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**Research Paper**

# A survey on the Nonlinear operations on a class of modulation spaces

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***Abstract***
*We follow the pioneers authors of [19] by showing a survey asan applicationon the nonlinear operation $f_i \mapsto F(f_i)$when maps the modulation space $M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)0 \le \epsilon \le \infty)$ to itself again. It is true that $M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)$ is a multiplication algebra when $s > 2n\epsilon/1+2\epsilon$, hence it is true and valid for this space if $F$ is fully entire. We also claim that it is still true for non-analytic $F$ when $\epsilon \ge 0$.*

***Keywords:*** *Modulation spaces, Paradifferential operators*

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## I. Introduction

We discuss nonlinear operations $f_i \mapsto F(f_i)$, that is, the composition of functions $F$ and $f_i$. Let $X$ be a function space. Then does the nonlinear operation map $X$ to the same space $X$ ? For the simplest case $F(z) = z^2$, that is, $F(f_i) = f_i^2$, the answer is yes when $X$ is a multiplication algebra. From this observation, we immediately obtain the affirmative answer to this question for any entire functions $F(z)$ and multiplication algebras $X$. The typical examples of multiplication algebras are $L^{1+\epsilon}$-Sobolev spaces $H_s^{1+\epsilon}(\mathbb{R}^n)(0 < \epsilon < \infty)$ with $s > n/1+\epsilon$ and Besov spaces $B_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)(0 \le \epsilon \le \infty)$ with $s > n/1+\epsilon$ (see Propositions the two follows).

When $F$ fails to satisfy the analyticity [19], answering this question is not so straightforward. Then however have an affirmative answer by virtue of the theory of paradifferential operators introduced by [3] and developed by [9]. The main argument is to write the composition $F(f_i)$ in the form of a linear operation

$$F(f_i) = M_{F,f_i}(x, D)f_i$$

(assuming that $F(0) = 0, f_i \in H_s^{1+\epsilon}(\mathbb{R}^n)$ is real-valued, and $s > n/1+\epsilon$ to be embedded in $L^\infty(\mathbb{R}^n)$), where $M_{F,f_i}(x, D)$ is a pseudo-differential operator of the Hörmander class $S_{1,1}^0$. Since pseudo-differential operators of this class are $H_s^{1+\epsilon}$-bounded for $s > 0$, we get the following result (see [19]):

**Theorem A([9, Theorem 1]).** Let $0 < \epsilon < \infty$ and $s > n/1+\epsilon$. Assume that $f_i: \mathbb{R}^n \to \mathbb{R}$, $f_i \in H_s^{1+\epsilon}(\mathbb{R}^n), F \in C^\infty(\mathbb{R})$ and $F(0) = 0$. Then, we have $F(f_i) \in H_s^{1+\epsilon}(\mathbb{R}^n)$.

**Remark 1.1 [19].** We can state Theorem A in an explicit form (see, e.g., Taylor [17, Section 3.1]):

$$\|\sum_i F(f_i)\|_{H_s^{1+\epsilon}} \le C\|F'\|_{C^{[s]+1}(\Omega)} \sum_i \left(1 + \|f_i\|_{L^\infty}^{[s]+1}\right)\|f_i\|_{H_s^{1+\epsilon}}$$

where $\Omega = \{t: |t| \le C'\|f_i\|_{L^\infty}\}$, and the constants $C$ and $C'$ are universal.

By the same argument, we have a similar conclusion for Besov spaces (see [13]), and a result for complex-valued sequence of functions $f_i: \mathbb{R}^n \to \mathbb{C}$ can be also stated considering the nonlinear operation $f_i \mapsto F(\mathrm{Re}f_i, \mathrm{Im}f_i)$ with two-variable functions $F(s,t)$, although here we only consider real-valued functions $f_i: \mathbb{R}^n \to \mathbb{R}$ just for the sake of simplicity.

We establish a similar result for modulation spaces $M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)$. Modulation spaces are relatively new function spaces introduced by [4] in 1980's to measure the decaying and regularity properties of a function or distribution in a way different from $L^{1+\epsilon}$-Sobolev spaces or Besov spaces. The main idea of modulation spaces is to consider the space variable and the variable of its Fourier transform simultaneously, while they are treated independently in $L^{1+\epsilon}$-Sobolev spaces and Besov spaces. Because of their nature, modulation spaces have wide several significant properties. For example, the Schödinger propagator $e^{it|D|^2}$ and the wave propagator $e^{it|D|}$ map the modulation space $M_s^{1+\epsilon,1+2\epsilon}$ to the same space ([1]), which means, we have no loss of regularity when we work on modulation spaces, while it is not true for $L^{1+\epsilon}$-Sobolev spaces $H_s^{1+\epsilon}$ or Besov spaces $B_s^{1+\epsilon,1+2\epsilon}$ (see [10]). When we try to utilize this advantage for nonlinear analysis, it is indispensable to ask whether the nonlinear operation also maps $M_s^{1+\epsilon,1+2\epsilon}$ to itself.

We know that modulation spaces $M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)(0 \le \epsilon \le \infty)$ with $s > \frac{2n\epsilon}{1+2\epsilon}$ (with $s = 0$ when $\epsilon = 0$ ) are multiplication algebras, where $\epsilon = 0$ (Proposition 3.3), hence nonlinear operation $f_i \mapsto F(f_i)$ maps these spaces to themselves when $F(z)$ is fully entire. Then it is natural to expect the same conclusion for non-analytic $F$ as is the case for $L^{1+\epsilon}$-Sobolev spaces and Besov spaces. Unfortunately, it is not obvious because the argument of paradifferential operators does not work in this case because pseudodifferential operators of class $S_{1,\delta}^0$ with $\delta > 0$ have exotic mapping property and are not $M_s^{1+\epsilon,1+2\epsilon}$-bounded (see [15]). Furthermore, if $F(z)$ is not necessarily analytic, a negative answer for $M_0^{1+\epsilon,1}$ is known. In fact, [2] established that the nonlinear operation $f_i \mapsto F(\mathrm{Re}f_i, \mathrm{Im}f_i)$ is a mapping on $M_0^{1,1}(\mathbb{R}^n)$ if and only if $F$ is real analytic and $F(0,0) = 0$, and [8] generalized this result to the case $M_0^{1+\epsilon,1}$ with $0 \le \epsilon < \infty$ although it is restricted to the case when $n = 1$. On the other hand, it is still possible for general $M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)$ with $0 < \epsilon \le \infty$ and $s > \frac{2n\epsilon}{1+2\epsilon}$ when $F$ is not analytic. Our main result states that it is affirmative for $(1+\epsilon)$ in a range away from $\epsilon = 0$(see [19]):

**Theorem 1.1 [19].** Let $0 \le \epsilon < \infty, 0 \le \epsilon < \infty$ (or $\epsilon = \infty$ ) and $s > \frac{2n\epsilon}{1+2\epsilon}$. Assume that $f_i: \mathbb{R}^n \to \mathbb{R}, f_i \in M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n), F \in C^\infty(\mathbb{R})$ and $F(0) = 0$. Then, we have $F(f_i) \in M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)$.

We remark that the condition $s > \frac{2n\epsilon}{1+2\epsilon}$ (with $s = 0$ when $\epsilon = 0$ ) is necessary for modulation spaces $M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)$ to be multiplication algebras ([6]). See also Appendix B. We also remark that Theorem 1.1 is reduced to the following result due to the local equivalence between the modulation spaces $M_s^{1+\epsilon,1+2\epsilon}$ and the Fourier Lebesgue spaces $\mathcal{F}L_s^{1+\epsilon}$ :

**Theorem 1.2 [19].** Let $0 \le \epsilon \le \infty$ and $s > \frac{2n\epsilon}{1+2\epsilon}$. Assume that $f_i: \mathbb{R}^n \to \mathbb{R}, f_i \in \mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$, $F \in C^\infty(\mathbb{R})$ and $F(0) = 0$. Then, we have $F(f_i) \in \mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$.

Finally, see [12] which also discusses the non-analytic nonlinear operations, but on modulation spaces with quasi-analytic regularity.

We introduce basic notations of function spaces and their properties which we used. We list examples of multiplication algebras as a starting point of our argument. We prove the theorem of nonlinear operation on Fourier Lebesgue spaces. We lift it to the case of modulation spaces by using the local equivalence between Fourier Lebesgue spaces and modulation spaces. This equivalence is a well-known fact but the proof is given in Appendix A for the sake of self-containedness. In Appendix B, necessity for Fourier Lebesgue spaces and modulation spaces to be multiplication algebras is considered (see [19]).

## II. Preliminaries

### 2.1. Basic notations

We denote by $\mathbb{R}, \mathbb{Z}$ and $\mathbb{Z}_+$the sets of reals, integers and non-negative integers, respectively. The notation $a \lesssim b$ means $a \le Cb$ with a constant $C > 0$ which may be different in each occasion, and $a \sim b$ means $a \lesssim b$ and $b \lesssim a$. For $0 \le \epsilon \le \infty, \frac{1+\epsilon}{\epsilon}$ is the dual number of $(1+\epsilon)$. We write $\langle x \rangle = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^n$ and $[s] = \max\{n \in \mathbb{Z}: n \le s\}$ for $s \in \mathbb{R}$.

We denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$ by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and its dual, the space of tempered distributions, by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$. The Fourier transform and the inverse Fourier transform of $f_i \in \mathcal{S}(\mathbb{R}^n)$ are given by

$$\mathcal{F}f_i(\xi) = \hat{f}_i(\xi) = \int_{\mathbb{R}^n} \sum_i e^{-i\xi\cdot x} f_i(x)dx \text{ and } \mathcal{F}^{-1}f_i(x) = \check{f}_i(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \sum_i e^{ix\cdot\xi} f_i(\xi)d\xi$$

respectively. For $m \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier multiplier operator is given by

$$m(D)f_i = \mathcal{F}^{-1}[m \cdot \mathcal{F}f_i] = (\mathcal{F}^{-1}m) * f_i$$

and for $s \in \mathbb{R}$ the Bessel potential by $(I - \Delta)^{s/2} f_i = \mathcal{F}^{-1}[\langle \cdot \rangle^s \cdot \mathcal{F}f_i]$ for $f_i \in \mathcal{S}(\mathbb{R}^n)$.

We will use some function spaces. The space of smooth functions with compact support on $\mathbb{R}^n$ is denoted by $C_0^\infty = C_0^\infty(\mathbb{R}^n)$. The Lebesgue space $L^{1+\epsilon} = L^{1+\epsilon}(\mathbb{R}^n)$ is equipped with the norm

$$\|f_i\|_{L^{1+\epsilon}} = \left(\int_{\mathbb{R}^n} \sum_i |f_i(x)|^{1+\epsilon} dx\right)^{1/1+\epsilon}$$

for $0 \le \epsilon < \infty$. If $\epsilon = \infty$, $\|f_i\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f_i(x)|$. Moreover, we denote the $L^{1+\epsilon}$-Sobolev space $H_s^{1+\epsilon}$ by $H_s^{1+\epsilon}(\mathbb{R}^n) = \left\{f_i \in \mathcal{S}'(\mathbb{R}^n): \|f_i\|_{H_s^{1+\epsilon}} = \left\|(I-\Delta)^{s/2} f_i\right\|_{L^{1+\epsilon}} < \infty\right\}$ for $0 < \epsilon < \infty$ and $s \in \mathbb{R}$, and the (weighted) Fourier Lebesgue space $\mathcal{F}L_s^{1+\epsilon}$ by $\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n) = \left\{f_i \in \mathcal{S}'(\mathbb{R}^n): \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}} = \left\|\langle\cdot\rangle^s \hat{f}_i\right\|_{L^{1+\epsilon}} < \infty\right\}$ for $0 \le \epsilon \le \infty$ and $s \in \mathbb{R}$. We remark that $H_s^2 = \mathcal{F}L_s^2$. Moreover, by the Hölder inequality, we have $\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n) \hookrightarrow H_{\tilde{s}}^2(\mathbb{R}^n)$ if $0 < \epsilon \le \infty$ and $n(\frac{1}{2} - \frac{1}{2+\epsilon}) < s - \tilde{s}$. Note that the second condition is equivalent to $\tilde{s} < n/2 + \left(s - \frac{2+\epsilon}{1+\epsilon}\right)$. From this relation, we immediately see the following.

