



Analytical and Numerical Solution of Boundary-Value Problems for Stationary Oscillations in Micropolar Elastic Layers Using the Fourier Integral Transform

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Abstract

This paper presents an analytical and numerical study of boundary-value problems associated with stationary oscillations in a homogeneous isotropic micropolar elastic layer subjected to harmonic surface loading. The formulation incorporates translational displacement, independent microrotation, stress, and couple-stress fields within a unified micropolar elasticity framework. The governing equations are reduced by the Fourier Integral Transform with respect to the horizontal coordinate, yielding a coupled three-field ordinary differential system in the thickness coordinate. The transformed system is solved through characteristic-root analysis and a six-mode representation, while the unknown modal coefficients are obtained from the transformed boundary conditions. The physical fields are then reconstructed through inverse Fourier transformation. Numerical verification is performed using boundary-condition residuals, independent physical-field checks, constitutive consistency, and monitoring of imaginary components. A systematic parametric study examines the effects of angular frequency, micropolar coupling, layer thickness, and the curvature parameter on the stationary response. The developed formulation provides a reproducible analytical reference solution for numerical verification and for further investigations of stationary responses in microstructured elastic media.

Keywords: Micropolar elasticity; stationary oscillations; Fourier integral transform; characteristic modes; microrotation; couple stress; boundary-value problems; elastic layers.

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I. Introduction

Classical elasticity theory provides the mathematical framework for describing deformation, stress distribution, and wave propagation in elastic solids. However, its classical continuum assumption does not explicitly account for independent rotations of material particles or intrinsic microstructural length scales. These effects become important in materials with pronounced internal structure and in problems where the mechanical response depends on the interaction between translational and rotational motions. [1], [2]

Micropolar elasticity extends the classical continuum framework by introducing independent microrotation degrees of freedom together with couple stresses. This formulation therefore provides a more comprehensive description of materials in which microstructural rotations contribute to the macroscopic response. [3]–[5]

Stationary or time-harmonic oscillation problems constitute an important class of dynamic boundary-value problems because they provide information on steady-state response, stress transmission, wave

propagation, and attenuation. Their analytical treatment is complicated by the coupling between displacement and microrotation fields and by the additional boundary conditions associated with couple stresses. [4]

The present study develops a Fourier Integral Transform formulation for a stationary oscillation problem in a finite micropolar elastic layer. The Fourier transformation reduces the governing partial differential equations to an ordinary differential system in the thickness coordinate, providing a systematic route to the characteristic equation, modal solution, boundary system, and physical field reconstruction. [6], [7]

The main contribution of this study is the development of a unified analytical and numerical framework for stationary oscillations in a finite micropolar elastic layer subjected to harmonic surface loading. The formulation integrates the Fourier Integral Transform of the coupled three-field governing system, characteristic-root analysis, construction of the corresponding modal vectors, assembly and solution of the boundary matrix, and inverse Fourier reconstruction of the physical fields. The resulting framework enables the evaluation of translational displacement, microrotation, stress, and couple-stress fields within the finite layer. The analytical solution is verified numerically through residual-based error measures and independent checks of the prescribed boundary conditions, followed by a systematic parametric investigation of the principal material, geometric, and dynamic parameters. The developed formulation provides a reproducible analytical reference solution that can be used for numerical verification and for further investigations of stationary responses in microstructured elastic media.

II. Mathematical Formulation of Micropolar Elasticity

2.1 Introduction

Micropolar elasticity considers both translational displacement and independent microrotation fields. The formulation introduces additional material constants and couple stresses that distinguish it from classical elasticity. [1], [8]

2.2 Kinematic Relations

The micropolar deformation measures are constructed from displacement gradients and the microrotation field. In compact tensor notation,

$$\gamma_{ij} = u_{j,i} + \epsilon_{ijk} \phi_k$$

with the corresponding curvature measure

$$\kappa_{ij} = \phi_{j,i}$$

2.3 Constitutive Relations

For a homogeneous isotropic micropolar solid, the force-stress and couple-stress tensors are related to the generalized strain and curvature tensors through the micropolar constitutive equations. The material constants used in the present formulation include the Lamé parameters λ and μ , the micropolar coupling parameter κ , and the curvature-related constants γ and η . The Fourier wave number is denoted throughout by β . [1], [3], [8]

2.4 Equations of Motion

The equations of motion follow from the balances of linear and angular momentum and couple the translational displacement fields to the independent microrotation field. The resulting dynamic system contains the inertial terms associated with the density ρ and the rotational inertia J . [1], [2]

2.5 Harmonic Representation

For stationary oscillations, the field variables are represented in harmonic form:

$$u(x,y,t) = u(x,y)e^{-i\omega t}$$

$$v(x,y,t) = v(x,y)e^{-i\omega t}$$

$$\varphi(x,y,t) = \varphi(x,y)e^{-i\omega t}$$

Consequently, $\partial^2/\partial t^2 \rightarrow -\omega^2$.

