{1}{r_j}\right) - \min_{1 \le j \le N}\left(\frac{1}{r_j}\right), r_j = \min(1, 1+\epsilon, 1+\epsilon) \tag{3.13}$$

and

$$R_N(q_1, \dots, q_N) = R_{1,N}(q_1, \dots, q_N) \tag{3.14}$$

Evidently, $R_{1+\epsilon,N}(q_1, \dots, q_N) = R_N(q_1, \dots, q_N)$ when $\epsilon \ge 0$.

Theorem 3.2 [38]. Let $I_N = \{1, \dots, N\}, (\omega_\rho)_j \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $0 \le \epsilon \le \infty, j \in I_N$, be such that (3.9), (3.11) and (3.13) hold. Then $((f_\rho)_1, \dots, (f_\rho)_N) \mapsto (f_\rho)_1 \cdots (f_\rho)_N$ in Definition 3.1 (1) restricts to a continuous, associative and symmetric map from $M^{p_1,q_1}_{((\omega_\rho)_1)}(\mathbf{R}^d) \times \cdots \times M^{p_N,q_N}_{((\omega_\rho)_N)}(\mathbf{R}^d)$ to $M^{p_0,q_0}_{((\omega_\rho)_0)}(\mathbf{R}^d)$, and

$$\left\|(f_\rho)_1 \cdots (f_\rho)_N\right\|_{M^{p_0,q_0}_{((\omega_\rho)_0)}} \lesssim \sum_\rho \prod_{j=1}^N \left\|(f_\rho)_j\right\|_{M^{p_j,q_j}_{((\omega_\rho)_j)}}, (f_\rho)_j \in M^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d), j \in I_N \tag{3.15}$$

Moreover, $(f_\rho)_1 \cdots (f_\rho)_N$ in (3.2) is independent of the choice of $(\phi_\rho)_0, \dots, (\phi_\rho)_N$ in Definition 3.1 (1).

Theorem 3.3 [38]. Let $I_N = \{1, \dots, N\}, (\omega_\rho)_j \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $0 \le \epsilon \le \infty, j \in I_N$, be such that (3.9), (3.12) and (3.14) hold. Then $((f_\rho)_1, \dots, (f_\rho)_N) \mapsto (f_\rho)_1 \cdots (f_\rho)_N$ in Definition 3.1 (1) restricts to a continuous, associative and symmetric map from $W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d) \times \cdots \times W^{p_N,q_N}_{((\omega_\rho)_N)}(\mathbf{R}^d)$ to $W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_0)}(\mathbf{R}^d)$, and

$$\left\|(f_\rho)_1 \cdots (f_\rho)_N\right\|_{W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_0)}} \lesssim \sum_\rho \prod_{j=1}^N \left\|(f_\rho)_j\right\|_{W^{p_j,q_j}_{((\omega_\rho)_j)}}, (f_\rho)_j \in W^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d), j \in I_N \tag{3.16}$$

Moreover, $(f_\rho)_1 \cdots (f_\rho)_N$ in (3.2) is independent of the choice of $(\phi_\rho)_0, \dots, (\phi_\rho)_N$ in Definition 3.1 (1).

The corresponding results for convolutions are the following. Here the conditions on the involved Lebesgue exponents are swapped as

$$\frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{p_j} - R_{1+\epsilon,N}(p_1, \dots, p_N), \frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{q_j} \tag{3.17}$$

or

$$\frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{p_j} - R_N(p_1, \dots, p_N), \frac{1}{1+\epsilon} \le \sum_{j=1}^N \frac{1}{q_j} \tag{3.18}$$

Theorem 3.4 (see [38]). Let $I_N = \{1, \dots, N\}, (\omega_\rho)_j \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $0 \le \epsilon \le \infty$, $j \in I_N$, be such that (3.10), (3.14) and (3.18) hold. Then $\left((f_\rho)_1, \dots, (f_\rho)_N\right) \mapsto (f_\rho)_1 * \cdots * (f_\rho)_N$ in Definition 3.1(2) restricts to a continuous, associative and symmetric map from $M^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d) \times \cdots \times M^{p_N,q_N}_{((\omega_\rho)_N)}(\mathbf{R}^d)$ to $M^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_0)}(\mathbf{R}^d)$, and

$$\left\|(f_\rho)_1 * \cdots * (f_\rho)_N\right\|_{M^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_0)}} \lesssim \sum_\rho \prod_{j=1}^{N} \left\|(f_\rho)_j\right\|_{M^{p_j,q_j}_{((\omega_\rho)_j)}}, \quad (f_\rho)_j \in M^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d), j \in I_N \tag{3.19}$$

Moreover, $(f_\rho)_1 * \cdots * (f_\rho)_N$ in (3.7) is independent of the choice of $(\phi_\rho)_0, \dots, (\phi_\rho)_N$ in Definition 3.1 (2).

Theorem 3.5(see [38]). Let $I_N = \{1, \dots, N\}, (\omega_\rho)_j \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $0 \le \epsilon \le \infty$, $j \in I_N$, be such that (3.10), (3.13) and (3.17) hold. Then $\left((f_\rho)_1, \dots, (f_\rho)_N\right) \mapsto (f_\rho)_1 * \cdots * (f_\rho)_N$ in Definition 3.1(2) restricts to a continuous, associative and symmetric map from $W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d) \times \cdots \times W^{p_N,q_N}_{((\omega_\rho)_N)}(\mathbf{R}^d)$ to $W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_0)}(\mathbf{R}^d)$, and

$$\left\|(f_\rho)_1 * \cdots * (f_\rho)_N\right\|_{W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_0)}} \lesssim \sum_\rho \prod_{j=1}^{N} \left\|(f_\rho)_j\right\|_{W^{p_j,q_j}_{((\omega_\rho)_j)}}, (f_\rho)_j \in W^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d), j \in I_N \tag{3.20}$$

Moreover, $(f_\rho)_1 * \cdots * (f_\rho)_N$ in (3.7) is independent of the choice of $(\phi_\rho)_0, \dots, (\phi_\rho)_N$ in Definition 3.1 (2).

For the proofs of Theorems $3.2\sqrt{3.5}$ we need the following proposition. Here recall [11,13,17,25,26] and Remark 1.4 for some facts concerning the operators P_{ϕ_ρ} and $V^*_{\phi_\rho}$.

Proposition 3.6(see [38]). Let $0 \le \epsilon \le \infty, \omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d}), \phi_\rho \in \Sigma_1(\mathbf{R}^d) \setminus 0$ and P_{ϕ_ρ} be the projection in Remark 1.4 . Then P_{ϕ_ρ} from $\Sigma_1'(\mathbf{R}^{2d})$ to $\Sigma_1'(\mathbf{R}^{2d})$, and $V^*_{\phi_\rho}$ from $\Sigma_1'(\mathbf{R}^{2d})$ to $\Sigma_1'(\mathbf{R}^d)$ restrict to continuous mappings

$$P_{\phi_\rho} : \mathrm{W}\left(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^{2d})\right) \to V_{\phi_\rho}\left(M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)\right) \hookrightarrow \mathrm{W}\left(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^{2d})\right), \tag{3.21}$$

$$P_{\phi_\rho} : \mathrm{W}\left(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon}_*(\mathbf{Z}^{2d})\right) \to V_{\phi_\rho}\left(W^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)\right) \hookrightarrow \mathrm{W}\left(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon}_*(\mathbf{Z}^{2d})\right), \tag{3.22}$$

$$V^*_{\phi_\rho} : \mathrm{W}\left(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^{2d})\right) \to M^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d) \tag{3.23}$$

and

$$V^*_{\phi_\rho} : \mathrm{W}\left(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon}_*(\mathbf{Z}^{2d})\right) \to W^{1+2\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d) \tag{3.24}$$

For $\epsilon \ge 0$, i. .e. the case when all spaces are Banach spaces, proofs of Proposition 3.6 can be found in e.g. [17] as well as in abstract forms in [11]. In the general case when $\epsilon \ge 0$, proofs of Proposition 3.6 are essentially given in [14,26]. In order to be self-contained we here present a short proof.