**Proposition 2.1 [19].** Let $0 < \epsilon \le \infty, s > \frac{2+\epsilon}{1+\epsilon}$ and $n/2 < \tilde{s} < n/2 + \left(s - \frac{2+\epsilon}{1+\epsilon}\right)$. Then, we have

$$\mathcal{F}L_s^{2+\epsilon}(\mathbb{R}^n) \hookrightarrow H_{\tilde{s}}^2(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$$

For $0 \le \epsilon \le \infty$ and $s \in \mathbb{R}$, we denote by $\ell_s^{1+\epsilon}$ the set of all complex number sequences $\{a_k\}_{k\in\mathbb{Z}^n}$ such that

$$\|\{a_k\}_{k\in\mathbb{Z}^n}\|_{\ell_s^{1+\epsilon}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s(1+\epsilon)} |a_k|^{1+\epsilon}\right)^{1/1+\epsilon} < \infty$$

if $\epsilon < \infty$, and $\|\{a_k\}_{k\in\mathbb{Z}^n}\|_{\ell_s^\infty} = \sup_{k\in\mathbb{Z}^n} \langle k\rangle^s |a_k| < \infty$ if $\epsilon = \infty$. For the sake of simplicity, we will write $\|a_k\|_{\ell_s^{1+\epsilon}}$ instead of the more correct notation $\|\{a_k\}_{k\in\mathbb{Z}^n}\|_{\ell_s^{1+\epsilon}}$.

We end by mentioning a key fact on the boundedness of Fourier multiplier operators invented by [7, Theorem 9].

**Proposition 2.2 [19].** Let $0 \le \epsilon < \infty$ and $s > n/2 + \epsilon$. Then, if $\frac{2(2+\epsilon)}{4+\epsilon} \le 1 + \epsilon \le \frac{2(2+\epsilon)}{\epsilon}$ when $\epsilon \ne 0$ or $0 \le \epsilon < \infty$ when $\epsilon = 0$, we have

$$\|\sum_i m(D) f_i\|_{L^{1+\epsilon}} \lesssim \|m\|_{H_s^{2+\epsilon}} \sum_i \|f_i\|_{L^{1+\epsilon}}$$

for all $m \in H_s^{2+\epsilon}(\mathbb{R}^n)$ and all $f_i \in L^{1+\epsilon}(\mathbb{R}^n)$.

**Remark 2.1 [19].** In Proposition 2.2, we excluded $\epsilon = \infty$ for the case $\epsilon = 0$. This comes from that $\mathcal{S}$ is not dense in $L^\infty$. In this case, we regard $m(D) f_i$ as the convolution $(\mathcal{F}^{-1} m) * f_i$. Then, this is well-defined since $H_s^2(\mathbb{R}^n) \hookrightarrow \mathcal{F}L_0^1(\mathbb{R}^n)$ for $s > n/2$, and thus Proposition 2.2 holds for $\epsilon = \infty$ and $\epsilon = 0$. In fact, if $s > n/2$,

$$\|\sum_i m(D) f_i\|_{L^\infty} = \left\|\sum_i (\mathcal{F}^{-1} m) * f_i\right\|_{L^\infty} \le \|\mathcal{F}^{-1} m\|_{L^1} \sum_i \|f_i\|_{L^\infty} \lesssim \|m\|_{H_s^2} \sum_i \|f_i\|_{L^\infty}$$

holds for all $m \in H_s^2(\mathbb{R}^n)$ and all $f_i \in L^\infty(\mathbb{R}^n)$.

### 2.2. Modulation spaces

We give the definition of modulation spaces which were introduced by [4] (see also [5]). We fix a functions (called a window functions) $g_i \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and denote the short-time Fourier transform of $f_i \in \mathcal{S}'(\mathbb{R}^n)$ with respect to $g_i$ by

$$V_{g_i} f_i(x, \xi) = \int_{\mathbb{R}^n} \sum_i e^{-i\xi\cdot t} \overline{g_i(t-x)} f_i(t) dt$$

We will sometimes write $V_{g_i}[f_i]$ when the form of $f_i$ is complicated. For $0 \le \epsilon \le \infty$ and $s \in \mathbb{R}$, the modulation space $M_s^{1+\epsilon,1+2\epsilon}$ is defined by

$$M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n) = \left\{f_i \in \mathcal{S}'(\mathbb{R}^n): \|\sum_i f_i\|_{M_s^{1+\epsilon,1+2\epsilon}} = \|\sum_i \|\langle\xi\rangle^s V_{g_i} f_i(x,\xi)\|_{L^{1+\epsilon}(\mathbb{R}_x^n)}\|_{L^{1+2\epsilon}(\mathbb{R}_\xi^n)} < \infty\right\}$$

We note that the definition of modulation spaces is independent of the choice of window functions. $M_s^{1+\epsilon,1+2\epsilon}$ are Banach spaces and $\mathcal{S} \subset M_s^{1+\epsilon,1+2\epsilon} \subset \mathcal{S}'$. In particular, $\mathcal{S}$ is dense in $M_s^{1+\epsilon,1+2\epsilon}$ if $0 \le \epsilon < \infty$. For $0 \le \epsilon < \infty$, the dual space of $M_s^{1+\epsilon,1+2\epsilon}$ can be seen as $\left(M_s^{1+\epsilon,1+2\epsilon}\right)' = M_{-s}^{\frac{1+\epsilon}{\epsilon},\frac{1+2\epsilon}{2\epsilon}}$. Moreover, we have the following complex interpolation theorem. If $0 < \theta < 1, s = (1-\theta)s_1 + \theta s_2, 1/1+\epsilon = (1-\theta)/1+\epsilon + \theta/1+2\epsilon$ and $1/1+2\epsilon = (1-\theta)/1+3\epsilon + \theta/1+4\epsilon$, we have $\left(M_{s_1}^{1+\epsilon,1+3\epsilon}, M_{s_2}^{1+2\epsilon,1+4\epsilon}\right)_\theta = M_s^{1+\epsilon,1+2\epsilon}$. As a further elementary property, we note the following embedding proved by [4, Proposition 6.5].

**Proposition 2.3 [19].** Let $0 \le \epsilon, 1+2\epsilon, 1+3\epsilon, 1+4\epsilon \le \infty$ and $s_1, s_2 \in \mathbb{R}$. Then, we have $M_{s_1}^{1+\epsilon,1+3\epsilon} \hookrightarrow M_{s_2}^{1+2\epsilon,1+4\epsilon}$ for $1+\epsilon \le 1+2\epsilon, 1+3\epsilon \le 1+4\epsilon$ and $s_1 \ge s_2$.

### 2.3. Besov spaces

We here give the definition of Besov spaces (see also [18, Section 2.3]). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy that $\varphi = 1$ on $\{\xi: |\xi| \le 1/2\}$ and $\operatorname{supp}\varphi \subset \{\xi: |\xi| \le 1\}$. We put $\psi = \varphi(\cdot/2) - \varphi(\cdot)$, and then see that $\operatorname{supp}\psi \subset$

$\{\xi: 1/2 \le |\xi| \le 2\}$. Moreover, we set $\varphi_j = \varphi(\cdot/2^j)$ and $\psi_j = \psi(\cdot/2^j)$ for $j \in \mathbb{Z}_+$, and denote the Fourier multiplier operators with respect to them by

$$S_j f_i = \varphi_j(D) f_i \text{ and } \Delta_j f_i = \psi_j(D) f_i$$

We remark that

$$\varphi + \sum_{j=0}^{\infty} \psi_j = 1 \text{ and } S_0 f_i + \sum_{j=0}^{\infty} \sum_i \Delta_j f_i = f_i$$

and also that $\Delta_j f_i = S_{j+1} f_i - S_j f_i$. By using these notations, for $0 \le \epsilon \le \infty$ and $s \in \mathbb{R}$, the Besov space $B_s^{1+\epsilon,1+2\epsilon}$ is defined by

$$B_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n) = \left\{ f_i \in \mathcal{S}'(\mathbb{R}^n): \|f_i\|_{B_s^{1+\epsilon,1+2\epsilon}} = \|S_0 f_i\|_{L^{1+\epsilon}} + \left( \sum_{j=0}^{\infty} \sum_i 2^{js(1+2\epsilon)} \|\Delta_j f_i\|_{L^{1+\epsilon}}^{1+2\epsilon} \right)^{1/1+2\epsilon} < \infty \right\}$$

Note that the norm of the Besov space is read with the usual modification for $\epsilon = \infty$. Besov spaces also have basic properties like modulation spaces, namely, completeness, density, duality and interpolation. However, we omit mentioning the details and see [18, Section 2.3].

## III. Multiplication algebras

We collect some properties called multiplication algebras. A function space $X$ is said to be a multiplication algebra if for all $f_i, g_i \in X$ the product $f_i \cdot g_i$ exists and belongs to $X$, and if the inequality $\|f_i \cdot g_i\|_X \lesssim \|f_i\|_X \cdot \|g_i\|_X$ holds for all $f_i, g_i \in X$. See [18, Section 2.8]. The following results on $L^{1+\epsilon}$-Sobolev and Besov spaces are well-known (see, e.g., [14, Chapter II, Theorem 2.1] and [18, Theorem 2.8.3]).

**Proposition 3.1 [19].** Let $0 < \epsilon < \infty$ and $s > n/1 + \epsilon$. Then, we have

$$\|\sum_i f_i \cdot g_i\|_{H_s^{1+\epsilon}} \lesssim \sum_i \|f_i\|_{H_s^{1+\epsilon}} \cdot \|g_i\|_{H_s^{1+\epsilon}}$$

for all $f_i, g_i \in H_s^{1+\epsilon}(\mathbb{R}^n)$.

**Proposition 3.2 [19].** Let $0 \le \epsilon \le \infty$ and $s > n/1 + \epsilon$. Then, we have

$$\|\sum_i f_i \cdot g_i\|_{B_s^{1+\epsilon,1+2\epsilon}} \lesssim \sum_i \|f_i\|_{B_s^{1+\epsilon,1+2\epsilon}} \cdot \|g_i\|_{B_s^{1+\epsilon,1+2\epsilon}}$$

for all $f_i, g_i \in B_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)$.

Some of modulation spaces are also multiplication algebras (see, e.g., [4, Remark 6.4 and Proposition 6.9] and [16, Proposition 3.2]).

**Proposition 3.3 [19].** Let $0 \le \epsilon \le \infty$ and $s > \frac{2n\epsilon}{1+2\epsilon}$. Then, we have

$$\|\sum_i f_i \cdot g_i\|_{M_s^{1+\epsilon,1+2\epsilon}} \lesssim \sum_i \|f_i\|_{M_s^{1+\epsilon,1+2\epsilon}} \cdot \|g_i\|_{M_s^{1+\epsilon,1+2\epsilon}}$$

for all $f_i, g_i \in M_s^{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)$, and

$$\|\sum_i f_i \cdot g_i\|_{M_0^{1+\epsilon,1}} \lesssim \sum_i \|f_i\|_{M_0^{1+\epsilon,1}} \cdot \|g_i\|_{M_0^{1+\epsilon,1}}$$

for all $f_i, g_i \in M_0^{1+\epsilon,1}(\mathbb{R}^n)$.

Finally, we give the following counterpart for Fourier Lebesgue spaces.

**Proposition 3.4 (see [19]).** Let $0 \le \epsilon \le \infty$ and $s > \frac{n\epsilon}{1+\epsilon}$. Then, we have

$$\|\sum_i f_i \cdot g_i\|_{\mathcal{F}L_s^{1+\epsilon}} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}} \cdot \|g_i\|_{\mathcal{F}L_s^{1+\epsilon}}$$

for all $f_i, g_i \in \mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$, and

$$\|\sum_i f_i \cdot g_i\|_{\mathcal{F}L_0^1} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_0^1} \cdot \|g_i\|_{\mathcal{F}L_0^1}$$

for all $f_i, g_i \in \mathcal{F}L_0^1(\mathbb{R}^n)$.

