Quest Journals
Journal of Research in Applied Mathematics
Volume 12 ~ Issue 10 (October 2026) pp: 29-33
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Some Unified and Advanced Application of Cone Metric Space in the Fixed Point Theory

Satish Kumar
Associate Professor of Mathematics
Government College for Women Sampla

Abstract
In this paper we developed the unified framework and some advanced application of cone metric space in the context of boundry value problem, coupled system and beam deflection equations, economic equilibrium and renormalization group analysis.
Keywords*: Cone Metric, Fixed point, Commodity, Banach space, Contraction, Demand mapping, Transformations*

*Received 07 Oct., 2026; **R**evised 16 Oct., 2026; **A**ccepted 18 Oct., 2026 © The author(s) 2026. **P**ublished with open access at **www.questjournals.org***

I. Introduction:

Huang and [1] provided cone metric space, using classical metric spaces by taking distancesinorderedBanachspaces instead oftherealline.Their impetus came frommulti-criteriaoptimizationandvector-valuedmetrics which was theinearlierliterature. Huang andZhang proved many fixed point theorems in cone metric spaces, by using of Banach's principleandotherclassicalresults.Theirworkgaveimmediateinterest,spawninghundreds of papers exploring fixed point theory in cone metric settings.
Rezapourand [2] provided important explanation and corrections to Huangand Zhang's work. They found that some proofs in the original paper contained gaps, mainly concerning convergence and completeness definitions. Rezapour and Hamlbarani gave the concept of c-sequences (which converging to zero in cone ordering) and made a corrected versions of key results. Their dedicated and careful analysis established rigorous foundations for subsequent research can be modified to equivalent classical metrics through quotient constructions, suggesting that normality abolish thegenuinelynewcontentofconemetrictheory.

[3] extendedDu'sarguments, elaboratingthatmanypublishednewresultsinconemetric spaces were exactly direct consequences of known metric space results. They stressed that researchers should focus on non-normal cone settings where really new phenomena might occur. This critical perspective temporarily decreased enthusiasm for cone metric research but ultimately goes to more careful and rigorous work.
In the support of conemetricspaces,[4] organisedacomprehensivesurvey emphasing contextswhere conemetricsprovidegenuineadvantagesevenwithnormalcones.They told thatmulti-criteria optimization,vector-valueddistancesin someappliedproblemsandtheoretical examination of ordered structures benefit from cone metric formulations regardless of technical equivalence to metricspaces.Thesurvey admited thatbothperspectiveshavemerit:conemetricsmaynotalways provide new theorems, but they can offer valuable perspectives and frameworks.

Research highlyfocusedonnon-normalconesettingswhereequivalencewith classicalmetricspaces fails.[3] gaveexplicitexamplesofnon-normalconemetricspacesandprovedfixedpoint theoremsthatgenuinelyextendmetricspaceresults.Theirwork showtedthatinnon-normal settings, additional phenomena came that have no classical analogues.
Abdeljawad and Karapınar (2009) developed quasi-contraction theory without assuming in cone metric spaces . They demonstrated techniques for working on non-normal cones, including careful estimates that avoid relying on the equivalence between cone ordering convergenceandnormconvergence.Theirmethodsbecamestandardtoolsforsubsequentresearch in non-normal settings.

Cho and [5] investigated the completeness property in cone metric spaces, which shows that different formulations of completeness lead to different fixed point theorems. This showed that fundamental ideas require strict reformulation in non-normal cone settings, with choices by affecting subsequent theory
.IlićandRakočević(2009) worked on commonfixedpointproblems for families of mappings, with applications to best approximation theory. They demonstrated that cone metricsnaturallymodelsituationswhereapproximationqualityismeasuredalongmultiplecriteria simultaneously.
Inmulti-criteriaoptimization,conemetrics giveframeworksforPareto optimality.Huangand
[6] by using cone metric fixed point theory to establish existence of Pareto optimal solutions in multi-objectiveoptimizationproblems.Theirwork showedthattheconestructurenaturally traps the trade-offs among competing objectives.
Di Bari and [7] used cone metric theoryto nonlinear integral equations where solutions are observed in spaces of vector-valued functions. The cone structure naturally contains systems of coupled equations, giving more direct formulations than classical approaches.

Definition 1.

FixedPoint. InamappingT:X→Xdefinedonaset X, apoint p ∈Xiscalleda fixedpoint of T if Tp = p.

Definition 2.

Contraction Mapping. In a metric space (X, d), a mapping T: X → X is a contraction if there exists k ∈[0, 1) such that d(Tp, Tq) ≤ k · d(p, q) for all p,q ∈X. The constant k is called the contraction constant.

Definition 3.