Proof. By Remark 1.4, the result follows if we prove (3.21) and (3.22), i.e., it suffices to prove

$$\left\|\sum_\rho P_{\phi_\rho} F_\rho\right\|_{\mathrm{W}(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon})} \lesssim \sum_\rho \|F_\rho\|_{\mathrm{W}(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon})}, F_\rho \in \mathrm{W}\left(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^{2d})\right) \tag{3.25}$$

and

$$\left\|\sum_\rho P_{\phi_\rho} F_\rho\right\|_{\mathrm{W}(\omega_\rho, \ell_*^{1+2\epsilon,1+\epsilon})} \lesssim \sum_\rho \|F_\rho\|_{\mathrm{W}(\omega_\rho, \ell_*^{1+2\epsilon,1+\epsilon})}, F_\rho \in \mathrm{W}\left(\omega_\rho, \ell_*^{1+2\epsilon,1+\epsilon}(\mathbf{Z}^{2d})\right) \tag{3.26}$$

We only prove (3.25). The estimate (3.26) follows by similar arguments and is left.

Let

$$a_j = \|F_\rho\|_{L^\infty(j+Q_{2d})} \text{ and } b_j = \left\|V_{\phi_\rho}\phi_\rho\right\|_{L^\infty(j+Q_{2d})}.$$

Since $V_{\phi_\rho}\phi_\rho \in \Sigma_1(\mathbf{R}^{2d})$, Proposition 1.10 gives

$$\left\|P_{\phi_\rho} F_\rho\right\|_{\mathrm{W}(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon})} \lesssim \|a * b\|_{\ell^{1+2\epsilon,1+\epsilon}} \lesssim \|b\|_{\ell^{\min(1,1+2\epsilon,1+\epsilon)}} \|a\|_{\ell^{1+2\epsilon,1+\epsilon}} \asymp \|F_\rho\|_{\mathrm{W}(\omega_\rho, \ell^{1+2\epsilon,1+\epsilon})}$$

Theorems 3.2 and 3.3 are Fourier transformations of Theorems 3.4 and 3.5. Hence it suffices to prove the last two theorems.

Proof of Theorems 3.4 and 3.5. First we prove (3.19). Suppose $(f_\rho)_j \in M^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$, and consider the cubes

$$Q_{d,1+\epsilon} = [0,1+\epsilon]^d \text{ and } Q = Q_{d,1} = [0,1]^d$$

Then

$$0 \le \chi_{k_1+Q} * \cdots * \chi_{k_N+Q} \le \chi_{k_1+\cdots+k_N+Q_{d,N}}, k_1, \dots k_N \in \mathbf{Z}^d$$

Let

$$G_1(x,\xi)=\left(V_{(\phi_\rho)_1}(f_\rho)_1(\cdot,\xi)*\cdots*V_{(\phi_\rho)_N}(f_\rho)_N(\cdot,\xi)\right)(x)$$
$$G_2(x,\xi)=\left(\left|V_{(\phi_\rho)_1}(f_\rho)_1(\cdot,\xi)\right|*\cdots*\left|V_{(\phi_\rho)_N}(f_\rho)_N(\cdot,\xi)\right|\right)(x)$$
$$a_j(k,\kappa)=\left\|V_{(\phi_\rho)_j}(f_\rho)_j\right\|_{L^\infty((k,\kappa)+Q_{2d,1})}$$

and

$$b(k,\kappa)=\|G_2\|_{L^\infty((k,\kappa)+Q_{2d,1})}$$

Then

$$\left\|V^*_{(\phi_\rho)_0}G_1\right\|_{M^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_0)}} \asymp \left\|P_{(\phi_\rho)_0}G_1\right\|_{\mathrm{W}((\omega_\rho)_0,\ell^{1+2\epsilon,1+\epsilon})} \lesssim \|G_1\|_{\mathrm{W}((\omega_\rho)_0,\ell^{1+2\epsilon,1+\epsilon})}$$
$$\leq \|G_2\|_{\mathrm{W}((\omega_\rho)_0,\ell^{1+2\epsilon,1+\epsilon})} \asymp \|b\|_{\ell^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_0)}}, \tag{3.27}$$

and

$$\left\|(f_\rho)_j\right\|_{M^{p_j,q_j}_{((\omega_\rho)_j)}} \asymp \|a_j\|_{\ell^{p_j,q_j}_{((\omega_\rho)_j)}} \tag{3.28}$$

in view of [32, Proposition 3.4] (see also Theorem 3.3 in [14]) and Proposition 3.6.

By (3.7) we have

$$G_2(x,\lambda) \leq \sum_{k_1,\dots,k_N\in\mathbf{Z}^d}\left(\prod_{j=1}^N a_j(k_j,\lambda)\right)(\chi_{k_1+Q}*\cdots\chi_{k_N+Q})(x)$$
$$\leq \sum_{k_1,\dots,k_N\in\mathbf{Z}^d}\left(\prod_{j=1}^N a_j(k_j,\lambda)\right)\chi_{k_1+\cdots+k_N+Q_{d,N}}(x) \tag{3.29}$$

We observe that

$$\chi_{k_1+\cdots+k_N+Q_{d,N}}(x)=0 \text{ when } x\notin l+Q_d, (k_1,\dots,k_N)\notin\Omega_l,$$

where

$$\Omega_l=\{(k_1,\dots,k_N)\in\mathbf{Z}^{Nd}; l_j-N\leq k_{1,j}+\cdots+k_{N,j}\leq l_j+1\}$$

and

$$k_n=(k_{n,1},\dots,k_{n,d})\in\mathbf{Z}^d \text{ and } l=(l_1,\dots,l_d)\in\mathbf{Z}^d, n=1,\dots,N.$$

Hence, if $x=l$ in (3.29), we get

$$b(l,\lambda)\leq \sum_{(k_1,\dots,k_N)\in\Omega_l}\left(\prod_{j=1}^N a_j(k_j,\lambda)\right)$$
$$\leq \sum_{m\in I_{N+1}}(a_1(\cdot,\lambda)*\cdots*a_N(\cdot,\lambda))(l-Ne_0+m) \tag{3.30}$$

where $e_0=(1,\dots,1)\in\mathbf{Z}^d$ and $I_N=\{0,\dots,N\}^d$. By multiplying with $(\omega_\rho)_0(l,\lambda)$, using (3.10), the fact that I_N is a finite set and that $(\omega_\rho)_0$ is moderate, we obtain

$$b(l,\lambda)(\omega_\rho)_0(l,\lambda)\leq \sum_{m\in I_{N+1}}\sum_\rho (a_1(\cdot,\lambda)*\cdots*a_N(\cdot,\lambda))(l-Ne_0+m)(\omega_\rho)_0(l,\lambda)$$
$$\leq \sum_{m\in I_{N+1}}\sum_\rho (a_1(\cdot,\lambda)*\cdots*a_N(\cdot,\lambda))(l-Ne_0+m)(\omega_\rho)_0(l-Ne_0+m,\lambda)$$
$$\lesssim \sum_{m\in I_{N+1}}\sum_\rho \left(\left(a_1(\cdot,\lambda)(\omega_\rho)_1(\cdot,\lambda)\right)*\cdots*\left(a_N(\cdot,\lambda)(\omega_\rho)_N(\cdot,\lambda)\right)\right)(l-Ne_0+m)$$

Hence (3.30) gives

$$b_{(\omega_\rho)_0}(l,\lambda)\lesssim \sum_{(k_1,\dots,k_N)\in\Omega_l}\sum_\rho\left(\prod_{j=1}^N a_{j,(\omega_\rho)_j}(k_j,\lambda)\right)$$
$$=\sum_{m\in I_{N+1}}\sum_\rho\left(a_{1,(\omega_\rho)_1}(\cdot,\lambda)*\cdots*a_{N,(\omega_\rho)_N}(\cdot,\lambda)\right)(l-Ne_0+m) \tag{3.30'}$$

where

$$a_{j,(\omega_\rho)_j}(k,\kappa) = a_j(k,\kappa)(\omega_\rho)_j(k,\kappa) \text{ and } b_{(\omega_\rho)_0}(k,\kappa) = b(k,\kappa)(\omega_\rho)_0(k,\kappa).$$

If we apply the $\ell^{1+2\epsilon}$ quasi-norm on (3.30) with respect to the l variable, then Proposition 1.10 (2) and the fact that I_{N+1} is a finite set give

$$\begin{aligned}\left\|b_{(\omega_\rho)_0}(\cdot,\lambda)\right\|_{\ell^{1+2\epsilon}} &\lesssim \left\|\sum_{m\in I_{N+1}}\sum_{\rho}\left(a_{1,(\omega_\rho)_1}(\cdot,\lambda)*\cdots*a_{N,(\omega_\rho)_N}(\cdot,\lambda)\right)(\cdot-Ne_0+m)\right\|_{\ell^{1+2\epsilon}}\\ &\lesssim \sum_{m\in I_{N+1}}\sum_{\rho}\left\|\left(a_{1,(\omega_\rho)_1}(\cdot,\lambda)*\cdots*a_{N,(\omega_\rho)_N}(\cdot,\lambda)\right)(\cdot-Ne_0+m)\right\|_{\ell^{1+2\epsilon}}\\ &\approx \left\|a_{1,(\omega_\rho)_1}(\cdot,\lambda)*\cdots*a_{N,(\omega_\rho)_N}(\cdot,\lambda)\right\|_{\ell^{1+2\epsilon}}\\ &\le \left\|a_{1,(\omega_\rho)_1}(\cdot,\lambda)\right\|_{\ell^{1+2\epsilon}}\cdots\left\|a_{N,(\omega_\rho)_N}(\cdot,\lambda)\right\|_{\ell^{p_N}}\end{aligned}$$