**Proof.** From the inequality $\langle\xi\rangle^s \lesssim \langle\xi - \eta\rangle^s + \langle\eta\rangle^s$ for any $\xi, \eta \in \mathbb{R}^n$ and $s \ge 0$, we have

$$\|f_i \cdot g_i\|_{\mathcal{F}L_s^{1+\epsilon}} \sim \left\| \langle\xi\rangle^s \int_{\mathbb{R}^n} \sum_i \hat{f}_i(\xi-\eta) \cdot \hat{g}_i(\eta) d\eta \right\|_{L^{1+\epsilon}\left(\mathbb{R}_\xi^n\right)}$$

$$\lesssim \sum_i \left\| \int_{\mathbb{R}^n} \langle\xi-\eta\rangle^s |\hat{f}_i(\xi-\eta)| \cdot |\hat{g}_i(\eta)| d\eta \right\|_{L^{1+\epsilon}\left(\mathbb{R}_\xi^n\right)}$$

$$+ \sum_i \left\| \int_{\mathbb{R}^n} |\hat{f}_i(\xi-\eta)| \cdot \langle\eta\rangle^s |\hat{g}_i(\eta)| d\eta \right\|_{L^{1+\epsilon}\left(\mathbb{R}_\xi^n\right)}$$

Then, we have by the Young and Hölder inequalities

$$\|\sum_i f_i \cdot g_i\|_{\mathcal{F}L_s^{1+\epsilon}} \lesssim \sum_i \|\langle\cdot\rangle^s \hat{f}_i\|_{L^{1+\epsilon}} \cdot \|\hat{g}_i\|_{L^1} + \sum_i \|\hat{f}_i\|_{L^1} \cdot \|\langle\cdot\rangle^s \hat{g}_i\|_{L^{1+\epsilon}} \cdot$$

$$\leq \sum_i \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}} \cdot \|\langle\cdot\rangle^{-s}\|_{L^{\frac{1+\epsilon}{\epsilon}}} \|g_i\|_{\mathcal{F}L_s^{1+\epsilon}} + \sum_i \|\langle\cdot\rangle^{-s}\|_{L^{\frac{1+\epsilon}{\epsilon}}} \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}} \cdot \|g_i\|_{\mathcal{F}L_s^{1+\epsilon}},$$

which yields from the assumption $s > \frac{n\epsilon}{1+\epsilon}$ that $\| \sum_i f_i \cdot g_i\|_{\mathcal{F}L_s^{1+\epsilon}} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}} \cdot \|g_i\|_{\mathcal{F}L_s^{1+\epsilon}}$. Here, we remark that, in the case $\epsilon = 0, \|\langle\cdot\rangle^{-s}\|_{L^{\frac{1+\epsilon}{\epsilon}}}$ is finite even if $s = 0$, which gives the conclusion for $\epsilon = 0$ and $s = 0$.

## IV. Proof of Theorem 1.2

We begin with an observation which will be used in the proof of Theorem 1.2. Put

$$G(t) = F(t) - \sum_{k=1}^{N} F^{(k)}(0) \frac{t^k}{k!} \tag{4.1}$$

for any $N \in \mathbb{N}$, where $F \in C^\infty(\mathbb{R})$ and $F(0) = 0$. Then, we see that $G(0) = G^{(1)}(0) = \cdots = G^{(N)}(0) = 0$, and have

$$F(f_i) = G(f_i) + \sum_{k=1}^{N} \sum_i F^{(k)}(0) \frac{f_i^k}{k!} \tag{4.2}$$

In order to obtain Theorem 1.2, we will prove that the right hand side of (4.2) belongs to $\mathcal{F}L_s^{1+\epsilon}$. However, it is trivial that the second term belongs to $\mathcal{F}L_s^{1+\epsilon}$, since $\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$ with $s > \frac{n\epsilon}{1+\epsilon}$ is a multiplication algebra (see Proposition 3.4). Hence, Theorem 1.2 is reduced to the following statement.

**Proposition 4.1 [19].** Let $0 \leq \epsilon \leq \infty$ and $s > \frac{n(4+3\epsilon)}{1+3\epsilon}$. Assume that $f_i: \mathbb{R}^n \to \mathbb{R}, f_i \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}(\mathbb{R}^n), G \in C^\infty(\mathbb{R})$ and $G(0) = G^{(1)}(0) = \cdots = G^{([s]+2)}(0) = 0$. Then, we have $G(f_i) \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}(\mathbb{R}^n)$.

Before starting the proof of Proposition 4.1, we transform $G(f_i)$ to a more manageable alternative expression, which was provided by [9, Section 2]. We first remark that $f_i \in \mathcal{F}L_s^{2-\epsilon}(\mathbb{R}^n) \hookrightarrow H_s^{\frac{2-\epsilon}{1-\epsilon}}(\mathbb{R}^n)$ holds for $\epsilon \leq 0$, and that $f_i \in \mathcal{F}L_s^{2+\epsilon}(\mathbb{R}^n) \hookrightarrow H_{\tilde{s}}^2(\mathbb{R}^n)$ for $\epsilon > 0$ and $\frac{n}{2} < \tilde{s} < \frac{n}{2} + \left(\frac{s-n}{2+\epsilon}\right)$ (see Proposition 2.1). They imply that $f_i$ belongs to $B_0^{\infty,1}(\mathbb{R}^n)$, hence to $L^\infty(\mathbb{R}^n)$, and so $f_i$ is a bounded uniformly continuous function. Then $S_j f_i$ converges uniformly to $f_i$ as $j \to \infty$, and $G(f_i) = G\left(\lim_{j\to\infty} S_j f_i\right) = \lim_{j\to\infty} G(S_j f_i)$. By the mean value theorem and the fact $S_{j+1} f_i = S_j f_i + \Delta_j f_i$, we have

$$G(f_i) = G(S_0 f_i) + \sum_{j=0}^{\infty} \sum_i \left[G(S_{j+1} f_i) - G(S_j f_i)\right]$$

$$= G(S_0 f_i) + \sum_{j=0}^{\infty} \int_0^1 \sum_i G^{(1)}(S_j f_i + t\Delta_j f_i) dt \cdot \Delta_j f_i = G(S_0 f_i) + \sum_{j=0}^{\infty} \sum_i m_j \cdot \Delta_j f_i,$$

where we set

$$m_j = \int_0^1 \sum_i G^{(1)}(S_j f_i + t\Delta_j f_i) dt \tag{4.3}$$

Moreover, we decompose $m_j$ into the low and high frequency parts. Recall from Section 2.3 that $\varphi(\xi) + \sum_{m=0}^{\infty} \psi(2^{-m}\xi) = 1$ for any $\xi \in \mathbb{R}^n$. Then, it follows that

$$\varphi\left(\frac{\xi}{C \cdot 2^j}\right) + \sum_{m=0}^{\infty} \psi\left(\frac{\xi}{C \cdot 2^{j+m}}\right) = 1$$

for any $\xi \in \mathbb{R}^n$, where $C$ is a sufficiently large constant. Using this decomposition, we have

$$m_j = \varphi\left(\frac{D}{C\cdot 2^j}\right)m_j + \sum_{m=0}^{\infty} \psi\left(\frac{D}{C\cdot 2^{j+m}}\right)m_j = q_j + \sum_{m=0}^{\infty} p_{j,m},$$

where we set

$$q_j = \varphi\left(\frac{D}{C\cdot 2^j}\right)m_j \text{ and } p_{j,m} = \psi\left(\frac{D}{C\cdot 2^{j+m}}\right)m_j \tag{4.4}$$

Therefore, $G(f_i)$ is expressed in the following form:

$$G(f_i) = G(S_0 f_i) + \sum_{j=0}^{\infty}\sum_i q_j\cdot\Delta_j f_i + \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_i p_{j,m}\cdot\Delta_j f_i \tag{4.5}$$

From now on, we give estimates for each term of the expression (4.5) without specifying constants explicitly. (We however remark that these implicit constants may depend on $\|f_i\|_{\mathcal{F}L_{\tilde{s}}^{2+\epsilon}}$.)

We start by stating two lemmas. The first one is for $q_j$ in (4.4).

**Lemma 4.1 (see [19]).** Let $0<\epsilon\le\infty, s>\frac{n\epsilon}{1+\epsilon}$ and $n/2<\tilde{s}<n/2+\left(s-\frac{n\epsilon}{1+\epsilon}\right)$. Suppose that $f_i\in\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$ and all the assumptions of $G$ are the same as in Proposition 4.1. Then, we have

$$\|q_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} \lesssim 1 \text{ if } 0<\epsilon\le 1$$

$$\|q_j\|_{H_{\tilde{s}}^2} \lesssim 1 \text{ if } 0<\epsilon\le\infty$$

for any $j\in\mathbb{Z}_+$. Here, the implicit constants are independent of $j\in\mathbb{Z}_+$.

**Proof.** We first consider the estimate with $0<\epsilon\le 1$. Set $(f_i)_{j,t} = S_j f_i + t\Delta_j f_i$. Recalling the definition of $m_j$ from (4.3), we have

$$\|q_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} \lesssim \|m_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} \le \int_0^1 \sum_i \left\|G^{(1)}\big((f_i)_{j,t}\big)\right\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} dt$$

Observe that

$$\begin{aligned}\|(f_i)_{j,t}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} &\lesssim \left(\|\mathcal{F}^{-1}\varphi_j\|_{L^1} + t\|\mathcal{F}^{-1}\psi_j\|_{L^1}\right)\sum_i \|f_i\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\\ &\lesssim (\|\mathcal{F}^{-1}\varphi\|_{L^1} + t\|\mathcal{F}^{-1}\psi\|_{L^1})\sum_i \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}},\end{aligned}$$

which means that $(f_i)_{j,t}\in H_s^{\frac{1+\epsilon}{\epsilon}}$ for any $j\in\mathbb{Z}_+$and any $t\in[0,1]$. Then, using Theorem A and Remark 1.1 together with the assumptions $G\in C^\infty(\mathbb{R})$ and $G^{(1)}(0)=0$, we have

$$\begin{aligned}\left\|\sum_i G^{(1)}\big((f_i)_{j,t}\big)\right\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} &\lesssim \|G^{(2)}\|_{C^{[s]+1}(\Omega)}\sum_i \left(1+\|(f_i)_{j,t}\|_{L^\infty}^{[s]+1}\right)\|(f_i)_{j,t}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\\ &\lesssim \|G\|_{C^{[s]+3}(\Omega)}\sum_i \left(1+\|f_i\|_{L^\infty}^{[s]+1}\right)\|f_i\|_{\mathcal{F}L_s^{1+\epsilon}}\end{aligned}$$

where $\Omega=\{t: |t|\lesssim \|f_i\|_{L^\infty}\}$. Note that the last quantity is finite since $f_i\in\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$ for $s>\frac{n\epsilon}{1+\epsilon}$ and the smooth function $G\in C^\infty(\mathbb{R})$ is measured by $C^{[s]+3}$ on the closed and bounded domain $\Omega$. Therefore, we have $\|q_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\lesssim 1$ for $0<\epsilon\le 1$.

We next consider the estimate with $0<\epsilon\le\infty$. This is, however, immediately given by the same argument as above. In fact, since we already know from Proposition 2.1 that $f_i\in\mathcal{F}L_s^{2+\epsilon}(\mathbb{R}^n)\hookrightarrow H_{\tilde{s}}^2(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$, we have by Theorem A and Remark 1.1

$$\begin{aligned}\left\|\sum_i G^{(1)}\big((f_i)_{j,t}\big)\right\|_{H_{\tilde{s}}^2} &\lesssim \|G^{(2)}\|_{C^{[\tilde{s}]+1}(\Omega)}\sum_i \left(1+\|(f_i)_{j,t}\|_{L^\infty}^{[\tilde{s}+1)}\right)\|(f_i)_{j,t}\|_{H_{\tilde{s}}^2}\\ &\lesssim \|G\|_{C^{[\tilde{s}]+3}(\Omega)}\sum_i \left(1+\|f_i\|_{L^\infty}^{[\tilde{s}]+1}\right)\|f_i\|_{H_{\tilde{s}}^2}\\ &\lesssim \|G\|_{C^{[\tilde{s}]+3}(\Omega)}\sum_i \left(1+\|f_i\|_{L^\infty}^{[\tilde{s}]+1}\right)\|f_i\|_{\mathcal{F}L_{\tilde{s}}^{2+\epsilon}}.\end{aligned}$$

Note that the last quantity is finite. Hence, we obtain $\|q_j\|_{H_{\tilde{s}}^2}\lesssim 1$ for $0<\epsilon\le\infty$.

The second one is concerned with $p_{j,m}$ in (4.4).