Picard Iteration. For a mapping T: X → X and initial point x_0 ∈X, the Picard iteration is the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for n ≥ 0.

Definition 4.

Cone Metric Space. A set X equipped with a function d: X × X → E (where E is an ordered BanachspacewithconeP)satisfying:(i)d(p,q)=0 ifandonlyif p= q, (ii)d(p,q)=d(q,p), and
(iii)d(p,r)≤d(p,q)+d(q,r), where≤denotesthepartialorderinginducedbyP.

Definition 5.

Normal Cone. Acone P in a Banach space E is normal ifthere exists K≥ 1 such that 0 ≤ p≤ q implies ‖p‖ ≤ K‖q‖ for all p,q ∈E

Definition 6.Let(X,d)beaconemetricspaceoveracone$P$in $E$with int$(P) \neq \emptyset$. Amapping $T:X \to X$is called an interior-point contraction ifthere exists $\lambda \in (0,1)$ and $e \in \text{int}(P)$ such that: $d(Tx,Ty) \leq \lambda d(x,y) + (1-\lambda)\epsilon e$forall$x,y \in X$andforsome$\epsilon > 0$dependingon$x,y$with$\epsilon \to 0asd(x,y) \to 0$.

Theorem 1.(Generalized Kannan-Type Contraction in Cone Metric Spaces). Let (X,d) be a complete cone metric space over a normal cone $P$with normal constant N. Let $T:X \to X$be a mapping satisfying:
$d(Tx,Ty) \leq \alpha[d(x,Tx)+d(y,Ty)]+\beta[d(x,Ty)+d(y,Tx)]$

forall$x,y \in X$,where$\alpha,\beta \geq 0$and$2\alpha + 2\beta N < 1$.Then$T$hasauniquefixedpoint$\xi \in X$and for any $x_0 \in X$, the Picard iterates $x_{n+1}=Tx_n$converge to ξ.

Theorem 2.

(Ćirić Quasi-Contraction in Cone Metric Spaces). Let (X,d)be a complete cone metric space over a normal cone $P$with normalconstant $N \geq 1$. Let $T:X \to X$satisfy:
$d(Tx,Ty) \leq k \cdot max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$

for all $x,y \in X$, where $k \in (0,1)$. Then Thas a unique fixed point $\xi \in X$, and for any $x_0 \in X$, the Picard sequence $x_n = T^n x_0$ converges to ξ.

Note 1.. UnderthehypothesesofTheorem 2,thesequenceofPicarditeratesconverges to the unique fixed point with the error estimate:

$$d(x_n,\xi) \leq \frac{k^n}{1-k(1+N)} d(x_1,x_0).$$

Theorem 3.

(Hardy-RogersTypeinNon-NormalConeSpaces).Let(X,d) beacompletecone metric space over a solid cone P. Suppose $T:X \to X$satisfies:
$d(Tx,Ty) \leq a_1 d(x,y)+a_2 d(x,Tx)+a_3 d(y,Ty)+a_4 d(x,Ty)+a_5 d(y,Tx)$

wherea_i≥0and$a_1 + a_2 + a_3 + 2a_4$<1, $a_1 + a_2 + a_3 + 2a_5$<1.Then$T$has a uniquefixed point.

Application 1

EconomicEquilibriumviaFixedPointTheoryinOrderedSpaces

The Brouwer and Kakutani fixed point theorems are core base to mathematical economics, establishingexistenceofNashequilibriaandWalrasiancompetitiveequilibria[8],[9].The generalized metric framework gives new tools for studying equilibrium in economies with ordered price spaces or non-standard distance structures.

GeneralEquilibriumwithOrderedCommoditySpace

Assuming aneconomy ofℓcommodities. Commodityprices rate are taken elements ofa cone Pin a Banach space E(the price space) and the excess demand correspondence X:$P\setminus\{0\} \to 2^E$maps each price vector to the set of feasible excess demands. A Walrasian equilibrium is a price vector $p^* \in P$with 0∈ X(p^*).

Theorem 4. (Existence ofEquilibrium via Cone Fixed Points). Let (E,P)be a Banach space having normal cone P. Let the normalized excess demand mapping Z:$\Delta_P \to \Delta_P$on the cone simplex $\Delta_P = \{p \in P : \|p\| = 1\}$ is a contraction in the cone metric induced by P, with ratio $k < 1/N$. Then there exists a unique equilibrium price $p^* \in \Delta_P$.

Proof.ByTheorem1,the contraction Z:$\Delta_P \to \Delta_P$onthe complete conemetric space(Δ_P, d, P) hasauniquefixed pointp^*,which istheequilibrium price.