By applying the $\ell^{1+\epsilon}$ quasi-norm and using Proposition 1.10 (1) we now get

$$\left\|b_{(\omega_\rho)_0}\right\|_{\ell^{1+2\epsilon,1+\epsilon}} \lesssim \left\|a_{1,(\omega_\rho)_1}\right\|_{\ell^{1+2\epsilon,1+\epsilon}}\cdots\left\|a_{N,(\omega_\rho)_N}\right\|_{\ell^{p_N,q_N}}$$

This is the same as

$$\|G_2\|_{L^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_0)}} \lesssim \|(F_\rho)_1\|_{L^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_1)}}\cdots\|(F_\rho)_N\|_{L^{p_N,q_N}_{((\omega_\rho)_N)}}$$

A combination of this estimate with (3.27) and (3.28) gives that $(f_\rho)_1*\cdots*(f_\rho)_N$ is well-defined and that (3.19) holds.

Next we prove (3.20). Let $(1+\epsilon)=\min(1,1+\epsilon)$. Then (3.30)' gives

$$\begin{aligned}b_{(\omega_\rho)_0}(l,\lambda)^{1+\epsilon} &\lesssim \sum_{(k_1,\ldots,k_N)\in\Omega_l}\sum_{\rho}\left(\prod_{j=1}^{N} a_{j,(\omega_\rho)_j}(k_j,\lambda)^{1+\epsilon}\right)\\ &= \sum_{m\in I_{N+1}}\sum_{\rho}\left(a_{1,(\omega_\rho)_1}(\cdot,\lambda)^{1+\epsilon}*\cdots*a_{N,(\omega_\rho)_N}(\cdot,\lambda)^{1+\epsilon}\right)(l-Ne_0+m)\end{aligned}$$

By applying the ℓ^1 norm with respect to the λ variable and using Minkowski's and Hölder's inequalities we obtain

$$\left\|b_{(\omega_\rho)_0}(l,\cdot)\right\|_{\ell^{1+\epsilon}}^{1+\epsilon} = \left\|b_{(\omega_\rho)_0}(l,\cdot)^{1+\epsilon}\right\|_{\ell^1} \lesssim \sum_{(k_1,\ldots,k_N)\in\Omega_l}\sum_{\rho}\left\|\prod_{j=1}^{N} a_{j,(\omega_\rho)_j}(k_j,\cdot)^{1+\epsilon}\right\|_{\ell^1}$$

$$\le \sum_{(k_1,\ldots,k_N)\in\Omega_l}\sum_{\rho}\left(\prod_{j=1}^{N} c_{j,(\omega_\rho)_j}(k_j)^{1+\epsilon}\right) = \sum_{m\in I_{N+1}}\sum_{\rho}\left(c_{1,(\omega_\rho)_1}^{1+\epsilon}*\cdots*c_{N,(\omega_\rho)_N}^{1+\epsilon}\right)(l-Ne_0+m),$$

where

$$c_{j,(\omega_\rho)_j}(k) = \left\|a_{j,(\omega_\rho)_j}(k,\cdot)^{1+\epsilon}\right\|_{\ell^{q_j/1+\epsilon}}^{1/1+\epsilon} = \left\|a_{j,(\omega_\rho)_j}(k,\cdot)\right\|_{\ell^{q_j}}$$

An application of the $\ell^{1+2\epsilon/1+\epsilon}$ quasi-norm on the last inequality and using Proposition 1.10 (2) now gives

$$\begin{aligned}\left\|b_{(\omega_\rho)_0}\right\|_{\ell_*^{1+\epsilon,1+2\epsilon}}^{1+\epsilon} &\lesssim \sum_{m\in I_{N+1}}\sum_{\rho}\left\|\left(c_{1,(\omega_\rho)_1}^{1+\epsilon}*\cdots*c_{N,(\omega_\rho)_N}^{1+\epsilon}\right)(\cdot-Ne_0+m)\right\|_{\ell^{1+2\epsilon/1+\epsilon}}\\ &\asymp \left\|c_{1,(\omega_\rho)_1}^{1+\epsilon}*\cdots*c_{N,(\omega_\rho)_N}^{1+\epsilon}\right\|_{\ell^{1+2\epsilon/1+\epsilon}} \le \left\|c_{1,(\omega_\rho)_1}^{1+\epsilon}\right\|_{\ell^{1+2\epsilon/1+\epsilon}}\cdots\left\|c_{N,(\omega_\rho)_N}^{1+\epsilon}\right\|_{\ell^{p_N/1+\epsilon}}\\ &= \left(\left\|c_{1,(\omega_\rho)_1}\right\|_{\ell^{1+2\epsilon}}\cdots\left\|c_{N,(\omega_\rho)_N}\right\|_{\ell^{p_N}}\right)^{1+\epsilon}\end{aligned}$$

which is the same as

$$\|G_2\|_{L^{1+2\epsilon,1+\epsilon}_{*,((\omega_\rho)_0)}} \lesssim \|(F_\rho)_1\|_{L^{1+2\epsilon,1+\epsilon}_{*,((\omega_\rho)_1)}}\cdots\|(F_\rho)_N\|_{L^{p_N,q_N}_{*,((\omega_\rho)_N)}}$$

which in particular shows that $(f_\rho)_1*\cdots*(f_\rho)_N$ is well-defined. Since $\left\|(f_\rho)_1*\cdots*(f_\rho)_N\right\|_{W^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_0)}} \asymp \|G\|_{L^{1+2\epsilon,1+\epsilon}_{*,((\omega_\rho)_0)}}$ and $\left\|(f_\rho)_j\right\|_{W^{p_j,q_j}_{((\omega_\rho)_j)}} \asymp \|(F_\rho)_j\|_{L^{p_j,q_j}_{*,((\omega_\rho)_j)}}$, $j=1,\ldots,N$, we get (3.20).

We need to prove the associativity, symmetry and invariance with respect to $(\phi_\rho)_0, \dots, (\phi_\rho)_N$ in Definition 3.1. We observe that if

$$r_j = \max(p_j, 1) \text{ and } s_j = \frac{q_j}{1+\epsilon}, 1+\epsilon = \min_{0\le j\le N}(q_j), j = 1, \dots, N,$$

then $M^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d) \subseteq M^{r_j,s_j}_{((\omega_\rho)_j)}(\mathbf{R}^d), j = 1, \dots, N$. By straight-forward computations it follows that if (3.17) or (3.18) hold, then (3.17) respectively (3.18) still hold with r_j and s_j in place of p_j and q_j, respectively, $j = 1, \dots, N$, for some $0 \le \epsilon \le \infty$. This reduce ourself to the case when $p_j, q_j \in [1, \infty]$ for every $j = 0, \dots, N$, in which case all modulation spaces are Banach spaces. We observe that Lemmas 5.2-5.4 and their proofs in [30] still hold true when $(\omega_\rho)_j$ are allowed to belong to the class $\mathcal{P}_E(\mathbf{R}^{2d})$, provided the involved window functions χ_j belong to $\Sigma_1(\mathbf{R}^d)$, and all distributions are allowed to belong to Σ_1' instead of $\mathcal{S}'$. The the associativity, symmetric assertions and invariant properties with respect to the choice of $(\phi_\rho)_0, \dots, (\phi_\rho)_N$ in Definition 3.1 now follows from these modified Lemmas 5.2-5.4 in [30] and their proofs. This gives the results.

Remark 3.7 [38]. Suppose that p_j, q_j and $(\omega_\rho)_j$ are the same as in Theorems 3.2 3.5, and that $p_j + q_j = \infty$ for at most one $j \in \{1, \dots, N\}$. Then it follows that extensions of the mappings $\left((f_\rho)_1, \dots, (f_\rho)_N\right) \mapsto (f_\rho)_1 \cdots (f_\rho)_N$ and $\left((f_\rho)_1, \dots, (f_\rho)_N\right) \mapsto (f_\rho)_1 * \cdots * (f_\rho)_N$ from $\Sigma_1(\mathbf{R}^d) \times \cdots \times \Sigma_1(\mathbf{R}^d)$ to $\Sigma_1(\mathbf{R}^d)$ in Theorems 3.2-3.5 are unique.