2.6 Fourier Integral Transform

The Fourier transform with respect to x is defined by

$$\hat{f}(\beta,y) = \int_{-\infty}^{\infty} f(x,y)e^{-i\beta x} dx$$

with inverse transformation

$$f(x,y) = (1/2\pi) \int_{-\infty}^{\infty} \hat{f}(\beta,y)e^{i\beta x} d\beta$$

Accordingly, $\partial/\partial x$ is replaced by $i\beta$, converting the original coupled partial differential system into a system of ordinary differential equations with respect to y . [9]

III. Fourier Integral Transform Methodology

After transformation, the three-field system is

$$(\mu+\kappa)\hat{U}'' - A_1\hat{U} - i\beta(\lambda+\mu)\hat{V}' - 2\kappa\hat{\Phi}' = 0$$

$$(\lambda+2\mu+\kappa)\hat{V}'' - A_2\hat{V} + i\beta(\lambda+\mu)\hat{U}' + 2i\beta\kappa\hat{\Phi} = 0$$

$$\gamma\hat{\Phi}'' - A_3\hat{\Phi} - 2\kappa\hat{U}' - 2i\beta\kappa\hat{V} = 0$$

where $A_1=(\lambda+2\mu+\kappa)\beta^2-\rho\omega^2$, $A_2=(\mu+\kappa)\beta^2-\rho\omega^2$, and $A_3=\eta\beta^2+4\kappa-J\omega^2$.

3.1 Transformed System

The transformed equations preserve the coupling among the two translational components and the independent microrotation field.

3.2 Characteristic System

With $\hat{U}=Ue^{-qy}$, $\hat{V}=Ve^{-qy}$, and $\hat{\Phi}=\Phi e^{-qy}$, the characteristic system is $K(q,\beta,\omega)a=0$, where $a=[U,V,\Phi]^T$.

3.3 Characteristic Equation and Modal Checks

A nontrivial solution requires $|K(q,\beta,\omega)|=0$, giving the characteristic polynomial $c_6q^6+c_4q^4+c_2q^2+c_0=0$ and six characteristic roots $q_1,\dots,q_6$, commonly arranged as $\pm q_1, \pm q_2, \pm q_3$. Root and modal-vector residuals are evaluated by $R_{\text{det}}(q)=|\det K(q)|/||K(q)||\infty^3$ and $R_{\text{vec}}(q)=||K(q)a||_2/(||K(q)||_2||a||_2)$, together with a conjugate-pair check.

3.4 Characteristic Matrix

$$K(q,\beta,\omega) = [[(\mu+\kappa)q^2-A_1, i\beta q(\lambda+\mu), 2\kappa q], [-i\beta q(\lambda+\mu), (\lambda+2\mu+\kappa)q^2-A_2, 2i\beta\kappa], [2\kappa q, -2i\beta\kappa, \gamma q^2-A_3]].$$

The mathematical sequence is: governing PDEs $\rightarrow$ harmonic form $\rightarrow$ Fourier transform $\rightarrow$ characteristic equation $\rightarrow$ modal solution $\rightarrow$ boundary system $\rightarrow$ inverse Fourier transformation.

IV. Analytical Solution of the Micropolar Elastic Boundary-Value Problem

4.1 Characteristic Roots and Modal Vectors

The six characteristic roots q_j are obtained from the characteristic equation, and the corresponding modal vectors a_j satisfy $K(q_j)a_j=0$. The modal vectors are normalized for numerical reconstruction.

4.2 Root and Modal Verification

Each computed root and its modal vector are checked using the characteristic and vector residuals, with conjugate-pair consistency monitored.

4.3 General Six-Mode Solution

$$[\hat{U}, \hat{V}, \hat{\Phi}]^T = \sum_{j=1}^6 C_j a_j e^{-q_j y}$$

$$\hat{U} = \sum C_j U_j e^{-q_j y}, \quad \hat{V} = \sum C_j V_j e^{-q_j y}, \quad \hat{\Phi} = \sum C_j \Phi_j e^{-q_j y}$$

$$\hat{U}' = -\sum q_j C_j U_j e^{-q_j y}, \quad \hat{V}' = -\sum q_j C_j V_j e^{-q_j y}, \quad \hat{\Phi}' = -\sum q_j C_j \Phi_j e^{-q_j y}.$$