**Lemma 4.2 [19].** Let $0<\epsilon\le\infty, s>\frac{n\epsilon}{1+\epsilon}$ and $n/2<\tilde{s}<n/2+\left(s-\frac{n\epsilon}{1+\epsilon}\right)$. Suppose that $f_i\in\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$ and all the assumptions of $G$ are the same as in Proposition 4.1. Then, we have

$$\|p_{j,m}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\lesssim 2^{-m([s]+1)} \text{ if } 0<\epsilon\le 1$$

$$\|p_{j,m}\|_{H_{\tilde{s}}^2}\lesssim 2^{-m([s]+1)} \text{ if } 0<\epsilon\le\infty$$

for any $j,m\in\mathbb{Z}_+$. Here, the implicit constants are independent of $j,m\in\mathbb{Z}_+$.

To prove Lemma 4.2, we prepare the following:

**Lemma 4.3 (see [19]).** Let $0<\epsilon\le\infty, s>\frac{n\epsilon}{1+\epsilon}$ and $n/2<\tilde{s}<n/2+\left(s-\frac{n\epsilon}{1+\epsilon}\right)$, and let $\alpha\in\mathbb{Z}_+^n$ satisfy that $|\alpha|=[s]+1$. Suppose that $f_i\in\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$ and all the assumptions of $G$ are the same as in Proposition 4.1. Then, we have

$$\|\partial^\alpha m_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\lesssim 2^{j([s]+1)} \text{ if } 0<\epsilon\le 1$$

$$\|\partial^\alpha m_j\|_{H_{\tilde{s}}^2}\lesssim 2^{j([s]+1)} \text{ if } 0<\epsilon\le\infty$$

for any $j\in\mathbb{Z}_+$. Here, the implicit constants are independent of $j\in\mathbb{Z}_+$.

**Proof.** We first consider the case $0<\epsilon\le 1$. Set $(f_i)_{j,t}=S_jf_i+t\Delta_jf_i$. Then we have by Proposition 3.1

$$\|\partial^\alpha m_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\le\int_0^1\sum_i\|\partial^\alpha[G^{(1)}((f_i)_{j,t})]\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}dt$$

$$\lesssim\sum_{\mu=1}^{|\alpha|}\sum_{\alpha_1+\cdots+\alpha_\mu=\alpha}\int_0^1\sum_i\|G^{(\mu+1)}((f_i)_{j,t})\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\cdot\|\partial^{\alpha_1}(f_i)_{j,t}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\cdots\|\partial^{\alpha_\mu}(f_i)_{j,t}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}dt$$

where $|\alpha|=[s]+1$. Observe that for $\beta\in\mathbb{Z}_+^n$

$$\left\|\sum_i\partial^\beta(f_i)_{j,t}\right\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\lesssim\left(\|\mathcal{F}^{-1}[\xi^\beta\cdot\varphi_j]\|_{L^1}+t\|\mathcal{F}^{-1}[\xi^\beta\cdot\psi_j]\|_{L^1}\right)\sum_i\|f_i\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\lesssim 2^{j|\beta|}\sum_i\|f_i\|_{\mathcal{F}L_s^{1+\epsilon}}$$

which also means that $(f_i)_{j,t}\in H_s^{\frac{1+\epsilon}{\epsilon}}$ for any $j\in\mathbb{Z}_+$and any $t\in[0,1]$. Therefore, by using Theorem A and Remark 1.1 together with the assumptions $G\in C^\infty(\mathbb{R})$ and $G^{(2)}(0)=\cdots=G^{([s]+2)}(0)=0$, we have for $\mu=1,\cdots,[s]+1$

$$\left\|\sum_i G^{(\mu+1)}((f_i)_{j,t})\right\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\lesssim\|G^{(\mu+2)}\|_{C^{[s]+1}(\Omega)}\sum_i\left(1+\|(f_i)_{j,t}\|_{L^\infty}^{[s]+1}\right)\|(f_i)_{j,t}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}$$

$$\lesssim\|G\|_{C^{\mu+[s]+3}(\Omega)}\sum_i\left(1+\|f_i\|_{L^\infty}^{[s]+1}\right)\|f_i\|_{\mathcal{F}L_s^{1+\epsilon}}$$

where $\Omega=\{t:|t|\lesssim\|f_i\|_{L^\infty}\}$. Note that the last quantity makes sense surely since $f_i\in\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$ for $s>\frac{n\epsilon}{1+\epsilon}$ and $G\in C^\infty(\mathbb{R})$ is considered on the closed and bounded domain $\Omega$. Hence, we obtain

$$\|\partial^\alpha m_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}\lesssim\sum_{\mu=1}^{[s]+1}\sum_{\alpha_1+\cdots+\alpha_\mu=\alpha}\sum_i\left(2^{j|\alpha_1|}\|f_i\|_{\mathcal{F}L_s^{1+\epsilon}}\right)\cdots\left(2^{j|\alpha_\mu|}\|f_i\|_{\mathcal{F}L_s^{1+\epsilon}}\right)\lesssim 2^{j([s]+1)}$$

which completes the proof for $0<\epsilon\le 1$.

We next consider the case $0<\epsilon\le\infty$. Repeating the same lines as above, since we already know from Proposition 2.1 that $f_i\in\mathcal{F}L_s^{2+\epsilon}(\mathbb{R}^n)\hookrightarrow H_{\tilde{s}}^2(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$, we have for $\beta\in\mathbb{Z}_+^n$

$$\left\|\sum_i\partial^\beta(f_i)_{j,t}\right\|_{H_{\tilde{s}}^2}\lesssim\left(\|\mathcal{F}^{-1}[\xi^\beta\cdot\varphi_j]\|_{L^1}+t\|\mathcal{F}^{-1}[\xi^\beta\cdot\psi_j]\|_{L^1}\right)\sum_i\|f_i\|_{H_{\tilde{s}}^2}\lesssim 2^{j|\beta|}\sum_i\|f_i\|_{\mathcal{F}L_s^{2+\epsilon}}$$

and by Theorem A and Remark 1.1 for $\mu=1,\cdots,[s]+1$

$$\left\|\sum_i G^{(\mu+1)}((f_i)_{j,t})\right\|_{H_{\tilde{s}}^2}\lesssim\|G^{(\mu+2)}\|_{C^{[\tilde{s}]+1}(\Omega)}\sum_i\left(1+\|(f_i)_{j,t}\|_{L^\infty}^{[\tilde{s}]+1}\right)\|(f_i)_{j,t}\|_{H_{\tilde{s}}^2}$$

$$\lesssim\|G\|_{C^{\mu+[\tilde{s}]+3}(\Omega)}\sum_i\left(1+\|f_i\|_{L^\infty}^{[\tilde{s}]+1}\right)\|f_i\|_{\mathcal{F}L_s^{2+\epsilon}}.$$

Hence, we obtain $\|\partial^\alpha m_j\|_{H_{\tilde{s}}^2}\lesssim 2^{j([s]+1)}$ for $0<\epsilon\le\infty$.

**Proof of Lemma 4.2.** By the moment condition of $\psi$ and a Taylor expansion, we have

$$p_{j,m}(x) = C^n \cdot 2^{(j+m)n} \int_{\mathbb{R}^n} \check{\psi}(C \cdot 2^{j+m} y) \cdot m_j(x-y) dy$$

$$= C^n \cdot 2^{(j+m)n} \int_{\mathbb{R}^n} \check{\psi}(C \cdot 2^{j+m} y) \left\{ m_j(x-y) - \sum_{|\alpha|<M} \frac{(-y)^\alpha}{\alpha!} (\partial^\alpha m_j)(x) \right\} dy$$

$$= C^n \cdot 2^{(j+m)n} \int_{\mathbb{R}^n} \check{\psi}(C \cdot 2^{j+m} y)$$

$$\cdot \left\{ M \sum_{|\alpha|=M} \frac{(-y)^\alpha}{\alpha!} \int_0^1 (1-t)^{M-1} \cdot (\partial^\alpha m_j)(x-ty) dt \right\} dy$$

where $M = [s] + 1$. Taking the $\left(\frac{1+\epsilon}{\epsilon}\right)$-norm of both sides, we have

$$\|p_{j,m}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}$$

$$\lesssim 2^{(j+m)n} \int_{\mathbb{R}^n} |\check{\psi}(C \cdot 2^{j+m} y)| \cdot |y|^{[s]+1} \left\{ \sum_{|\alpha|=[s]+1} \int_0^1 \|(\partial^\alpha m_j)(x-ty)\|_{H_s^{\frac{1+\epsilon}{\epsilon}}(\mathbb{R}_x^n)} dt \right\} dy$$

$$\sim 2^{-(j+m)([s]+1)} \left( \int_{\mathbb{R}^n} |\check{\psi}(y)| \cdot |y|^{[s]+1} dy \right) \sum_{|\alpha|=[s]+1} \|\partial^\alpha m_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}$$

$$\sim 2^{-(j+m)([s]+1)} \sum_{|\alpha|=[s]+1} \|\partial^\alpha m_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}}$$

Since we have $\|\partial^\alpha m_j\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} \lesssim 2^{j([s]+1)}$ for $0 < \epsilon \le 1$ by Lemma 4.3, we obtain $\|p_{j,m}\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} \lesssim 2^{-m([s]+1)}$. By the same manner as above, we also have $\|p_{j,m}\|_{H_s^2} \lesssim 2^{-m([s]+1)}$ for $0 < \epsilon \le \infty$.

We prove Proposition 4.1.

**Proof of Proposition 4.1.** We recall the alternative form of $G(f_i)$ given in (4.5), that is,

$$G(f_i) = G(S_0 f_i) + \sum_{j=0}^{\infty} \sum_i q_j \cdot \Delta_j f_i + \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_i p_{j,m} \cdot \Delta_j f_i$$

and prove that the function $G(f_i)$ belongs to $\mathcal{F}L_s^{1+\epsilon}$, which will be archived by three steps. In the first and second steps, we consider the second and third summations, and then consider $G(S_0 f_i)$ in the last step.

**Step 1:** We first consider the case $\epsilon < \infty$. Taking the $\mathcal{F}L_s^{1+\epsilon}$-norm of the second summation in (4.5), we have

$$\left\| \sum_i \langle \cdot \rangle^s \sum_{j=0}^{\infty} \mathcal{F}[q_j \cdot \Delta_j f_i] \right\|_{L^{1+\epsilon}} = \left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} \sum_i \langle \xi \rangle^{s(1+\epsilon)} \left| \sum_{j=0}^{\infty} \mathcal{F}[q_j \cdot \Delta_j f_i](\xi) \right|^{1+\epsilon} d\xi \right)^{1/1+\epsilon} \tag{4.6}$$

where $\Omega_\ell = \{\xi : 2^\ell < |\xi| \le 2^{\ell+1}\}$ if $\ell \neq 0$ and $\Omega_0 = \{\xi : |\xi| \le 2\}$. We remark that

$$\mathrm{supp}\mathcal{F}[q_j \cdot \Delta_j f_i] \subset \{\xi : |\xi| \le C \cdot 2^{j+1}\}$$

since $\mathcal{F}[q_j \cdot \Delta_j f_i] = \left[\varphi\left(\frac{\cdot}{C \cdot 2^j}\right) \widehat{m}_j\right] * [\psi_j \hat{f}_i]$. This means that on the domain $\Omega_\ell, \mathcal{F}[q_j \cdot \Delta_j f_i]$ always vanishes unless $j \ge \ell - N (j \ge 0$ if $\ell = 0, \cdots, N)$, where $N$ is a constant which depends only on $C \gg 1$ (roughly, $2^N \sim C$ ). Hence, the right hand side of (4.6) is equal to

$$\left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} \sum_i \langle \xi \rangle^{s(1+\epsilon)} \left| \sum_{j=\ell-N}^{\infty} \mathcal{F}[q_j \cdot \Delta_j f_i](\xi) \right|^{1+\epsilon} d\xi \right)^{1/1+\epsilon} \tag{4.7}$$

where the inner summation should be read as $\sum_{j=0}^{\infty}$ if $\ell = 0, \cdots, N$. Then, using the Hölder inequality to the inner summation, we have

$$(4.7) \lesssim \left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} 2^{\ell s(1+\epsilon)} \left( \sum_{j=\ell-N}^{\infty} \sum_i 2^{js(1+\epsilon)} |\mathcal{F}[q_j \cdot \Delta_j f_i](\xi)|^{1+\epsilon} \right) \cdot \left( \sum_{j=\ell-N}^{\infty} 2^{-js\left(\frac{1+\epsilon}{\epsilon}\right)} \right)^{\frac{1}{\epsilon}} d\xi \right)^{1/1+\epsilon}$$

$$\lesssim \left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} \sum_{j=\ell-N}^{\infty} \sum_i 2^{js(1+\epsilon)} |\mathcal{F}[q_j \cdot \Delta_j f_i](\xi)|^{1+\epsilon} d\xi \right)^{1/1+\epsilon}$$

$$\lesssim \left( \sum_{j=0}^{\infty} 2^{js(1+\epsilon)} \int_{\mathbb{R}^n} \sum_{i} \left| \mathcal{F}[q_j \cdot \Delta_j f_i](\xi) \right|^{1+\epsilon} d\xi \right)^{\frac{1}{1+\epsilon}} \tag{4.8}$$