In fact, by the proof of Theorem 3.8 below, we may assume that $p_j, q_j \ge 1$ for every j. If $p_j, q_j < \infty$ for every $j \in \{1, \dots, N\}$, then the uniquenesses follow from (3.15), (3.16), (3.19), (3.20) and the fact that $\Sigma_1(\mathbf{R}^d)$ is dense in each $M^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$ and $W^{p_j,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$ for $j \in \{1, \dots, N\}$. For the general situation, the assertion follows from the previous case and duality.

Evidently, Theorems 3.2, 3.5 show that multiplications and convolutions on $\Sigma_1(\mathbf{R}^d)$ can be extended to involve suitable quasi-Banach modulation spaces. Remark 3.7 shows that in most situations, these extensions from products on $\Sigma_1(\mathbf{R}^d)$ are unique. For the multiplication and convolution mappings in Theorems 3.2 and 3.5 we can say more.

Theorem 3.8(see [38]). Let $(\omega_\rho)_j \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $p_j, q_j \in (0, \infty]$ and $j \in \{0, \dots, N\}$. Then the following is true:

(1) if (3.9), (3.11) and (3.13) hold, then $\left((f_\rho)_1, \dots, (f_\rho)_N\right) \mapsto (f_\rho)_1 \cdots (f_\rho)_N$ from $\Sigma_1(\mathbf{R}^d) \times \cdots \times \Sigma_1(\mathbf{R}^d)$ to $\Sigma_1(\mathbf{R}^d)$ is uniquely extendable to a continuous map from $M^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d) \times \cdots \times M^{p_N,q_N}_{((\omega_\rho)_N)}(\mathbf{R}^d)$ to $M^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_0)}(\mathbf{R}^d)$, and (3.15) holds;

(2) if (3.10), (3.13) and (3.17) hold, then $\left((f_\rho)_1, \dots, (f_\rho)_N\right) \mapsto (f_\rho)_1 * \cdots * (f_\rho)_N$ from $\Sigma_1(\mathbf{R}^d) \times \cdots \times \Sigma_1(\mathbf{R}^d)$ to $\Sigma_1(\mathbf{R}^d)$ is uniquely extendable to a continuous map from $W^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d) \times \cdots \times W^{p_N,q_N}_{((\omega_\rho)_N)}(\mathbf{R}^d)$ to $W^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_0)}(\mathbf{R}^d)$, and (3.20) holds;

The problems with uniqueness in Theorem 3.8 appear when one or more Lebesgue exponents are equal to infinity, since $\Sigma_1(\mathbf{R}^d)$ fails to be dense in corresponding modulation spaces. In these situations we shall use narrow convergence, introduced in [28], and is a weaker form of convergence than the norm convergence.

Definition 3.9 [38]. Let $\omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d}), 0 \le \epsilon \le \infty, f_\rho, (f_\rho)_j \in M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d), j \ge 1$ and let

$$(H_\rho)_{f_\rho,\omega_\rho,1+\epsilon}(\xi) \equiv \left\| V_{\phi_\rho} f_\rho(\cdot, \xi)\omega_\rho(\cdot, \xi) \right\|_{L^{1+\epsilon}(\mathbf{R}^d)}$$

Then $(f_\rho)_j$ is said to converge to f_ρ narrowly as $j \to \infty$, if the following conditions are fulfilled:

(1) $(f_\rho)_j \to f_\rho$ in $\Sigma_1'(\mathbf{R}^d)$ as $j \to \infty$;

(2) $(H_\rho)_{(f_\rho)_j,\omega_\rho,1+\epsilon} \to (H_\rho)_{f_\rho,\omega_\rho,1+\epsilon}$ in $L^{1+\epsilon}(\mathbf{R}^d)$ as $j \to \infty$.

The following result is a special case of Theorem 4.17 in [31]. The proof is therefore omitted.

Proposition 3.10 [38]. Let $\omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $0 \le \epsilon \le \infty$ be such that $\epsilon < \infty$. Then $\Sigma_1(\mathbf{R}^d)$ is dense in $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ with respect to the narrow convergence.

We also need the following generalization of Lebesgue's theorem, which follows by a straight-forward application of Fatou's lemma.

Lemma 3.11 [38]. Let μ be a positive measure on a measurable set $\Omega, \{(f_\rho)_j\}_{j=1}^\infty$ and $\{(g_\rho)_j\}_{j=1}^\infty$ be sequences in $L^1(d\mu)$ such that $(f_\rho)_j \to f_\rho$ a. e., $(g_\rho)_j \to g_\rho$ in $L^1(d\mu)$as j tends to infinity, and that $|(f_\rho)_j| \le (g_\rho)_j$ for every $j \in \mathbf{N}$. Then $(f_\rho)_j \to f_\rho$ in in $L^1(d\mu)$ as j tends to infinity.

Remark 3.12 [38]. The narrow convergence is especially interesting when $\epsilon = \infty$. Let $\omega_\rho \in \mathcal{P}_E(\mathbf{R}^{2d}), 0 \leq \epsilon \leq \infty, \phi_\rho \in \Sigma_1(\mathbf{R}^d)$ and $f_\rho \in M^{\infty,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d), (f_\rho)_j \in \Sigma_1(\mathbf{R}^d)$ converges to f_ρ narrowly as $j \to \infty$, and let $(H_\rho)_{f_\rho,\omega_\rho,\infty}$ be the same as in Definition 3.9, Then we may choose these $(f_\rho)_j$ such that

$$\lim_{j\to\infty} V_{\phi_\rho}(f_\rho)_j(x,\xi) = V_{\phi_\rho} f_\rho(x,\xi), \left|V_{\phi_\rho} f_\rho(x,\xi)\omega_\rho(x,\xi)\right| \leq (H_\rho)_{f_\rho,\omega_\rho,\infty}(\xi)$$

$$\text{and} \lim_{j\to\infty} \left\|(H_\rho)_{(f_\rho)_j,\omega_\rho,\infty} - (H_\rho)_{f_\rho,\omega_\rho,\infty}\right\|_{L^{1+\epsilon}} = 0 \quad (3.31)$$

(See [31, Theorem 4.17] and its proof.) It is then possible to apply Lemma 3.11 in integral expressions containing $V_{\phi_\rho}(f_\rho)_j(x,\xi)$ and $V_{\phi_\rho} f_\rho(x,\xi)$ and perform suitable limit processes.

Proof of Theorem 3.8. Since (2) is the Fourier transform of (1), it suffices to prove (1).

The existence of the extension follows from Theorem 3.2. Since $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}(\mathbf{R}^d)$ increases with $(1+\epsilon)$, we may assume that equality is attained in (3.11) and that $p_0 = \cdots p_N = \infty$. By replacing q_j with

$$r_j = \max(1, q_j)$$

it follows from (3.11) that for $r_0 = \frac{q_0}{1+\epsilon} \geq 1$ and some $\epsilon \geq 0$,

$$\frac{1}{1+\epsilon} \leq \sum_{j=1}^{N} \frac{1}{r_j} - N + 1$$

and that

$$M^{\infty,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d) \subseteq M^{\infty,r_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$$

Suppose $(g_\rho)_1, (g_\rho)_2 \in M^{\infty,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$ are such that $(g_\rho)_1$ equals $(g_\rho)_2$ as elements in $M^{\infty,r_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$. Then $(g_\rho)_1$ is also equal to $(g_\rho)_2$ as elements in $M^{\infty,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$. Hence it suffices to prove the uniqueness of the product $(f_\rho)_1 \cdots (f_\rho)_N \in M^{\infty,1+\epsilon}_{((\omega_\rho)_0)}(\mathbf{R}^d)$ of $(f_\rho)_j \in M^{\infty,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d), j = 1, \dots, N$, when additionally $\epsilon \geq 0$, i.e.,

$$\frac{1}{q_1} + \cdots + \frac{1}{q_N} = N - 1 + \frac{1}{q_0}, q_0, \dots, q_N \in [1, \infty] \quad (3.32)$$

In particular, all involved modulation spaces are Banach spaces.

Let $j_0 \in \{1, \dots, N\}$ be chosen such that $q_j \leq q_{j_0}$ for every $j \in \{1, \dots, N\}$. Then $j < \infty$ when $j \neq j_0$.