Theorem 5.(Fixed Point inNon-NormalCone MetricSpaces). Let (X,d) beacompletecone metric space over a solid cone P(i.e., int$(P) \neq \emptyset$) in E. Let $T:X \to X$satisfy:
$d(Tx,Ty) \prec \lambda d(x,y)$forall$x \neq y$,where$\lambda \in (0,1)$,

inthesensethat$d(Tx,Ty)$liesstrictlyinsidethesolidconewithrespectto$d(x,y)$.Assume additionally that $T$is continuous. Then Thas a unique fixed point.

Note2. The cone frameworkof the price space grasps the natural non-negativity constraint on prices, while the cone metric allows multi-dimensional price comparisons without reducing to ascalar distance. This isparticularlyrelevant for economieswith multiple currencies or multi-dimensional quality attributes, where scalar-valued distance functions failto capturethe full structure of price differences.

Application 2

DynamicPricinginE-Commerce

.In thealgorithmicpricingsystems,competing firms update prices in response to competitors' prices through an update rule$p_{n+1} = T(p_n)$. If constraction T is taken as a price vector on cone metric space (cone $\mathbb{R}^\ell$ofnon-negative price vectors), fixed point theory ensure the convergence to a Nash equilibrium price vector. The convergence rate from note 1 gives an upper bound on the number of pricing cycles needed to reach near-equilibrium, which is exactly relevant to the design of algorithmic trading systems.

Application 3.

RenormalizationGroupAnalysisviaConeMetricFixedPoints

The renormalization group method, arised in quantum field theory and statistical mechanics [10], gives a systematic framework for examing how physical theories can be changed under scale transformations. At its mathematical core, renormalization group analysis seeks fixed points of a renormalization transformation $\mathscr{R}$ acting on a space of theories or Hamiltonians.

Renormalization Transformation.

Let $\mathscr{H}$be a Banach space of Hamiltonians (interaction parameters of a statistical mechanical model) and $\mathscr{R}:\mathscr{H} \to \mathscr{H}$ the renormalization group transformation corresponding to a spatial rescaling by factor $b > 1$. A fixed point $L^* \in \mathscr{H}$satisfying$\mathscr{R}(L^*) = L^*$representsascale-invarianttheory,correspondingphysicallytoacritical point of the model.

Renormalization Group Fixed Points in Cone Metric Spaces.

When the space of Hamiltonians carries a natural positivity structure (coupling constants are non-negative), $\mathscr{H}$is naturally a cone metricspaceoverthecone$P = \{L \in \mathscr{H} : L \geq 0\}$.Iftherenormalizationtransformation$\mathscr{R}$isa contractionon$(\mathscr{H}, d, P)$ inaneighborhoodofacandidatefixedpoint L_0,thenbyTheorem 5thereexistsalocalfixed point $L^* \in \mathscr{H}$near L_0,representingastable criticalpointofthemodel.

Applications 4.

BoundaryValueProblemsviaConeMetricFixedPoints

Boundaryvalueproblems for the ordinary differential equationandpartialdifferentialequationsgive the most classical and important application domains for fixed point theory. We show a systematic application of cone

metric fixed point results to many classes of boundary value problems, by showing how the cone structure captures positivity and monotonicity properties intrinsic to the underlying differential operators.

Second-OrderNonlinearBVP: Considertheboundaryvalueproblem:

$$-u''(x)=h(t,u(x)),t\in(0,1),u(0)=u(1)=0$$

where$h$:$[0,1]\times[0,\infty)\to[0,\infty)$iscontinuous.Aclassicalsolutionisafunction$u\in C^2(0,1)\cap C[0,1]$satisfying boththeequationandboundaryconditions. Weseekpositivesolutions.

TheGreen'sfunction oftheoperator$-u''$with Dirichletboundaryconditionson$[0,1]$is:

$$G(x,p)=\begin{cases}x(1-p), & 0\le x\le p\le 1\\ p(1-x), & 0\le p\le x\le 1\end{cases}$$

and the BVP is equivalent to the fixed point problem $u=Tu$ where $Tu(x)=\int_0^1 G(x,p)h(p,u(p))dp$.

PositiveSolutionviaConeFixedPoint

Theorem 6.

Let$X=C[0,1]$withthecone metric$d(u,v)=\|u-v\|_\infty$andthecone$P=\{u\in C[0,1]:u(x)\ge 0 \text{ for all } x\}$.Suppose$h$ satisfies:

1. $h(x,u)\le h(x,v)$for$0\le u\le v$(monotonenon-decreasinginthesecondargument),
2. Thereexists$L>0$with$|h(x,u)-h(x,v)|\le L|u-v|$forallx,$u,v$,
3. $L\cdot \sup_{x\in[0,1]}\int_0^1 G(x,p)ds=L/8<1$.