Here, in the last inequality, we used the fact that $\mathbb{R}^n = \cup_{\ell=0}^{\infty} \Omega_\ell$. Now, we observe that

$$\left\| \sum_i \mathcal{F}[q_j \cdot \Delta_j f_i] \right\|_{L^{1+\epsilon}} = \left\| \sum_i \widetilde{q_j}(D)[\psi_j \cdot \hat{f}_i] \right\|_{L^{1+\epsilon}}$$

where $\widetilde{q_j}(x) = q_j(-x)$. Then, we see that the last quantity of (4.8) is equal to

$$\left( \sum_{j=0}^{\infty} \sum_{i} 2^{js(1+\epsilon)} \left\| \widetilde{q_j}(D)[\psi_j \cdot \hat{f}_i] \right\|_{L^{1+\epsilon}}^{1+\epsilon} \right)^{1/1+\epsilon} \tag{4.9}$$

Apply Proposition 2.2 with $\epsilon = 1$ for $0 \le \epsilon \le \frac{2}{3}$ and with $\epsilon = 0$ for $0 < \epsilon < \infty$ to (4.9). Here, we note that the assumption $0 \le \epsilon \le \frac{2}{3}$ is used to assure the conditions $\frac{2(2+\epsilon)}{(4+\epsilon)} \le \frac{4}{3} + \epsilon \le \frac{2(2+\epsilon)}{(\epsilon)}$ and $\epsilon \ge 0$ in Proposition 2.2. Then, we have

$$(4.9) \lesssim \begin{cases} \left( \sum_{j=0}^{\infty} \sum_{i} 2^{js(4/3+\epsilon)} \left\| q_j \right\|_{H_s^{\frac{4+3\epsilon}{1+3\epsilon}}}^{4/3+\epsilon} \left\| \psi_j \cdot \hat{f}_i \right\|_{L^{\frac{4}{3}+\epsilon}}^{\frac{4}{3}+\epsilon} \right)^{\frac{3}{4+3\epsilon}} & \text{if } 0 \le \epsilon \le 2/3 \\ \left( \sum_{j=0}^{\infty} \sum_{i} 2^{js(2+\epsilon)} \left\| q_j \right\|_{H_{\tilde{s}}^2}^{2+\epsilon} \left\| \psi_j \cdot \hat{f}_i \right\|_{L^{2+\epsilon}}^{2+\epsilon} \right)^{1/2+\epsilon} & \text{if } 0 < \epsilon < \infty \end{cases}$$

where $\tilde{s}$ is the number satisfying that $n/2 < \tilde{s} < n/2 + \left( s - \frac{n(1+\epsilon)}{2+\epsilon} \right)$. Thus, we obtain from Lemma 4.1

$$(4.9) \lesssim \left( \sum_{j=0}^{\infty} \sum_{i} 2^{js(2+\epsilon)} \left\| \psi_j \cdot \hat{f}_i \right\|_{L^{2+\epsilon}}^{2+\epsilon} \right)^{1/2+\epsilon}$$

Since it follows that $\sum_{j=0}^{\infty} |\psi_j|^{2+\epsilon} \lesssim 1$ (if $\epsilon < \infty$ ) and $2^j \sim \langle \xi \rangle$ on the support of $\psi_j$, we realize that

$$\left( \sum_{j=0}^{\infty} \sum_{i} 2^{js(2+\epsilon)} \left\| \psi_j \cdot \hat{f}_i \right\|_{L^{2+\epsilon}}^{2+\epsilon} \right)^{1/2+\epsilon} \sim \left( \sum_{j=0}^{\infty} \sum_{i} \left\| \psi_j \cdot \langle \xi \rangle^s \hat{f}_i \right\|_{L^{2+\epsilon}}^{2+\epsilon} \right)^{1/2+\epsilon} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^{2+\epsilon}}$$

for $0 \le \epsilon < \infty$, which gives the desired result for the case $0 \le \epsilon < \infty$.

We next consider the case $\epsilon = \infty$. However, this case is obtained similarly to the above. In fact, we have

$$\sup_{\xi \in \mathbb{R}^n} \left| \langle \xi \rangle^s \sum_{j=0}^{\infty} \sum_{i} \mathcal{F}[q_j \cdot \Delta_j f_i](\xi) \right| \lesssim \sup_{\ell \in \mathbb{Z}_+} \left( \sup_{\xi \in \Omega_\ell} 2^{\ell s} \sum_{j=\ell-N}^{\infty} \sum_{i} \left| \mathcal{F}[q_j \cdot \Delta_j f_i](\xi) \right| \right), \tag{4.10}$$

since each $\Omega_\ell$ is disjoint. Recalling from Lemma 4.1 that $\|q_j\|_{H_{\tilde{s}}^2} \lesssim 1$ holds independently of $j \in \mathbb{Z}_+$, we have by Remark 2.1

$$\sum_i \left| \mathcal{F}[q_j \cdot \Delta_j f_i](\xi) \right| \lesssim \sum_i \|q_j\|_{H_s^{\frac{2}{s}}} \|\psi_j \langle \cdot \rangle^{-s} \cdot \langle \cdot \rangle^s \hat{f}_i\|_{L^\infty} \lesssim 2^{-js} \sum_i \|f_i\|_{\mathcal{F}L_s^\infty}$$

Hence, we obtain

$$2^{\ell s} \sum_{j=\ell-N}^{\infty} \sum_{i} \left| \mathcal{F}[q_j \cdot \Delta_j f_i](\xi) \right| \lesssim 2^{\ell s} \sum_{j=\ell-N}^{\infty} \sum_{i} 2^{-js} \|f_i\|_{\mathcal{F}L_s^\infty} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^\infty}$$

for any $\ell \in \mathbb{Z}_+$, where all the implicit constants above are independent of $\ell \in \mathbb{Z}_+$. Substituting this estimate into (4.10), we have the desired result for the case $\epsilon = \infty$.

Combining all the calculations above, we obtain for $0 \le \epsilon \le \infty$

$$\left\| \sum_{j=0}^{\infty} \sum_{i} q_j \cdot \Delta_j f_i \right\|_{\mathcal{F}L_s^{2+\epsilon}} < \infty \tag{4.11}$$

**Step 2:** We first consider the case $\epsilon < \infty$. As in Step 1, we take the $\mathcal{F}L_s^{1+\epsilon}$-norm of the third summation in (4.5). Then, using the dyadic decomposition, we have

$$\left\|\langle\xi\rangle^s \sum_{j=0}^{\infty}\sum_i \mathcal{F}\left[\sum_{m=0}^{\infty} p_{j,m}\cdot\Delta_j f_i\right]\right\|_{L^{1+\epsilon}} \lesssim \sum_{m=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\int_{\Omega_\ell} 2^{\ell s(1+\epsilon)}\left|\sum_{j=0}^{\infty}\sum_i \mathcal{F}\left[p_{j,m}\cdot\Delta_j f_i\right](\xi)\right|^{1+\epsilon} d\xi\right)^{1/1+\epsilon}, \quad (4.12)$$

where $\Omega_\ell=\{\xi: 2^\ell<|\xi|\le 2^{\ell+1}\}$ if $\ell\neq 0$ and $\Omega_0=\{\xi:|\xi|\le 2\}$. Considering the support of $\mathcal{F}[p_{j,m}\cdot\Delta_j f_i]$, since we have

$$\text{supp}\mathcal{F}p_{j,m}\subset\{\xi: C\cdot 2^{j+m-1}\le|\xi|\le C\cdot 2^{j+m+1}\} \text{ and}$$
$$\text{supp}\mathcal{F}[\Delta_j f_i]\subset\{\xi: 2^{j-1}\le|\xi|\le 2^{j+1}\}$$

we see that

$$\text{supp}\mathcal{F}[p_{j,m}\cdot\Delta_j f_i]\subset\{\xi: C\cdot 2^{j+m-2}\le|\xi|\le C\cdot 2^{j+m+2}\}$$

This implies that on the domain $\Omega_\ell$, the function $\mathcal{F}[p_{j,m}\cdot\Delta_j f_i]$ always vanishes unless $j,\ell,m\in\mathbb{Z}_+$satisfy that $j+m+N-2\le\ell\le j+m+N+1$, where $N$ is the constant which depends only on $C\gg 1$. Put $\Lambda=\{j\in\mathbb{Z}_+:\ell-m-N-1\le j\le\ell-m-N+2\}$, where this set is read as $\Lambda=\emptyset$ if $\ell-m-N+2<0$. Then, $0\le\#\Lambda\le 4$. Hence, the right hand side of (4.12) is equivalent to

$$\sum_{m=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\int_{\Omega_\ell}\sum_{j\in\Lambda}\sum_i 2^{(j+m)s(1+\epsilon)}\left|\mathcal{F}[p_{j,m}\cdot\Delta_j f_i](\xi)\right|^{1+\epsilon}d\xi\right)^{1/1+\epsilon} \quad (4.13)$$

Then, we have by the Fubini-Tonelli theorem

$$(4.13)\le\sum_{m=0}^{\infty}2^{ms}\left(\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}\sum_i 2^{js(1+\epsilon)}\left|\mathcal{F}[p_{j,m}\cdot\Delta_j f_i](\xi)\right|^{1+\epsilon}d\xi\right)^{1/1+\epsilon}$$
$$=\sum_{m=0}^{\infty}2^{ms}\left(\sum_{j=0}^{\infty}2^{js(1+\epsilon)}\sum_i\left\|\mathcal{F}[p_{j,m}\cdot\Delta_j f_i]\right\|_{L^{1+\epsilon}}^{1+\epsilon}\right)^{1/1+\epsilon} \quad (4.14)$$

Using the identity $\|\mathcal{F}[p_{j,m}\cdot\Delta_j f_i]\|_{L^{1+\epsilon}}=\|\widetilde{p_{j,m}}(D)[\psi_j\cdot\hat{f}_i]\|_{L^{1+\epsilon}}$, where $\widetilde{p_{j,m}}(x)=p_{j,m}(-x)$, we see that the last quantity of (4.14) is equal to

$$\sum_{m=0}^{\infty}2^{ms}\left(\sum_{j=0}^{\infty}\sum_i 2^{js(1+\epsilon)}\left\|\widetilde{p_{j,m}}(D)[\psi_j\cdot\hat{f}_i]\right\|_{L^{1+\epsilon}}^{1+\epsilon}\right)^{1/1+\epsilon} \quad (4.15)$$

As in Step 1, we have

$$(4.9)\lesssim\begin{cases}\displaystyle\sum_{m=0}^{\infty}2^{ms}\left(\sum_{j=0}^{\infty}\sum_i 2^{js(\frac{4}{3}+\epsilon)}\|p_{j,m}\|_{H_s^{\frac{4+3\epsilon}{1+3\epsilon}}}^{\frac{4}{3}+\epsilon}\|\psi_j\cdot\hat{f}_i\|_{L^{\frac{4}{3}+\epsilon}}^{\frac{4}{3}+\epsilon}\right)^{1/2+\epsilon} & \text{if } 0\le\epsilon\le 2/3\\ \displaystyle\sum_{m=0}^{\infty}2^{ms}\left(\sum_{j=0}^{\infty}\sum_i 2^{js(2+\epsilon)}\|p_{j,m}\|_{H_{\tilde{s}}^2}^{2+\epsilon}\|\psi_j\cdot\hat{f}_i\|_{L^{2+\epsilon}}^{2+\epsilon}\right)^{1/2+\epsilon} & \text{if } 0<\epsilon<\infty\end{cases}$$

for $n/2<\tilde{s}<n/2+\left(s-\frac{n(1+\epsilon)}{2+\epsilon}\right)$. Hence, recalling the properties that $\sum_{j=0}^{\infty}|\psi_j|^{2+\epsilon}\lesssim 1$ (if $\epsilon<\infty$) and $2^j\sim\langle\xi\rangle$ on supp$\psi_j$, we have by Lemma 4.2

$$(4.15)\lesssim\sum_{m=0}^{\infty}2^{ms}\cdot 2^{-m([s]+1)}\left(\sum_{j=0}^{\infty}\sum_i 2^{js(2+\epsilon)}\|\psi_j\cdot\hat{f}_i\|_{L^{2+\epsilon}}^{2+\epsilon}\right)^{1/2+\epsilon}\lesssim\sum_i\|f_i\|_{\mathcal{F}L_s^{2+\epsilon}}$$

for $0\le\epsilon<\infty$, which gives the desired result for the case $0\le\epsilon<\infty$.