The product $(f_\rho)_1 \cdots (f_\rho)_N$ is uniquely defined and can be obtained through (3.2) for every $\varphi_\rho \in \Sigma_1(\mathbf{R}^d)$ when $(f_\rho)_j \in \Sigma_1(\mathbf{R}^d)$ and $(f_\rho)_{j_0} \in M^{\infty,q_{j_0}}_{((\omega_\rho)_{j_0})}(\mathbf{R}^d), j \neq j_0$. For general $(f_\rho)_j \in M^{\infty,q_j}_{((\omega_\rho)_j)}(\mathbf{R}^d)$, choose $(f_\rho)_{j,k} \in \Sigma_1(\mathbf{R}^d), k = 1,2, \dots$ such that $(f_\rho)_{j,k}$ converges to $(f_\rho)_j$ narrowly as k tends to infinity, and that (3.31) holds with $(f_\rho)_j$ and $(f_\rho)_{j,k}$ in place of f_ρ and $(f_\rho)_j$, respectively. Then it follows by replacing $(f_\rho)_j$ by $(f_\rho)_{j,k}$ when $j \neq j_0$ in (3.2) and applying Lemma 3.11 on the integral in (3.2) that

$$\ell(\varphi_\rho) \equiv \lim_{k\to\infty} \left((f_\rho)_{1,k} \cdots (f_\rho)_{N,k}, \varphi_\rho\right)$$

exists and defines an element in $f_\rho \in \Sigma_1'(\mathbf{R}^d)$. This shows that the only possibility to define $(f_\rho)_1 \cdots (f_\rho)_N$ in a continuous way is to put $(f_\rho)_1 \cdots (f_\rho)_N = f_\rho$, and the asserted uniqueness follows.

IV. Extensions and variations

We extend the results on step and Fourier step multipliers to certain so-called curve step and Fourier curve step multipliers. That is a generalized form of step and Fourier step multipliers, where the constants $a_0(j)$ in the definition of M_{b,a_0} and $M_{\mathcal{F},b,a_0}$ are replaced by certain nonconstant functions or even distributions. In the end we are able to generalize Theorems 2.1 and 2.3 to such multipliers. These achievements are based on Hölder-Young relations for multiplications and convolutions in Section 3, In the case of trivial weights and all modulation spaces are Banach spaces, our results are similar to [3, Theorem 6] and [27, Proposition 4.12].

The multipliers and Fourier multipliers which we consider are given in the following.

Definition 4.1 [38]. Let $b \in \mathbf{R}^d_+$ be fixed, Λ_b and Q_b be given by (1.43) and (1.44), and let

$$a_0 \equiv \{a_0(j,\cdot)\}_{j\in\Lambda_b} \subseteq C^\infty(\mathbf{R}^d) \quad (4.1)$$

be such that

$$\left(\sum_{j\in\Lambda_b} a_0(j,\cdot)\chi_{j+Q_b}\right) \in \Sigma_1'(\mathbf{R}^d).$$

Then the multiplier

$$M_{b,a_0}: f_\rho \mapsto \sum_{j\in\Lambda_b} a_0(j,\cdot)\chi_{j+Q_b} f_\rho, \qquad (1.46)'$$

from $C^\infty(\mathbf{R}^d)$ to $L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ is called slope step multiplier with respect to b and a_0. The Fourier multiplier

$$M_{\mathcal{F},b,a_0} \equiv \mathcal{F}^{-1} \circ M_{b,a_0} \circ \mathcal{F}, \qquad (1.45)'$$

is called slope step Fourier multiplier with respect to b and a_0.

First we perform some studies of

$$T_{\psi_\rho} a_0 \equiv \sum_{j\in\Lambda_b} a_0(j,\cdot)\psi_\rho(\cdot - j) \qquad (4.2)$$

where $\psi_\rho \in \mathcal{S}(\mathbf{R}^d)$ is suitable. The conditions on the sequence (4.1) that we have in mind are that for fixed $(\omega_\rho)_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and $0 \le \epsilon \le \infty$, the functions

$$\mathfrak{b}_{a_0,\alpha}(x) \equiv \sup_{\beta\le\alpha}\left(\sup_{j\in\Lambda_b}\left|(\partial_x^\beta a_0)(j,x)\right|\right) \qquad (4.3)$$

should belong to $L^{1+\epsilon}(\mathbf{R}^d)$ for every $\alpha \in \mathbf{N}^d$, or that for some or for every $\epsilon \ge 0$, the function

$$\mathfrak{b}_{a_0,1+\epsilon}(x) \equiv \sup_{\alpha\in\mathbf{N}^d}\left(\sup_{j\in\Lambda_b}\left|\frac{(\partial_x^\alpha a_0)(j,x)}{(1+\epsilon)^{|\alpha|}\alpha!^{1+\epsilon}}\right|\right) \qquad (4.4)$$

should belong to $L^{1+\epsilon}(\mathbf{R}^d)$.

Proposition 4.2(see [38]). Let $b \in \mathbf{R}^d_+$ be fixed, Λ_b be given by (1.43), $\epsilon > 0$, (4.1) be a sequence of functions on $C^\infty(\mathbf{R}^d)$, $\psi_\rho \in \mathcal{S}(\mathbf{R}^d)$, and let $T_{\psi_\rho}a_0$, $\mathfrak{b}_{a_0,\alpha}$ and $\mathfrak{b}_{a_0,1+\epsilon}$ be given by (4.2)-(4.4) when $\alpha \in \mathbf{N}^d$ and $\epsilon \ge 0$. Then the following is true:

(1) if $\mathfrak{b}_{a_0,\alpha} \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ for $\alpha = (0,\dots,0) \in \mathbf{N}^d$, then the series in (4.2) is locally uniformly convergent and defines an element in $C(\mathbf{R}^d)$;

(2) if $\mathfrak{b}_{a_0,\alpha} \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ for every $\alpha \in \mathbf{N}^d$, then $T_{\psi_\rho}a_0 \in C^\infty(\mathbf{R}^d)$ and

$$\left|\left(\partial^\alpha T_{\psi_\rho}a_0\right)(x)\right| \lesssim \mathfrak{b}_{a_0,\alpha}(x), x \in \mathbf{R}^d$$

for every $\alpha \in \mathbf{N}^d$;

(3) if in addition $\psi_\rho \in \mathcal{S}^{1+\epsilon}_{1+\epsilon}(\mathbf{R}^d)$ and $\mathfrak{b}_{a_0,h_0} \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$, then

$$\left|\left(\partial^\alpha T_{\psi_\rho}a_0\right)(x)\right| \lesssim (1+\epsilon)^\alpha \alpha!^{1+\epsilon}\, \mathfrak{b}_{a_0,h_0}(x), x \in \mathbf{R}^d$$

for some $\epsilon \ge 0$;

(4) if in addition $\psi_\rho \in \Sigma^{1+\epsilon}_{1+\epsilon}(\mathbf{R}^d)$, $c > 1$ and $\mathfrak{b}_{a_0,h_0} \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$, then

$$\left|\left(\partial^\alpha T_{\psi_\rho}a_0\right)(x)\right| \lesssim (ch_0)^\alpha \alpha!^{1+\epsilon}\, \mathfrak{b}_{a_0,h_0}(x), x \in \mathbf{R}^d$$

Proof. We only prove (1) and (4). The other assertions follow by similar arguments and are left for the reader.

Let $\Lambda = \Lambda_b$, $\alpha = (0,\dots,0) \in \mathbf{N}^d$ and suppose that $\mathfrak{b}_{a_0,\alpha} \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$. We have

$$\sum_{j\in\Lambda}\sum_\rho \left|a_0(j,x)\psi_\rho(x-j)\right| \le \mathfrak{b}_{a_0,\alpha}(x)\sum_{j\in\Lambda}\sum_\rho |\psi_\rho(x-j)| \asymp \mathfrak{b}_{a_0,\alpha}(x),$$

which shows that (4.2) is locally uniformly convergent. Since $a_0(j,\cdot)$ and $\psi_\rho(\cdot - j)$ are continuous functions, it follows that $T_{\psi_\rho}a_0$ in (4.2) is continuous.