4.4 Boundary Matrix Formulation

At $y=0$: $\bar{\sigma}_{xy}=0$, $\bar{\sigma}_{yy}=\bar{P}(\beta)$, $\bar{m}_{yz}=0$. At $y=h$: $\bar{\sigma}_{xy}=0$, $\bar{\sigma}_{yy}=0$, $\bar{m}_{yz}=0$.

$$\bar{\sigma}_{xy} = \mu(\hat{U}' + i\beta\hat{V}) + \kappa(\hat{U}' - i\beta\hat{V} - 2\hat{\Phi})$$

$$\bar{\sigma}_{yy} = \lambda(i\beta\hat{U} + \hat{V}') + 2\mu\hat{V}'$$

$$\bar{m}_{yz} = \gamma\hat{\Phi}'$$

The six conditions are assembled as $B(\beta, \omega, h)C = F(\beta)$, with $C = [C_1, \dots, C_6]^T$ and $F(\beta) = [0, \bar{P}(\beta), 0, 0, 0, 0]^T$. The modal coefficients are obtained by direct solution of the boundary system.

4.5 Boundary-System Verification

The boundary system is solved directly after checking that B has finite entries, full numerical rank, and is not numerically singular. The algebraic residual is $R_B = \|BC - F\|_2 / \|F\|_2$. The six physical boundary conditions are then independently evaluated from the reconstructed fields.

4.6 Field Reconstruction

The transformed displacement and microrotation fields are reconstructed from the six modal contributions. The stress and couple-stress fields are subsequently evaluated from the constitutive relations given above.

4.7 Inverse Fourier Transformation

For each reconstructed field $\hat{f}(\beta, y)$, the physical field is recovered from $f(x, y) = (1/2\pi) \int_{-\infty}^{\infty} \hat{f}(\beta, y) e^{i\beta x} d\beta$. The procedure is applied to u , v , ϕ , σ_{xy} , σ_{yy} , and m_{yz} over the prescribed finite wave-number range.

4.8 Numerical Verification Chain

Verification proceeds through characteristic-root and modal-vector residuals, the boundary residual, independent physical boundary checks, constitutive/field consistency, and monitoring of imaginary components.

4.9 Computational Implementation and Reproducibility

For each parameter set, the transformed system is assembled, the six roots are solved, normalized modal vectors are constructed, the boundary matrix is assembled, $BC=F$ is solved directly, the fields are reconstructed, and the inverse Fourier transform is evaluated. Field, mode, boundary-condition, and parameter ordering are kept consistent throughout the implementation.

V. Numerical Results and Discussion

5.1 Reference Parameters and Numerical Setup

The numerical solution considers a homogeneous isotropic micropolar elastic layer occupying $0 \leq y \leq h$. The reference material, geometric, loading, and computational parameters are listed in Table 1. The normalized thickness coordinate is $0 \leq y/h \leq 1$. The reference loading is $\bar{T}=0$, $\bar{P}=1$, and $\bar{M}=0$. The Fourier computational range is $0 \leq \beta \leq 40 \text{ m}^{-1}$ with 400 sampling points, together with 201 x-points and 101 y-points. The reference verification value is $\beta=2 \text{ m}^{-1}$.

5.2 Numerical Verification

The reference solution is verified through the boundary-condition residual, independent physical-field error, constitutive-field error, and maximum imaginary component. The results in Table 2 demonstrate numerical consistency to near machine precision.

5.3 Displacement and Microrotation Response

The reconstructed displacement components U and V and the microrotation Φ exhibit distinct depth-dependent behavior within the finite layer. Representative verified values are given in Table 3. At the selected locations, V is larger than U , while Φ remains of the same order after normalization, confirming the coupled translational–rotational character of the response.

5.4 Stress and Couple-Stress Response

The stress and couple-stress fields are evaluated from the reconstructed fields. At the loaded surface, $\sigma_{yy}=1$ while σ_{xy} is essentially zero and m_{yz} is negligible. At the free surface, σ_{xy} , σ_{yy} , and m_{yz} are all essentially zero. Nonzero interior shear and normal stresses demonstrate that satisfying zero tangential traction at the surfaces does not imply zero shear stress throughout the layer.

5.5 Parametric Study

5.5.1 Effect of Angular Frequency ω

For $\omega=5, 10$, and 15 rad/s , the computed response changes only slightly over the investigated range.

5.5.2 Effect of Micropolar Coupling Parameter κ

For $\kappa=0.3, 0.6$, and $0.9 \times 10^9 \text{ Pa}$, the maximum displacement amplitudes and maximum shear stress decrease as κ increases.

5.5.3 Effect of Layer Thickness h

For $h=0.5, 1$, and 2 m , the field extrema and spatial distributions change with thickness. The stress response is not monotonic for every quantity.

5.5.4 Effect of Curvature Parameter γ

For $\gamma=0.004$, 0.008 , and 0.012 N, the principal displacement and stress measures remain nearly unchanged, whereas the maximum couple stress increases with γ .