We next consider the case $\epsilon=\infty$, which is obtained similarly to the above. In fact, we have

$$\sup_{\xi\in\mathbb{R}^n}\left|\langle\xi\rangle^s\sum_{j=0}^{\infty}\mathcal{F}\left[\sum_{m=0}^{\infty}\sum_i p_{j,m}\cdot\Delta_j f_i\right](\xi)\right|\lesssim\sup_{\ell\in\mathbb{Z}_+}\left(\sup_{\xi\in\Omega_\ell}\sum_{m=0}^{\infty}2^{\ell s}\sum_{j\in\Lambda}\sum_i\left|\mathcal{F}[p_{j,m}\cdot\Delta_j f_i](\xi)\right|\right)$$

(see above for the definition of the sets $\Omega_\ell$ and $\Lambda$ ). Recalling from Lemma 4.2 that $\|p_{j,m}\|_{H_{\tilde{s}}^2}\lesssim 2^{-m([s]+1)}$ holds independently of $j,m\in\mathbb{Z}_+$and following the same lines as in Step 1, we have

$$\left|\sum_i\mathcal{F}[p_{j,m}\cdot\Delta_j f_i](\xi)\right|\lesssim\|p_{j,m}\|_{H_{\tilde{s}}^2}\sum_i\|\psi_j\hat{f}_i\|_{L^\infty}\lesssim 2^{-m([s]+1)}\cdot 2^{-js}\sum_i\|f_i\|_{\mathcal{F}L_s^\infty}$$

Hence, we obtain

$$\sum_{m=0}^{\infty} 2^{\ell s}\sum_{j\in\Lambda}\sum_{i} \left|\mathcal{F}[p_{j,m}\cdot\Delta_j f_i](\xi)\right| \lesssim \sum_{m=0}^{\infty} 2^{-m([s]+1)}\cdot 2^{\ell s}\sum_{j\in\Lambda}\sum_{i} 2^{-js}\|f_i\|_{\mathcal{F}L_s^{\infty}} \sim \sum_{i} \|f_i\|_{\mathcal{F}L_s^{\infty}}$$

for any $\ell \in \mathbb{Z}_+$. This gives the desired result for the case $\epsilon = \infty$.

Combining all the calculations above, we obtain for $0 \le \epsilon \le \infty$

$$\left\|\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i} p_{j,m}\cdot\Delta_j f_i\right\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}} < \infty \tag{4.16}$$

**Step 3:** Lastly, we prove that $G(S_0 f_i) \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}$. Observe that

$$G(S_0 f_i) = \int_0^1 \sum_{i} G^{(1)}(t\cdot S_0 f_i)dt\cdot S_0 f_i = \sum_{i} m_{f_i}\cdot S_0 f_i$$

where $m_{f_i} = \int_0^1 \sum_i G^{(1)}(t\cdot S_0 f_i)dt$. Then, since $\mathcal{F}L_r^{\frac{4}{3}+\epsilon} \hookrightarrow \mathcal{F}L_s^{\frac{4}{3}+\epsilon}$ for $r \ge s$ and $\langle\xi\rangle^r \lesssim 1 + |\xi_1|^r \cdots + |\xi_n|^r$ for $r \ge 0$, we have by Proposition 2.2 for $0 \le \epsilon \le 2/3$

$$\begin{aligned}\left\|\sum_{i} \langle\xi\rangle^s\mathcal{F}[G(S_0 f_i)]\right\|_{L^{\frac{4}{3}+\epsilon}} &\lesssim \sum_{i} \left\|\mathcal{F}[m_{f_i}\cdot S_0 f_i]\right\|_{L^{\frac{4}{3}+\epsilon}} + \sum_{i}\sum_{\ell=1}^{n} \left\|\mathcal{F}\left[\partial_\ell^{[s]+1}(m_{f_i}\cdot S_0 f_i)\right]\right\|_{L^{\frac{4}{3}+\epsilon}} \\ &\lesssim \sum_{i} \left\|\widetilde{m_{f_i}}(D)[\varphi\cdot\hat{f}_i]\right\|_{L^{\frac{4}{3}+\epsilon}} + \sum_{\ell=1}^{n}\sum_{\mu=0}^{[s]+1}\sum_{i} \left\|\widetilde{\partial_\ell^{\mu} m_{f_i}}(D)\left[\xi_\ell^{[s]+1-\mu}\varphi\cdot\hat{f}_i\right]\right\|_{L^{\frac{4}{3}+\epsilon}} \\ &\lesssim \sum_{i} \left\|m_{f_i}\right\|_{H_s^{\frac{4+3\epsilon}{1+3\epsilon}}}\|\varphi\cdot\hat{f}_i\|_{L^{\frac{4}{3}+\epsilon}} + \sum_{\ell=1}^{n}\sum_{\mu=0}^{[s]+1}\sum_{i} \left\|\partial_\ell^{\mu} m_{f_i}\right\|_{H_s^{\frac{4+3\epsilon}{1+3\epsilon}}}\left\|\xi_\ell^{[s]+1-\mu}\varphi\cdot\hat{f}_i\right\|_{L^{\frac{4}{3}+\epsilon}} \\ &\lesssim \sum_{i} \|f_i\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}\sum_{\ell=1}^{n}\sum_{\mu=0}^{[s]+1} \left\|\partial_\ell^{\mu} m_{f_i}\right\|_{H_s^{\frac{4+3\epsilon}{1+3\epsilon}}}\end{aligned}$$

and have for $0 < \epsilon \le \infty$

$$\left\|\sum_{i} \langle\xi\rangle^s\mathcal{F}[G(S_0 f_i)]\right\|_{L^{\frac{4}{3}+\epsilon}} \lesssim \sum_{i} \|f_i\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}\sum_{\ell=1}^{n}\sum_{\mu=0}^{[s]+1} \left\|\partial_\ell^{\mu} m_{f_i}\right\|_{H_s^2},$$

where we used the notation $\tilde{h}(\xi) = h(-\xi)$. Moreover, as in the proof of Lemma 4.3, Theorem A yields that for $\mu = 0,1,\cdots,[s]+1$

$$\left\|\sum_{i} \partial_\ell^{\mu} m_{f_i}\right\|_{H_s^{\frac{1+\epsilon}{\epsilon}}} \lesssim \sum_{\nu=0}^{\mu}\sum_{i} \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}}^{\nu} < \infty \text{ if } 0 < \epsilon \le 1$$

$$\left\|\sum_{i} \partial_\ell^{\mu} m_{f_i}\right\|_{H_s^2} \lesssim \sum_{\nu=0}^{\mu}\sum_{i} \|f_i\|_{\mathcal{F}L_s^{2+\epsilon}}^{\nu} < \infty \text{ if } 0 < \epsilon \le \infty$$

since the assumption $f_i \in \mathcal{F}L_s^{2+\epsilon}$ with $s > \frac{n\epsilon}{2+\epsilon}$ gives that for $\beta \in \mathbb{Z}_+^n$, $\left\|\sum_i \partial^\beta(S_0 f_i)\right\|_{H_s^{\frac{2+\epsilon}{\epsilon}}} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^{2+\epsilon}}$ if $0 < \epsilon \le 1$, and $\left\|\sum_i \partial^\beta(S_0 f_i)\right\|_{H_s^2} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^{2+\epsilon}}$ if $0 < \epsilon \le \infty$. Hence, we obtain $\|G(S_0 f_i)\|_{\mathcal{F}L_s^{2+\epsilon}} < \infty$. By Steps 1-3, we conclude that $G(f_i) \in \mathcal{F}L_s^{2+\epsilon}$ if $f_i \in \mathcal{F}L_s^{2+\epsilon}$.

Now, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** As is stated at the beginning, $F(f_i)$ with $F \in C^\infty(\mathbb{R})$ and $F(0) = 0$ is given by

$$F(f_i) = G(f_i) + \sum_{k=1}^{N}\sum_{i} F^{(k)}(0)\frac{f_i^k}{k!}$$

for any $N \ge 0$, where $G \in C^\infty(\mathbb{R})$ and $G(0) = G^{(1)}(0) = \cdots = G^{(N)}(0) = 0$. Choosing $N = [s]+2$, we obtain from Proposition 4.1 that $G(f_i) \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}$ if $f_i \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}$. The second one is shown by Proposition 3.4. In fact, since $F \in C^\infty(\mathbb{R})$, we have $\left|F^{(k)}(0)\right| \lesssim 1$, so that it follows that

$$\left\|\sum_{k=1}^{N}\sum_{i} F^{(k)}(0)\frac{f_i^k}{k!}\right\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}} \lesssim \sum_{k=1}^{N}\sum_{i} \|f_i\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}^k < \infty$$

if $f_i \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}$. Hence, we obtain that $F(f_i) \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}$ if $f_i \in \mathcal{F}L_s^{\frac{4}{3}+\epsilon}$.

## V. Proof of Theorem 1.1

As in Section 4, $F(f_i)$ is expressed in the following form:

$$F(f_i)=G(f_i)+\sum_{k=1}^{N}\sum_i F^{(k)}(0)\frac{f_i^k}{k!}, \tag{5.1}$$

for any $N\in\mathbb{N}$, where $G(0)=G^{(1)}(0)=\cdots=G^{(N)}(0)=0$. Applying a Taylor expansion to $G$, we have

$$G(f_i)=f_i^N\cdot H(f_i),\ \text{where } H(f_i)=\frac{1}{(N-1)!}\int_0^1\sum_i (1-\theta)^{N-1}G^{(N)}(\theta f_i)d\theta \tag{5.2}$$

Note that $H\in C^\infty(\mathbb{R})$ and $H(0)=0$. Hence, we mainly prove that $G(f_i)$ in (5.2) belongs to $M_s^{1+\epsilon,\frac{4}{3}+\epsilon}$ if $f_i\in M_s^{1+\epsilon,\frac{4}{3}+\epsilon}$. In order to prove this, we prepare the following lemma:

**Lemma 5.1 (see [19]).** Let $0\le\epsilon\le\infty$ and $s>\frac{4+3\epsilon}{1+3\epsilon}$, and let $N$ be an arbitrary natural number. Suppose that $G$ is the function in (5.2), $f_i\in M_s^{1+\epsilon,\frac{4}{3}+\epsilon}$ and real-valued functions $\phi,\tilde{\phi}\in C_0^\infty(\mathbb{R}^n)$ satisfy that $\tilde{\phi}\equiv 1$ on supp$\phi$. Then, we have

$$\left\|\sum_i \langle\xi\rangle^s V_\phi[G(f_i)](x,\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}\lesssim\sum_i \left\|\langle\xi\rangle^s V_{\tilde{\phi}}f_i(x,\xi)\right\|^N_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}$$

for any $x\in\mathbb{R}^n$. Here, the implicit constant is independent of $x\in\mathbb{R}^n$.