Next suppose additionally that $\psi_\rho \in \Sigma^{1+\epsilon}_{1+\epsilon}(\mathbf{R}^d)$ and consider $f_\rho = T_{\psi_\rho}a_0$. For every $\alpha \in \mathbf{N}^d, \varepsilon > 0$ and $\epsilon > -1$, we have

$$\begin{aligned}\left|(\partial^\alpha f_\rho)(x)\right| &\le \sum_{j\in\Lambda}\sum_{\gamma\le\alpha}\sum_\rho \binom{\alpha}{\gamma}\left|\partial^{\alpha-\gamma}a_0(j,x)\right|\left|\partial^\gamma\psi_\rho(x-j)\right| \\ &\lesssim \mathfrak{b}_{a_0,h_0}(x)\sum_{j\in\Lambda}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma} h_0^{|\alpha-\gamma|}(\alpha-\gamma)!^{1+\epsilon}\,\varepsilon^{|\gamma|}\gamma' e^{-(1+\epsilon)|x-j|^{\frac{1}{1+\epsilon}}} \\ &\le (h_0+\varepsilon)^{|\alpha|}\alpha!^{1+\epsilon}\,\mathfrak{b}_{a_0,h_0}(x)\sum_{j\in\Lambda} e^{-(1+\epsilon)|x-j|^{\frac{1}{1+\epsilon}}} \asymp (h_0+\varepsilon)^{|\alpha|}\alpha!^{1+\epsilon}\,\mathfrak{b}_{a_0,h_0}(x)\end{aligned}$$

and the result follows.

In the next result we show that if $\mathfrak{b}_{a_0,\alpha}$ or $\mathfrak{b}_{a_0,1+\epsilon}$ in the previous proposition belong to $\mathrm{W}^1\left((\omega_\rho)_0, \ell^{1+\epsilon}\right)$, then for $T_{\psi_\rho}a_0$ in (4.2) we have

$$T_{\psi_\rho}a_0 \in M^{1+\epsilon,1+\epsilon}_{\left((\omega_\rho)_{1+\epsilon}\right)}(\mathbf{R}^d) \bigcap W^{1+\epsilon,1+\epsilon}_{\left((\omega_\rho)_{1+\epsilon}\right)}(\mathbf{R}^d) \qquad (4.5)$$

and

$$\left\|T_{\psi_\rho}a_0\right\|_{M^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_{1+\epsilon})}}+\left\|T_{\psi_\rho}a_0\right\|_{W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_{1+\epsilon})}}\lesssim \max_{|\alpha|\le N}\|\mathfrak{b}_{a_0,\alpha}\|_{\mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)} \tag{4.6}$$

or

$$\left\|T_{\psi_\rho}a_0\right\|_{M^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_{1+\epsilon})}}+\left\|T_{\psi_\rho}a_0\right\|_{W^{1+\epsilon,1+\epsilon}_{((\omega_\rho)_{1+\epsilon})}}\lesssim \|\mathfrak{b}_{a_0,1+\epsilon}\|_{\mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)} \tag{4.7}$$

See also Remark 1.9 for notations.

Proposition 4.3(see [38]). Let $(\omega_\rho)_0\in\mathcal{P}_E(\mathbf{R}^d),\Lambda\subseteq\mathbf{R}^d,\epsilon>0,T_{\psi_\rho}a_0,\mathfrak{b}_{a_0,\alpha}$, $\mathfrak{b}_{a_0,1+\epsilon}$ and ψ_ρ be the same as in Proposition 4.2, and let $0\le\epsilon\le\infty$. Then the following is true:

(1) if in addition $(\omega_\rho)_0\in\mathcal{P}(\mathbf{R}^d),\mathfrak{b}_{a_0,\alpha}\in\mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)$ for every $\alpha\in\mathrm{N}^d$, and $\vartheta_{1+\epsilon}(x,\xi)=(\omega_\rho)_0(x)\langle\xi\rangle^{1+\epsilon}$ when $\epsilon\ge0$, then (4.5) and (4.6) hold;

(2) if in addition $(\omega_\rho)_0\in\mathcal{P}^0_{E,1+\epsilon}(\mathbf{R}^d),\psi_\rho\in\mathcal{S}^{1+\epsilon}_{1+\epsilon}(\mathbf{R}^d),\mathfrak{b}_{a_0,1+\epsilon}\in\mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)$ for some $\epsilon\ge0$, and $\vartheta_{1+\epsilon}(x,\xi)=(\omega_\rho)_0(x)e^{(1+\epsilon)|\xi|^{\frac{1}{1+\epsilon}}}$ when $\epsilon\ge0$, then (4.5) and (4.7) hold for some $\epsilon\ge0$;

(3) if in addition $(\omega_\rho)_0\in\mathcal{P}_{E,1+\epsilon}(\mathbf{R}^d),\psi_\rho\in\Sigma^{1+\epsilon}_{1+\epsilon}(\mathbf{R}^d),\mathfrak{b}_{a_0,1+\epsilon}\in\mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)$ for every $\epsilon\ge0$, and $\vartheta_{1+\epsilon}(x,\xi)=(\omega_\rho)_0(x)e^{(1+\epsilon)|\xi|^{\frac{1}{1+\epsilon}}}$ when $\epsilon\ge0$, then (4.5) and (4.7) hold for every $\epsilon>-1$.

Proof. We only prove (2). The other assertions follow by similar arguments and are left for the reader.

Let $f_\rho=T_{\psi_\rho}a_0,(\psi_\rho)_j=\psi_\rho(\cdot-j)$ and $\tilde{\phi}_\rho(x)=\overline{\phi_\rho(-x)}$. Since

$$\left(V_{\phi_\rho}f_\rho\right)(x,\xi)=e^{-i\langle x,\xi\rangle}\left(V_{\widehat{\phi}_\rho}\hat{f}_\rho\right)(\xi,-x)$$

we get

$$\begin{aligned}\left(V_{\phi_\rho}f_\rho\right)(x,\xi)&=e^{-i\langle x,\xi\rangle}\sum_{j\in\Lambda}\left(V_{\widehat{\phi}_\rho}\mathcal{F}\big((\psi_\rho)_ja_0(j,\cdot)\big)\right)(\xi,-x)\\&=e^{-i\langle x,\xi\rangle}\sum_{j\in\Lambda}\mathcal{F}^{-1}\left(\mathcal{F}\big((\psi_\rho)_ja_0(j,\cdot)\big)\cdot\overline{\widehat{\phi}_\rho(\cdot-\xi)}\right)(x)\\&=(2\pi)^{-\frac{d}{2}}e^{-i\langle x,\xi\rangle}\sum_{j\in\Lambda}\left(\big((\psi_\rho)_ja_0(j,\cdot)\big)*\big(\tilde{\phi}_\rho\cdot e^{i\langle\cdot,\xi\rangle}\big)\right)(x)\end{aligned}$$

Hence, Leibnitz rule, integrations by parts and Proposition 1.1 give

$$\begin{aligned}&\left|\xi^\alpha\left(V_{\phi_\rho}f_\rho\right)(x,\xi)\right|\lesssim\sum_{j\in\Lambda}\left|\left(\left((\psi_\rho)_ja_0(j,\cdot)\right)*\left(\tilde{\phi}_\rho\cdot\left(D_x^\alpha e^{i\langle\cdot,\xi\rangle}\right)\right)\right)(x)\right|\\&\le\sum\frac{\alpha!}{\gamma_1!\gamma_2!\gamma_3!}\left(\left|\left(\partial^{\gamma_1}(\psi_\rho)_j\right)(\partial^{\gamma_2}a_0(j,\cdot))\right|*\left|\partial^{\gamma_3}\tilde{\phi}_\rho\right|\right)(x)\\&\lesssim\sum 3^{|\alpha|}1+\epsilon^{|\gamma_1+\gamma_2+\gamma_3|}(\gamma_1!\gamma_2!\gamma_3!)^{1+\epsilon}\left(\left(e^{-(1+\epsilon)|\cdot-j|^{\frac{1}{1+\epsilon}}}\mathfrak{b}_{a_0,1+\epsilon}\right)*e^{-(1+\epsilon)|\cdot|^{\frac{1}{1+\epsilon}}}\right)(x)\\&\le(9(1+\epsilon))^{|\alpha|}\alpha!^{1+\epsilon}\sum_{j\in\Lambda}\left(\left(e^{-(1+\epsilon)|\cdot-j|^{\frac{1}{1+\epsilon}}}\mathfrak{b}_{a_0,1+\epsilon}\right)*e^{-(1+\epsilon)|\cdot|^{\frac{1}{1+\epsilon}}}\right)(x)\\&(9(1+\epsilon))^{|\alpha|}\alpha!^{1+\epsilon}\left(\mathfrak{b}_{a_0,1+\epsilon}*e^{-(1+\epsilon)|\cdot|^{\frac{1}{1+\epsilon}}}\right)(x)\end{aligned}$$

Here the second and third sums are taken with respect to all $j\in\Lambda$ and all $\gamma_1,\gamma_2,\gamma_3\in\mathbf{N}^d$ such that $\gamma_1+\gamma_2+\gamma_3=\alpha$.