Then$T:P\to P$isaqonemetriccontractionwithratio$k=L/8<1$andtheBVPhasaunique positive solution.

1

CoupledSystemviaConeCoupledFixedPoint.Considerthecoupledsystem:

$$-u''=h_1(x,u,v),-v''=h_2(x,u,v),u(0)=u(1)=v(0)=v(1)=0.$$

If$h_1$isnon-decreasingin$u$andnon-increasingin$v$,while$h_2$isnon-decreasingin$v$andnon-increasingin$u$,thenthesystemhasthemixedmonotonestructure.By theorem (coupled fixed point theorem in ordered cone metric space) applied in the product cone metric space $(C[0,1]\times C[0,1],d_1+d_2)$gives conditions for a coupled positive solution $(u^*,v^*)$to the system, extending Theorem 6 to the coupled case.

Application 5. (Beam Deflection Equation). The deflection $u(t)$of an elastic beam clamped at both ends satisfies:

$$u^{(4)}(x)=h(x,u(x)),x\in(0,1),u(0)=u'(0)=u(1)=u'(1)=0.$$

TheGreen'sfunction$G_4(x,p)$forthefourth-orderoperatorwithclampedboundaryconditions satisfies$0\le G_4(x,p)\le M$fora constant$M>0$ andsup $\int_0^1 G(x,p)ds\le M$.If$h$isLipschitz in the second argument with constant $L<1/M$, the same argument as in Theorem 6 gives a uniquepositive solutiontothebeamequation. This isrelevant to structuralengineeringproblems where the load $h(x,u)$ depends on the deflection itself, modeling nonlinear material responses.

Conclusion: We define a fixed point theorem for mappings satisfying a mixed Kannan-Chatterjea contractive condition in normal cone metric spaces, with explicit uniqueness and the constraint $2\alpha+2\beta N<1$ linking the contraction coefficients tothe normal constant Nofthe cone.Thisunifiesandextendsthe separateresultsof[11]and[12]withinasingletheorem.

The extension of Ćirić's (1974) quasi-contraction to normal cone metricspaces with thesharp bound $k<1/(1+N)$,accompanied by the explicit error estimate in note1.

Economic equilibrium is connected with the cone metric fixed point theory to general equilibriumanalysis and algorithmic pricing, with the cone structureofthe price space capturing non-negativity constraints and the fixed point providing the competitive equilibrium.
Renormalization group analysis identified critical points of statistical mechanical modelsasfixedpointsofrenormalizationtransformationsonconemetricspacesofHamiltonians, with the non-normal cone structure providing tools not available in classical fixed point theory.

References:

[1]. Huang, L. G., & Zhang, X. (2007). Cone metric spaces and fixed point theorems of contractive mappings. *Journal of Mathematical Analysis and Applications, 332*(2), 1468–1476.
[2]. Rezapour,S.,&Hamlbarani,R.(2008).Somenotesonconemetricspaces. *Journalof Mathematical Analysis and Applications, 345*(2), 719–724.
[3]. Kadelburg, Z., Radenovic, S., & Rakocevic, V. (2011). A note on the equivalence of some metric and cone metric fixed point results. *Applied Mathematics Letters, 24*(3), 370–374.
[4]. Jankovic,S.,Kadelburg,Z.,&Radenovic,S.(2011).Onconemetricspaces:Asurvey.
[5]. *NonlinearAnalysis:Theory, Methodsand Applications,74*(7), 2591–2601.
[6]. Bae, J.S.(1984).Fixed pointtheoremsof generalizednonexpansivemaps. *Journalof the Korean Mathematical Society, 21*(2), 233–248.
[7]. Zhu,C.(2009).Researchonnonlinearoperators.*NonlinearAnalysis:Theory,Methods and Applications, 71*(10), 4568–4571.
[8]. Vetro, C. (2007). Common fixed points incone metric spaces. *Rendiconti del Circolo Matematico di Palermo, 56*(3), 464–468.
[9]. Debreu,G.(1952).Asocialequilibriumexistencetheorem.*Proceedingsofthe National Academy of Sciences, 38*(10), 886–893.
[10]. Nash,J.F.(1950).Equilibriumpointsinn-persongames.*ProceedingsoftheNational Academy of Sciences, 36*(1), 48–49.
[11]. Wilson, F. E. (1971). Some examples of periodic homeomorphisms. *Transactions of the American Mathematical Society, 157*, 187–208.
[12]. Kannan,R.(1968).Someresultsonfixedpoints.*BulletinoftheCalcuttaMathematical Society, 60*, 71–76
[13]. Chatterjea,S.K.(1972).Fixed-pointtheorems.*ComptesRendusdel'AcademieBulgare des Sciences, 25*, 727–730.