No independent parametric sweep of β is included; $\beta=2 \text{ m}^{-1}$ is retained as the reference verification value.

5.6 Classical Elasticity Comparison and Limiting Case

No numerical classical-elasticity comparison is introduced because an independently computed classical reference is not part of the present numerical study. The theoretical limiting case provides the consistency check: as $\kappa \rightarrow 0$, $\gamma \rightarrow 0$, and $J \rightarrow 0$, the microrotation field satisfies $\Phi \rightarrow 0$ and the classical Navier equations are recovered.

5.7 Engineering Interpretation

The results show that the micropolar formulation captures independent microrotation and couple-stress effects that are absent from classical elasticity. The increase of κ reduces the maximum displacement and shear-stress measures, whereas γ mainly affects the couple-stress response while leaving the principal displacement and stress measures nearly unchanged. Layer thickness modifies the spatial distribution and extrema of the fields, and the investigated frequency range produces comparatively small changes. These features are relevant to microstructured materials and components such as composites, foams, granular and cellular materials, and micro- and nano-structural systems.

5.8 Concluding Remarks

The numerical results validate the analytical framework for a finite micropolar elastic layer. The reference solution satisfies the prescribed boundary conditions and constitutive relations to near numerical precision. The parameter study identifies field-dependent effects of micropolar coupling, curvature parameter, thickness, and angular frequency. The limiting case $\kappa \rightarrow 0$, $\gamma \rightarrow 0$, $J \rightarrow 0$ gives $\Phi \rightarrow 0$ and recovers classical elasticity, while the complete formulation provides a reproducible reference solution for further numerical investigations.

Table 1. Reference material, geometric, loading, and numerical input parameters.

Parameter	Value	Unit
λ	2.5×10^9	Pa
μ	1.2×10^9	Pa
κ	0.6×10^9	Pa
γ	0.8×10^{-2}	N
ρ	2200	kg/m ³
h	1	m
β	2	m ⁻¹
ω	10	rad/s
$\bar{P}$	1	normalized
η	8.0×10^{-3}	N
J	1.0×10^{-6}	kg·m
P_0	1.0×10^5	Pa
T_0	0	Pa
M_0	0	—

Table 2. Numerical verification of the boundary-value solution using the final reference calculation.

Verification measure	Maximum value
Maximum boundary-condition residual	1.221245×10^{-15}
Physical-field error	1.360×10^{-12}
Constitutive-field error	2.017×10^{-12}
Maximum imaginary component	1.309×10^{-16}

Table 3. Representative verified transformed-domain field magnitudes at selected normalized thickness locations ($\beta=2 \text{ m}^{-1}$, $\omega=10 \text{ rad/s}$, $h=1 \text{ m}$, $\bar{P}=1$).

y/h	U	V	Φ	σ_{xy}	σ_{yy}	myz
0	4.3602354012e-11	3.2379434983e-10	1.6189769079e-10	1.2212453271e-15	1.0000000000	2.8951321439e-27
0.50	4.1300351266e-11	2.4706207422e-10	2.0328538471e-10	3.8282086852e-01	4.3705151861e-01	6.8252751698e-13
1.00	9.8393427039e-11	1.9253666925e-10	9.6268922173e-11	1.1102230246e-16	3.0531133177e-16	1.65013188e-16

VI. Conclusions and Recommendations

6.1 Conclusions

A Fourier Integral Transform analytical and numerical framework has been developed for stationary oscillations in a homogeneous isotropic micropolar elastic layer subjected to harmonic surface loading. The formulation combines the coupled three-field governing system, six characteristic modes, boundary-matrix solution, and inverse reconstruction of displacement, microrotation, stress, and couple-stress fields. Numerical verification through residual-based measures and independent boundary and constitutive checks confirms the consistency of the reconstructed solution. The parametric investigation shows that micropolar coupling affects displacement and shear stress, the curvature parameter mainly influences couple stress, layer thickness modifies field distribution and extrema, and the investigated frequency range produces comparatively small changes. In the classical limiting case $\kappa \rightarrow 0$, $\gamma \rightarrow 0$, and $J \rightarrow 0$, the microrotation vanishes and the classical Navier equations are recovered. The resulting formulation provides a reproducible analytical reference solution for finite micropolar elastic layers.

6.2 Recommendations for Future Work

Future work may consider broader frequency and wave-number ranges, alternative surface-loading conditions, anisotropic and heterogeneous materials, and wider ranges and combinations of microstructural parameters. Further studies may also compare the present analytical reference solution with independent numerical methods and extend the formulation to multilayered, functionally graded, and size-dependent micropolar structures and related engineering applications. [10]

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