This implies that for some constant C which is independent of $1+\epsilon$, and α we have

$$\left(\frac{|\xi|^{\frac{1}{1+\epsilon}}}{(C(1+\epsilon))^{\frac{1}{1+\epsilon}}}\right)^k\left|V_{\phi_\rho}f_\rho(x,\xi)\right|^{\frac{1}{1+\epsilon}}\lesssim 2^{-k}\left(\left(\mathfrak{b}_{a_0,1+\epsilon}*e^{-(1+\epsilon)|\cdot|^{\frac{1}{1+\epsilon}}}\right)(x)\right)^{\frac{1}{1+\epsilon}}$$

and by taking the sum over all $k\ge0$ we land on

$$\left|V_{\phi_\rho}f_\rho(x,\xi)e^{r_{(1+\epsilon)}|\xi|^{\frac{1}{1+\epsilon}}}\right|\lesssim\left(\mathfrak{b}_{a_0,1+\epsilon}*e^{-(1+\epsilon)|\cdot|^{\frac{1}{1+\epsilon}}}\right)(x),r_{(1+\epsilon)}=\frac{1+\epsilon}{(C(1+\epsilon))^{\frac{1}{1+\epsilon}}}$$

By multiplying with $(\omega_\rho)_0$ and using that $(\omega_\rho)_0(x+y)\lesssim(\omega_\rho)_0(x)e^{(1+\epsilon)|y|^{\frac{1}{1+\epsilon}}}$ for every $\epsilon\ge0$, we obtain

$$\left|V_{\phi_\rho} f_\rho(x,\xi)\vartheta_{r_{(1+\epsilon)}}(x,\xi)\right| \lesssim \left(\left(\mathfrak{b}_{a_0,1+\epsilon}\left(\omega_\rho\right)_0\right) * e^{-(1+\epsilon)|\cdot|^{\frac{1}{1+\epsilon}}}\right)(x), r_{(1+\epsilon)} = \frac{1+\epsilon}{(C(1+\epsilon))^{\frac{1}{1+\epsilon}}} \quad (4.8)$$

for some $\epsilon \geq 0$.

By applying [32, Proposition 2.5] on the last inequality we obtain

$$\left\|V_{\phi_\rho} f_\rho\right\|_{\mathrm{W}\left(\vartheta_{r_{(1+\epsilon)}},\ell^{1+\epsilon,\infty}\right)} + \left\|V_{\phi_\rho} f_\rho\right\|_{\mathrm{W}\left(\vartheta_{r_{(1+\epsilon)}},\ell_*^{1+\epsilon,\infty}\right)} \lesssim \left\|e^{-(1+\epsilon)|\cdot|^{\frac{1}{1+\epsilon}}}\right\|_{\mathrm{W}\left(1,\ell^{\min(1,1+\epsilon)}\right)} \left\|\mathfrak{b}_{a_0,1+\epsilon}\right\|_{\mathrm{W}\left(\vartheta_{r_{(1+\epsilon)}},\ell^{1+\epsilon}\right)}$$

The result now follows for general $0 \leq \epsilon \leq \infty$ from the relations $\left\|V_{\phi_\rho} f_\rho\right\|_{\mathrm{W}\left(\vartheta_{r_{(1+\epsilon)}},\ell^{1+\epsilon,\infty}\right)} \asymp \|f_\rho\|_{M^{1+\epsilon,\infty}}$, and $M^{1+\epsilon,\infty}_{(\vartheta_{2(1+\epsilon)})}(\mathbf{R}^d) \hookrightarrow M^{1+\epsilon,1+\epsilon}_{(\vartheta_{1+\epsilon})}(\mathbf{R}^d) \hookrightarrow M^{1+\epsilon,\infty}_{(\vartheta_{1+\epsilon})}(\mathbf{R}^d)$, and similarly with $W^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}$ and $\ell_*^{1+\epsilon,1+\epsilon}$ spaces in place of $M^{1+\epsilon,1+\epsilon}_{(\omega_\rho)}$ and $\ell^{1+\epsilon,1+\epsilon}$ spaces.

We have now the following extension of Theorem 2.1. Here involved Lebesgue exponents and weight functions should fullfil

$$\frac{1}{1+3\epsilon} - \frac{1}{1+2\epsilon} \leq \frac{1}{1+\epsilon}, \frac{1}{1+2\epsilon} - \frac{1}{1+3\epsilon} \geq \max\left(\frac{1}{1+3\epsilon} - 1{,}0\right) \quad (4.9)$$

and

$$\left(\omega_\rho\right)_{0,2}(x) \lesssim \left(\omega_\rho\right)_{0,1}(x)\left(\omega_\rho\right)_0(x) \quad (4.10)$$

Theorem 4.4(see [38]). Let $0 \leq \epsilon \leq \infty, 0 < \epsilon \leq \infty, 1+2\epsilon, 1+3\epsilon \in (\min(1,1+3\epsilon), \infty)$ be such that (4.9) holds, $b > 0, \left(\omega_\rho\right)_0, \left(\omega_\rho\right)_{0,j}$ be weights on $\mathbf{R}^d$ such that (4.10) holds true, $\omega_\rho(x,\xi) = \left(\omega_\rho\right)_0(x)$ and $\left(\omega_\rho\right)_j(x,\xi) = \left(\omega_\rho\right)_{0,j}(x), j = 1,2, x,\xi \in \mathbf{R}^d$. Let a_0 in (4.1) be such that $\Lambda = \Lambda_b$ and $a_0(j,\cdot) \in C^\infty(\mathbf{R}^d)$ for every $j \in \Lambda_b$, and let $\mathfrak{b}_{a_0,\alpha}$ and $\mathfrak{b}_{a_0,1+\epsilon}$ be given by (4.3) and (4.4). Also suppose that one of the following conditions hold true:

(i) $\mathfrak{b}_{a_0,\alpha} \in \mathrm{W}^1\left(\left(\omega_\rho\right)_0, \ell^{1+\epsilon}\right)$ for every $\alpha \in \mathbf{N}^d$, and $\left(\omega_\rho\right)_0, \left(\omega_\rho\right)_{0,j} \in \mathcal{P}(\mathbf{R}^d), j = 1,2$;

(ii) $\mathfrak{b}_{a_0,1+\epsilon} \in \mathrm{W}^1\left(\left(\omega_\rho\right)_0, \ell^{1+\epsilon}\right)$ for some $\epsilon \geq 0$, and $\left(\omega_\rho\right)_0, \left(\omega_\rho\right)_{0,j} \in \mathcal{P}^0_{E,1+\epsilon}(\mathbf{R}^d), j = 1,2$;

(iii) $\mathfrak{b}_{a_0,1+\epsilon} \in \mathrm{W}^1\left(\left(\omega_\rho\right)_0, \ell^{1+\epsilon}\right)$ for every $\epsilon \geq 0$, and $\left(\omega_\rho\right)_0, \left(\omega_\rho\right)_{0,j} \in \mathcal{P}_{E,1+\epsilon}(\mathbf{R}^d), j = 1,2$.

Then the following is true:

(1) M_{b,a_0} is continuous from $W^{1+2\epsilon,1+\epsilon}_{\left(\left(\omega_\rho\right)_1\right)}(\mathbf{R}^d)$ to $W^{1+3\epsilon,1+\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$;

(2) M_{b,a_0} is continuous from $M^{1+2\epsilon,1+2\epsilon}_{\left(\left(\omega_\rho\right)_1\right)}(\mathbf{R}^d)$ to $M^{1+3\epsilon,1+3\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$.

Proof. We only prove the result when (iii) holds. The other cases follow by similar arguments and is left for the reader.