**Proof.** We first observe from (5.2) and the assumption $\tilde{\phi}(\cdot-x)\equiv 1$ on supp$\phi(\cdot-x)$ that

$$\begin{aligned}V_\phi[G(f_i)](x,\xi)&=\int_{\mathbb{R}^n}\sum_i e^{-i\xi\cdot t}\phi(t-x)\cdot G(\tilde{\phi}(t-x)f_i(t))dt\\&=\int_{\mathbb{R}^n}\sum_i e^{-i\xi\cdot t}\phi(t-x)\cdot(\tilde{\phi}(t-x)f_i(t))^N\cdot H(\tilde{\phi}(t-x)f_i(t))dt\\&=\mathcal{F}\sum_i\left[\phi(\cdot-x)\cdot(\tilde{\phi}(\cdot-x)f_i)^N\cdot H(\tilde{\phi}(\cdot-x)f_i)\right](\xi)\end{aligned}$$

Multiplying the weight $\langle\xi\rangle^s$ to both sides and taking the $L^{\frac{4}{3}+\epsilon}$-norm with respect to the $\xi$-variable, we have by Proposition 3.4

$$\begin{aligned}&\left\|\sum_i \langle\xi\rangle^s V_\phi[G(f_i)](x,\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}\\&=\sum_i\left\|\langle\xi\rangle^s\mathcal{F}\left[\phi(\cdot-x)\cdot(\tilde{\phi}(\cdot-x)f_i)^N\cdot H(\tilde{\phi}(\cdot-x)f_i)\right](\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}\\&\lesssim\sum_i\|\phi(\cdot-x)\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}\cdot\|\tilde{\phi}(\cdot-x)f_i\|^N_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}\cdot\|H(\tilde{\phi}(\cdot-x)f_i)\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}\cdot\end{aligned}$$

It obviously follows that $\|\phi(\cdot-x)\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}\sim 1$ and $\|\tilde{\phi}(\cdot-x)f_i\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}=\left\|\langle\xi\rangle^s V_{\tilde{\phi}}f_i(x,\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}$. We only consider $\|H(\tilde{\phi}(\cdot-x)f_i)\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}$ to obtain the conclusion. By Lemma A. 1 and Proposition 3.3, we have

$$\|\sum_i \tilde{\phi}(\cdot-x)f_i\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}\sim\|\sum_i \tilde{\phi}(\cdot-x)f_i\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}}\lesssim\sum_i \|\tilde{\phi}\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}}\cdot\|f_i\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}}<\infty$$

where the implicit constants are both independent of $x\in\mathbb{R}^n$. Then, recalling that $H\in C^\infty(\mathbb{R})$ and $H(0)=0$, we have $\sup_{x\in\mathbb{R}^n}\|H(\tilde{\phi}(\cdot-x)f_i)\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}<\infty$ by Theorem 1.2 if $0\le\epsilon\le\infty$.

Hence, we obtain

$$\left\|\sum_i \langle\xi\rangle^s V_\phi[G(f_i)](x,\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}\lesssim\left\|\sum_i \langle\xi\rangle^s V_{\tilde{\phi}}f_i(x,\xi)\right\|^N_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}$$

Here, recalling all the proofs in Section 4, we see that $\|H(\tilde{\phi}(\cdot-x)f_i)\|_{\mathcal{F}L_s^{\frac{4}{3}+\epsilon}}$ can be estimated by a polynomial of $\left\|\langle\xi\rangle^s V_{\tilde{\phi}}f_i(x,\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}$. This implies that the explicit order of the power in the right hand side can be actually taken larger than $N$. However, the explicit expression is not important, since it is sufficient to understand that the order can be chosen arbitrarily large as we want. Hence, we here omitted the details.

Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We recall the expressions (5.1) and have by Proposition 3.3

$$\|\sum_i F(f_i)\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}} \lesssim \|G(f_i)\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}} + \sum_{k=1}^{N}\sum_i \|f_i\|^k_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}} \tag{5.3}$$

Here, we choose $N \in \mathbb{N}$ such that $N \geq [\max(\frac{3(1+\epsilon)}{4+3\epsilon}, \frac{4+3\epsilon}{3(1+\epsilon)})] + 1$, and it should be remarked that we exclude the cases $\epsilon = \infty$ and $\epsilon < \infty$, or $\epsilon < \infty$ and $\epsilon = \infty$ in Theorem 1.1, since such $N$ cannot be taken in those cases.

We first consider $\|G(f_i)\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}}$ for the case $1+\epsilon \leq \frac{4}{3}+\epsilon$. Let real-valued functions $\phi, \tilde{\phi} \in C_0^\infty(\mathbb{R}^n)$ satisfy that $\tilde{\phi} \equiv 1$ on $\mathrm{supp}\phi$. Then, we have by the Minkowski inequality for integrals and Lemma 5.1

$$\|\sum_i G(f_i)\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}} \lesssim \sum_i \left\| \|\langle\xi\rangle^s V_\phi[G(f_i)](x,\xi)\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)} \right\|_{L^{1+\epsilon}(\mathbb{R}^n_x)}$$

$$\lesssim \sum_i \left\| \|\langle\xi\rangle^s V_{\tilde{\phi}} f_i(x,\xi)\|^N_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)} \right\|_{L^{1+\epsilon}(\mathbb{R}^n_x)} = \sum_i \left\| \|\langle\xi\rangle^s V_{\tilde{\phi}} f_i(x,\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)} \Big\|^N_{L^{N\frac{4}{3}+\epsilon}(\mathbb{R}^n_x)}$$

Since $N_{\frac{4}{3}+2\epsilon} > \frac{4}{3}+\epsilon \geq 1+\epsilon$, we have by Proposition 2.3

$$\|\sum_i G(f_i)\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}} \lesssim \sum_i \left\| \|\langle\xi\rangle^s V_{\tilde{\phi}} f_i(x,\xi)\|_{L^{N\frac{4}{3}+2\epsilon}(\mathbb{R}^n_x)} \right\|^N_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)}$$

$$\lesssim \sum_i \left\| \|\langle\xi\rangle^s V_{\tilde{\phi}} f_i(x,\xi)\|_{L^{1+\epsilon}(\mathbb{R}^n_x)} \right\|^N_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)} \sim \sum_i \|f_i\|^N_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}}$$

We next assume that $\frac{4}{3}+\epsilon < \frac{4}{3}+2\epsilon < \infty$. As above, Proposition 2.3 and Lemma 5.1 yield that

$$\|\sum_i G(f_i)\|_{M_s^{\frac{4}{3}+2\epsilon,\frac{4}{3}+\epsilon}} \lesssim \sum_i \|G(f_i)\|_{M_s^{\frac{4}{3}+\epsilon,\frac{4}{3}+\epsilon}} \lesssim \sum_i \left\| \|\langle\xi\rangle^s V_{\tilde{\phi}} f_i(x,\xi)\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)} \right\|^N_{L^{N\frac{4}{3}+\epsilon}(\mathbb{R}^n_x)}$$

Since $N_{\frac{4}{3}+\epsilon} > \frac{4}{3}+2\epsilon > \frac{4}{3}+\epsilon$, we use Proposition 2.3 again and obtain

$$\|\sum_i G(f_i)\|_{M_s^{\frac{4}{3}+2\epsilon,\frac{4}{3}+\epsilon}} \lesssim \sum_i \left\| \|\langle\xi\rangle^s V_{\tilde{\phi}} f_i(x,\xi)\right\|_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_x)} \Big\|^N_{L^{\frac{4}{3}+\epsilon}(\mathbb{R}^n_\xi)} \sim \sum_i \|f_i\|^N_{M_s^{\frac{4}{3}+2\epsilon,\frac{4}{3}+\epsilon}}$$

Therefore, for $0 \leq \epsilon < \infty$ (or $\epsilon = \infty$ ), we have $\|G(f_i)\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}} \lesssim \|f_i\|^N_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}}$.

Collecting all the estimates above, we obtain $\|F(f_i)\|_{M_s^{1+\epsilon,\frac{4}{3}+\epsilon}} < \infty$. This is the desired conclusion.

**Appendix A. Local equivalence between modulation and Fourier Lebesgue spaces**

We state that modulation spaces are locally equivalent to Fourier Lebesgue spaces. The corresponding result for $s = 0$ was already proved by [11, Lemma 1], and the weighted case is obtained by following the same argument. However, we give a proof.

**Lemma A.1 (see [19]).** Let $0 \leq \epsilon \leq \infty$ and $s \in \mathbb{R}$. Suppose that $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ satisfies that $\mathrm{supp}\chi \subset \{x: |x - x_0| \leq R\}$. Then, we have $\|\chi \cdot f_i\|_{M_s^{1+\epsilon,1+2\epsilon}} \sim \|\chi \cdot f_i\|_{\mathcal{F}L_s^{1+2\epsilon}}$. Here, the implicit constant is independent of $x_0 \in \mathbb{R}^n$, but depends on $R > 0$.

**Proof.** Put $(f_i)_\chi = \chi \cdot f_i$. We first prove the $\lesssim$ part. Choose $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ satisfying that $\mathrm{supp}\phi \subset \{x: |x| \leq R\}$. Then, we see that $V_\phi[(f_i)_\chi](x,\xi)$ always vanishes unless $x \in \mathbb{R}^n$ satisfies that $|x - x_0| \leq 2R$. Using the identity $|V_\phi[(f_i)_\chi](x,\xi)| = \left|\hat{\bar{\phi}}(D-\xi)(f_i)_\chi(x)\right|$, we have by the Hölder and Hausdorff-Young inequalities

$$\left\|\sum_i V_\phi[(f_i)_\chi](x,\xi)\right\|_{L^{1+2\epsilon}(\mathbb{R}^n_x)} = \left\|\sum_i \chi_{B_{2R}(x_0)}(x) \cdot V_\phi[(f_i)_\chi](x,\xi)\right\|_{L^{1+2\epsilon}(\mathbb{R}^n_x)}$$

$$\lesssim R^{n/1+\epsilon} \sum_i \left\|\overline{\hat{\phi}(D-\xi)}(f_i)_\chi(\cdot)\right\|_{L^\infty} \lesssim R^{n/1+\epsilon} \sum_i \left\|\hat{\phi}(t-\xi) \cdot \mathcal{F}[(f_i)_\chi](t)\right\|_{L^1(\mathbb{R}^n_t)}$$

Multiplying the weight $\langle\xi\rangle^s$ to both sides, using the inequality $\langle\xi\rangle^s \lesssim \langle t\rangle^s \langle t-\xi\rangle^{|s|}$ and taking the $L^{1+2\epsilon}$-norm with respect to the $\xi$-variable, we have by the Young inequality

$$\left\|\left\|\sum_i \langle\xi\rangle^s V_\phi[(f_i)_\chi](x,\xi)\right\|_{L^{1+\epsilon}(\mathbb{R}^n_x)}\right\|_{L^{1+2\epsilon}(\mathbb{R}^n_\xi)} \lesssim R^{n/1+\epsilon}\sum_i \left\|\left\|\langle t-\xi\rangle^{|s|}\hat{\phi}(t-\xi)\cdot\langle t\rangle^s\mathcal{F}[(f_i)_\chi](t)\right\|_{L^1(\mathbb{R}^n_t)}\right\|_{L^{1+2\epsilon}(\mathbb{R}^n_\xi)}$$
$$= R^{n/1+\epsilon}\sum_i \left\|\left(\langle\cdot\rangle^{|s|}|\hat{\phi}|\right)*\left(\langle\cdot\rangle^s|\mathcal{F}[(f_i)_\chi]|\right)\right\|_{L^{1+2\epsilon}}$$
$$\lesssim R^{n/1+\epsilon}\sum_i \left\|\langle\cdot\rangle^s\mathcal{F}[(f_i)_\chi]\right\|_{L^{1+2\epsilon}}$$

We next prove the $\gtrsim$ part. Choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\mathrm{supp}\phi \equiv 1$ on $\{x: |x| \le 2R\}$. Then, $\phi(\cdot - x) \equiv 1$ on $\mathrm{supp}\chi$ if $x \in \mathbb{R}^n$ satisfies that $|x - x_0| \le R$. Hence, it follows that

$$R^{n/1+\epsilon}\sum_i \left|\mathcal{F}[(f_i)_\chi](\xi)\right| \sim \sum_i \left\|\chi_{B_R(x_0)}(x)\cdot\mathcal{F}[(f_i)_\chi](\xi)\right\|_{L^{1+\epsilon}(\mathbb{R}^n_x)}$$
$$= \left\|\chi_{B_R(x_0)}(x)\cdot\int_{\mathbb{R}^n}\sum_i e^{-i\xi\cdot t}\phi(t-x)\cdot\chi(t)f_i(t)dt\right\|_{L^{1+\epsilon}(\mathbb{R}^n_x)}$$
$$\le \sum_i \left\|V_\phi[(f_i)_\chi](x,\xi)\right\|_{L^{1+\epsilon}(\mathbb{R}^n_x)}$$

Multiplying the weight $\langle\xi\rangle^s$ to both sides and taking the $L^{1+2\epsilon}$-norm with respect to the $\xi$-variable, we have

$$\left\|\sum_i \langle\cdot\rangle^s\mathcal{F}[(f_i)_\chi]\right\|_{L^{1+2\epsilon}} \lesssim R^{-n/1+\epsilon}\sum_i \left\|\left\|\langle\xi\rangle^s V_\phi[(f_i)_\chi](x,\xi)\right\|_{L^{1+\epsilon}(\mathbb{R}^n_x)}\right\|_{L^{1+2\epsilon}(\mathbb{R}^n_\xi)}$$

Therefore, recalling the property that the modulation space norm is independent of the choice of window functions, we obtain $\|(f_i)_\chi\|_{M_s^{1+\epsilon,1+2\epsilon}} \sim \|(f_i)_\chi\|_{\mathcal{F}L_s^{1+2\epsilon}}$.