Let $\psi_\rho \in \Sigma_{1+\epsilon}^{1+\epsilon}(\mathbf{R}^d)$ be such that $\psi_\rho = 1$ on Q_b and supported in a neighbourhood of Q_b, and let $\Lambda_1, \ldots, \Lambda_N$ be sublattices of $\Lambda = \Lambda_b$ such that $\bigcup_{j=1}^N \Lambda_j = \Lambda$ and $\mathrm{supp}\psi_\rho(\cdot - k_1) \cap \mathrm{supp}\psi_\rho(\cdot - k_2) = \emptyset, k_1, k_2 \in \Lambda_j, k_1 \neq k_2$, for every $j = 1, \ldots, N$. Then

$$M_{b,a_0} = \sum_{j=1}^N S_j,$$

where $S_j = S_{2,j} \circ S_{1,j}$, with $S_{1,j}$ and $S_{2,j}$ being the multiplication operators with the functions

$$(\varphi_\rho)_{1,j} \equiv \sum_{k\in\Lambda_j} \sum_\rho a_0(k,\cdot)\psi_\rho(\cdot - k) \text{ and } (\varphi_\rho)_{2,j} \equiv \sum_{k\in\Lambda_j} \chi_{Q_b}(\cdot - k)$$

respectively. The result follows if we prove the asserted continuity properties for S_j in place of M_{b,a_0}.

By Proposition 4.3 it follows that $(\varphi_\rho)_{1,j} \in M^{1+\epsilon,1-\epsilon}_{(\vartheta_{1+\epsilon})}(\mathbf{R}^d) \cap W^{1+\epsilon,1-\epsilon}_{(\vartheta_{1+\epsilon})}(\mathbf{R}^d)$ for every $0 \leq \epsilon \leq 1$ and $\epsilon \geq 0$. Hence, if we choose $(1-\epsilon)$ small enough, Theorems 3.2 and 3.3 show that $S_{1,j}$ is continuous from $W^{1+2\epsilon,1-\epsilon}_{\left(\left(\omega_\rho\right)_1\right)}(\mathbf{R}^d)$ to $W^{1+3\epsilon,1-\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$, and from $M^{1+2\epsilon,1+2\epsilon}_{\left(\left(\omega_\rho\right)_1\right)}(\mathbf{R}^d)$ to $M^{1+3\epsilon,1+2\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$. In view of Theorem 2.1 one has that $S_{2,j}$ is continuous on $W^{1+3\epsilon,1-\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$, and from $M^{1+3\epsilon,1+2\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$ to $M^{1+3\epsilon,1+3\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$, for every j. By combining these mapping properties it follows that S_j is continuous from $W^{1+2\epsilon,1-\epsilon}_{\left(\left(\omega_\rho\right)_1\right)}(\mathbf{R}^d)$ to $W^{1+3\epsilon,1-\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$, and from $M^{1+2\epsilon,1+2\epsilon}_{\left(\left(\omega_\rho\right)_1\right)}(\mathbf{R}^d)$ to $M^{1+2\epsilon,1+3\epsilon}_{\left(\left(\omega_\rho\right)_2\right)}(\mathbf{R}^d)$ for every j, and the result follows.

By Fourier transforming the latter result we obtain the following extension of Theorem 2.3, The details are left for the reader. Here

$$\frac{1}{1+3\epsilon}-\frac{1}{1+2\epsilon}\le\frac{1}{1-\epsilon}\text{ and }\frac{1}{1+2\epsilon}-\frac{1}{1+3\epsilon}\ge\max\left(\frac{1}{1+3\epsilon}-1,0\right) \tag{4.11}$$

Theorem 4.5 [38]. Let $0\le\epsilon\le\infty, 0<\epsilon<\infty, 1+2\epsilon, 1+3\epsilon\in(\min(1,1+3\epsilon),\infty)$ be such that (4.11) holds, $b>0, (\omega_\rho)_0, (\omega_\rho)_{0,j}$ be weights on $\mathbf{R}^d$ such that (4.10) holds true, $\omega_\rho(x,\xi)=(\omega_\rho)_0(\xi)$ and $(\omega_\rho)_j(x,\xi)=(\omega_\rho)_{0,j}(\xi), j=1,2, x,\xi\in\mathbf{R}^d$. Let $(f_\rho)_0$ in (4.1) be such that $\Lambda=\Lambda_b$ and $a_0(j,\cdot)\in C^\infty(\mathbf{R}^d)$ for every $j\in\Lambda_b$, and let $\mathfrak{b}_{a_0,\alpha}$ and $\mathfrak{b}_{a_0,1+\epsilon}$ be given by (4.3) and (4.4). Also suppose that one of the following conditions hold true:

(i) $\mathfrak{b}_{a_0,\alpha}\in \mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)$ for every $\alpha\in\mathrm{N}^d$, and $(\omega_\rho)_0,(\omega_\rho)_{0,j}\in\mathcal{P}(\mathbf{R}^d), j=1,2$;

(ii) $\mathfrak{b}_{a_0,1+\epsilon}\in \mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)$ for some $\epsilon\ge0$, and $(\omega_\rho)_0,(\omega_\rho)_{0,j}\in\mathcal{P}^0_{E,1+\epsilon}(\mathbf{R}^d), j=1,2$;

(iii) $\mathfrak{b}_{a_0,1+\epsilon}\in \mathrm{W}^1\left((\omega_\rho)_0,\ell^{1+\epsilon}\right)$ for every $\epsilon\ge0$, and $(\omega_\rho)_0,(\omega_\rho)_{0,j}\in\mathcal{P}_{E,1+\epsilon}(\mathbf{R}^d), j=1,2$.

Then the following is true:

(1) $M_{\mathcal{F},b,a_0}$ is continuous from $M^{1+\epsilon,1+2\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d)$ to $M^{1+\epsilon,1+3\epsilon}_{((\omega_\rho)_2)}(\mathbf{R}^d)$;

(2) $M_{\mathcal{F},b,a_0}$ is continuous from $W^{1+2\epsilon,1+2\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d)$ to $W^{1+2\epsilon,1+3\epsilon}_{((\omega_\rho)_2)}(\mathbf{R}^d)$.

We observe that Theorems 4.4 and 4.5 include the following extensions of Theorems 2.1 and 2.3.

Corollary 4.6 [38]. Let $0\le\epsilon\le\infty, 0<\epsilon\le\infty, 1+2\epsilon, 1+3\epsilon\in(\min(1,1+\epsilon),\infty)$ be such that (4.9) hold, $b>0, (\omega_\rho)_0, (\omega_\rho)_{0,j}\in\mathcal{P}_E(\mathbf{R}^d)$ be such that (4.10), $(\omega_\rho)_j(x,\xi)=(\omega_\rho)_{0,j}(x), j=1,2, x,\xi\in\mathbf{R}^d, x,\xi\in\mathbf{R}^d$, and let $a_0\in\ell^{1+\epsilon}_{((\omega_\rho)_0)}(\Lambda_b)$. Then the following is true:

(1) M_{b,a_0} is continuous from $W^{1+2\epsilon,1+\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d)$ to $W^{1+3\epsilon,1+\epsilon}_{((\omega_\rho)_2)}(\mathbf{R}^d)$;

(2) M_{b,a_0} is continuous from $M^{1+2\epsilon,1+2\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d)$ to $M^{1+3\epsilon,1+3\epsilon}_{((\omega_\rho)_2)}(\mathbf{R}^d)$.

Corollary 4.7 [38]. Let $0\le\epsilon\le\infty, 1+2\epsilon, 1+3\epsilon\in(\min(1,1+\epsilon),\infty), 0<\epsilon\le\infty$ be such that (4.11) hold, $b>0, (\omega_\rho)_0, (\omega_\rho)_{0,j}\in\mathcal{P}_E(\mathbf{R}^d)$ be such that (4.10) holds, $(\omega_\rho)_j(x,\xi)=(\omega_\rho)_{0,j}(\xi), j=1,2, x,\xi\in\mathbf{R}^d, x,\xi\in\mathbf{R}^d$, and let $a_0\in\ell^{1+\epsilon}_{((\omega_\rho)_0)}(\Lambda_b)$. Then the following is true:

(1) $M_{\mathcal{F},b,a_0}$ is continuous from $M^{1+\epsilon,1+2\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d)$ to $M^{1+\epsilon,1+3\epsilon}_{((\omega_\rho)_2)}(\mathbf{R}^d)$;

(2) $M_{\mathcal{F},b,a_0}$ is continuous from $W^{1+2\epsilon,1+2\epsilon}_{((\omega_\rho)_1)}(\mathbf{R}^d)$ to $W^{1+3\epsilon,1+3\epsilon}_{((\omega_\rho)_2)}(\mathbf{R}^d)$.

Proof of Corollaries 4.6 and 4.7. Let $\psi_\rho\in\Sigma_1^{1+\epsilon}(\mathbf{R}^d)$ be compactly supported and chosen such that $\psi_\rho=1$ on Q_b. Then the results follow by letting $a_0(j,\cdot)=a_0(j)\psi_\rho(\cdot-j)$ in Theorems 4.4 and 4.5. The details are left for the reader.

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