**Appendix B. Conditions for modulation spaces and Fourier Lebesgue spaces to be multiplication algebras**

We first consider necessary and sufficient conditions for modulation spaces to be multiplication algebras, that is, for the estimate

$$\|\sum_i f_i\cdot g_i\|_{M_s^{1+\epsilon,1+2\epsilon}} \lesssim \sum_i \|f_i\|_{M_s^{1+\epsilon,1+2\epsilon}}\cdot\|g_i\|_{M_s^{1+\epsilon,1+2\epsilon}}$$

to hold. They are given as follows.

**Proposition B.1 [19].** Let $0 \le \epsilon \le \infty, 0 < \epsilon \le \infty$ and $s \in \mathbb{R}$. Then, the modulation space $M_s^{1+\epsilon,1+\epsilon}(\mathbb{R}^n)$ is a multiplication algebra if and only if the condition $s > \frac{n\epsilon}{1+\epsilon}$ is satisfied.

Actually, this proposition is immediately obtained from [6, Theorem 1.5]. In fact, in [6], necessary and sufficient conditions for the more general estimate

$$\|\sum_i f_i\cdot g_i\|_{M_s^{1+\epsilon,1+\epsilon}} \lesssim \sum_i \|f_i\|_{M_{s_1}^{1+\epsilon,1+3\epsilon}}\cdot\|g_i\|_{M_{s_2}^{1+2\epsilon,1+4\epsilon}}$$

were established, so that Proposition B. 1 is given by setting $\epsilon = 0$ and $s = s_1 = s_2$. (We remark that, although only the case $\epsilon > 0$ is considered in Proposition B.1, the whole case $\epsilon \ge 0$ is treated in [6].) However, we give a proof of Proposition B. 1 where the following two lemmas are essential:

**Lemma B.1** ([6, Proposition 5.1]). Let $0 \le \epsilon \le \infty$ and $s \in \mathbb{R}$. Then, if the modulation space $M_s^{1+\epsilon,1+2\epsilon}$ is a multiplication algebra, we have $\ell_s^{1+2\epsilon} * \ell_s^{1+2\epsilon} \hookrightarrow \ell_s^{1+2\epsilon}$.

**Lemma B.2 (see [19]).** Let $0 < \epsilon \le \infty$ and $s \in \mathbb{R}$. Then, if $\ell_s^{1+\epsilon} * \ell_s^{1+\epsilon} \hookrightarrow \ell_s^{1+\epsilon}$ holds, we have $s > \frac{n\epsilon}{1+\epsilon}$.

**Proof.** We assume towards a contradiction that $s \le \frac{n(1+\epsilon)}{1+\epsilon}$. Since $\epsilon > 0$, we can take $\varepsilon > 0$ such that $1 - 1/1 + \epsilon - \varepsilon > 0$. For this $\varepsilon > 0$, we define the sequences

$$a_{k,N} = \begin{cases} \langle k\rangle^{\left(-\frac{n}{1+\epsilon}\right)-s}(1+\epsilon+\log\langle k\rangle)^{\left(-\frac{1}{1+\epsilon}\right)-\varepsilon}, & \text{if } |k| \le N, \\ 0, & \text{otherwise} \end{cases}$$

$$b_{k,N} = \begin{cases} 1, & \text{if } N \le |k| \le 5N, \\ 0, & \text{otherwise} \end{cases}$$

in $k \in \mathbb{Z}^n$, where $N > 0$ is a sufficiently large integer and $\epsilon > 0$ is a suitable constant which depends only on the dimension $n$.

We first estimate each sequence on $\ell_s^{1+\epsilon}$. For the case $\epsilon < \infty$, the spherical coordinate transform yields that

$$\|a_{k,N}\|_{\ell_s^{1+\epsilon}}^{1+\epsilon} = \sum_{|k|\le N} \langle k\rangle^{-n}(1+\epsilon+\log\langle k\rangle)^{-1-\varepsilon(1+\epsilon)}$$

$$\lesssim \int_{|x|\le 2N} \langle x\rangle^{-n}\left(\frac{1+\epsilon}{\epsilon}+\log\langle x\rangle\right)^{-1-\varepsilon(1+\epsilon)} dx \lesssim \int_0^{2N} (1+r)^{-1}(1+\log(1+r))^{-1-\varepsilon(1+\epsilon)}dr$$

By the change of variable $t = 1 + \log(1+r)$, we have

$$\|a_{k,N}\|_{\ell_s^{1+\epsilon}}^{1+\epsilon} \lesssim \int_1^{1+\log(1+2N)} t^{-1-\varepsilon(1+\epsilon)}dt \lesssim 1$$

For the case $\epsilon = \infty$, we have $\|a_{k,N}\|_{\ell_s^\infty} \le 1$, since $\varepsilon > 0$. On the other hand, we have $\|b_{k,N}\|_{\ell_s^{1+\epsilon}} \sim N^{s+n/1+\epsilon}$ holds for $0 < \epsilon \le \infty$.

Next, we consider the convolution $\{a_{\cdot,N} * b_{\cdot,N}\}_{k\in\mathbb{Z}^n}$. For $2N \le |k| \le 4N$, we have

$$\sum_{\ell\in\mathbb{Z}^n} a_{\ell,N}b_{k-\ell,N} = \sum_{N\le|k-\ell|\le 5N} a_{\ell,N} = \sum_{|\ell|\le N} a_{\ell,N},$$

since $\{\ell\in\mathbb{Z}^n : |\ell|\le N\} \subset \{\ell\in\mathbb{Z}^n : N \le |k-\ell| \le 5N\}$ and $a_{\ell,N} = 0$ if $|\ell| > N$. Then by $s \le \frac{n\epsilon}{1+\epsilon}$ we have

$$\sum_{\ell\in\mathbb{Z}^n} a_{\ell,N}b_{k-\ell,N} = \sum_{|\ell|\le N} \langle\ell\rangle^{\left(-\frac{n}{1+\epsilon}\right)-s}(1+\epsilon+\log\langle\ell\rangle)^{\left(-\frac{1}{1+\epsilon}\right)-\varepsilon}$$

$$\ge \sum_{|\ell|\le N} \langle\ell\rangle^{-n}(1+\epsilon+\log\langle\ell\rangle)^{\left(-\frac{1}{1+\epsilon}\right)-\varepsilon}$$

$$\gtrsim \int_{|x|\le N/2} \langle x\rangle^{-n}\left(\frac{1+\epsilon}{\epsilon}+\log\langle x\rangle\right)^{\left(-\frac{1}{1+\epsilon}\right)-\varepsilon} dx$$

$$\sim \int_0^{N/2} r^{n-1}(1+r)^{-n}(1+\log(1+r))^{\left(-\frac{1}{1+\epsilon}\right)-\varepsilon}dr$$

$$\gtrsim \int_1^{N/2} (1+r)^{-1}(1+\log(1+r))^{\left(-\frac{1}{1+\epsilon}\right)-\varepsilon}dr$$

and hence by the same change of variable as above we have

$$\sum_{\ell\in\mathbb{Z}^n} a_{\ell,N}b_{k-\ell,N} \gtrsim \int_{1+\log 2}^{1+\log(1+N/2)} t^{\left(-\frac{1}{1+\epsilon}\right)-\varepsilon} \gtrsim (1+\log(1+N/2))^{\left(1-\frac{1}{1+\epsilon}\right)-\varepsilon}$$

This concludes that

$$\left\|\{a_{\cdot,N} * b_{\cdot,N}\}_{k\in\mathbb{Z}^n}\right\|_{\ell_s^{1+\epsilon}} \ge \left\|\{a_{\cdot,N} * b_{\cdot,N}\}_{k\in\{2N\le|k|\le 4N\}}\right\|_{\ell_s^{1+\epsilon}}$$

$$\gtrsim N^{s+n/1+\epsilon}(1+\log(1+N/2))^{\left(1-\frac{1}{1+\epsilon}\right)-\varepsilon}$$

Collecting the estimates above, we have by the assumption $\ell_s^{1+\epsilon} * \ell_s^{1+\epsilon} \hookrightarrow \ell_s^{1+\epsilon}$

$$\left\|\{a_{\cdot,N} * b_{\cdot,N}\}_{k\in\mathbb{Z}^n}\right\|_{\ell_s^{1+\epsilon}} \lesssim \|a_{k,N}\|_{\ell_s^{1+\epsilon}} \cdot \|b_{k,N}\|_{\ell_s^{1+\epsilon}}$$

$$\Longrightarrow N^{s+n/1+\epsilon}(1+\log(1+N/2))^{\left(1-\frac{1}{1+\epsilon}\right)-\varepsilon} \lesssim 1\cdot N^{s+n/1+\epsilon}$$

$$\Leftrightarrow (1+\log(1+N/2))^{\left(1-\frac{1}{1+\epsilon}\right)-\varepsilon} \lesssim 1$$

However, the last estimate fails when we choose a sufficiently large number $N > 0$, since $\left(\frac{\epsilon}{\epsilon-1}\right) - \varepsilon > 0$. This contradicts to the assumption $\ell_s^{1+\epsilon} * \ell_s^{1+\epsilon} \hookrightarrow \ell_s^{1+\epsilon}$. Therefore, we obtain $s > \frac{n\epsilon}{1+\epsilon}$.

**Proof of Proposition B.1.** The "IF" part is given by Proposition 3.3, and the "ONLY IF" part is an immediate conclusion of Lemmas B. 1 and B.2.

We also have a similar optimality for Fourier Lebesgue spaces:

**Proposition B.2 [19].** Let $0 < \epsilon \le \infty$ and $s \in \mathbb{R}$. Then, the Fourier Lebesgue space $\mathcal{F}L_s^{1+\epsilon}(\mathbb{R}^n)$ is a multiplication algebra if and only if the condition $s > \frac{n\epsilon}{1+\epsilon}$ is satisfied.

For the proof of Proposition B.2, we use the following lemma instead of Lemma B.1:

**Lemma B.3** ([6, Proposition 4.1]). Let $0 \le \epsilon \le \infty$ and $s \in \mathbb{R}$. Then, if the estimate

$$\left\|\sum_i \langle\cdot\rangle^s(f_i * g_i)\right\|_{L^{1+\epsilon}} \lesssim \sum_i \|\langle\cdot\rangle^s f_i\|_{L^{1+\epsilon}} \cdot \|\langle\cdot\rangle^s g_i\|_{L^{1+\epsilon}}$$

holds, we have $\ell_s^{1+\epsilon} * \ell_s^{1+\epsilon} \hookrightarrow \ell_s^{1+\epsilon}$.

**Proof of Proposition B.2.** The "IF" part is given by Proposition 3.4. The "ONLY IF" part is an immediate conclusion of Lemmas B. 2 and B. 3 if we notice the equivalence

$$\|\sum_i f_i \cdot g_i\|_{\mathcal{F}L_s^{1+\epsilon}} \lesssim \sum_i \|f_i\|_{\mathcal{F}L_s^{1+\epsilon}} \cdot \|g_i\|_{\mathcal{F}L_s^{1+\epsilon}}$$

$$\Leftrightarrow \left\|\sum_i \langle\cdot\rangle^s(\hat{f}_i * \hat{g}_{\mathrm{i}})\right\|_{L^{1+\epsilon}} \lesssim \sum_i \left\|\langle\cdot\rangle^s\hat{f}_i\right\|_{L^{1+\epsilon}} \cdot \|\langle\cdot\rangle^s\hat{g}_i\|_{L^{1+\epsilon}}.$$

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