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Research Paper

On Local Limit Theorems for Free Groups

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Abstract
Richard Sharp [24] obtain a local limit theorem for elements of a free group G*under the abelianization map* $[\cdot] : G \to G/[G,G]$.*It obtained by an analysis involving subshifts of finite type, where he obtain a result of independent interest. The case of fundamental groups of compact surfaces of genus* $g^s \geq 2$ *is also discussed. As an application we raised the elements of* G.

Keywords: *Twisted matrices, Local limit theorem, Free Group, Markov Groups.*

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I. INTRODUCTION

For$G$ denote the free group on $\epsilon \geq 0$ generators $\{a_1^s, \dots, a_{2+\epsilon}^s\}$. For $g^s \in G$, let $|g^s|$ denote its word length, i.e., $|g^s| = \inf\left\{n \geq 0: g^s = g_1^s \cdots g_n^s, g_i^s \in \left\{(a^s)_1^{\pm 1}, \dots, (a^s)_{2+\epsilon}^{\pm 1}\right\}\right\}$, and let $[g^s]$ denote the image of g^s under the abelianization map $[\cdot]: G \to G/[G,G] \cong \mathbb{Z}^{2+\epsilon}$. Let $\mathcal{W}_s(n) = \{g^s \in G: |g^s| = n\}$ and observe that $\#\mathcal{W}_s(n) = 2(2+\epsilon)(2(2+\epsilon)-1)^{n-1}$. We shall be interested in the distribution of the elements of $\mathcal{W}_s(n)$ in $\mathbb{Z}^{2+\epsilon}$by the mapping $[\cdot]$, as $n \to \infty$. In particular, defining $\mathcal{W}_s(n, \alpha^s) = \{g^s \in \mathcal{W}_s(n): [g^s] = \alpha^s\}$, we wish to examine the dependence of $\#\mathcal{W}_s(n, \alpha^s)$ on α^s as well as on n.

We intend to regard $\#\mathcal{W}_s(n, \alpha^s)/\#\mathcal{W}_s(n)$ as a probability distribution on $\mathbb{Z}^{2+\epsilon}$ and to ask about its limiting behaviour as $n \to \infty$. Rivin has shown that a central limit theorem is satisfied, i.e., for $A_s \subset \mathbb{R}^{2+\epsilon}$,

$$\lim_{n\to\infty} \frac{1}{\#\mathcal{W}_s(n)} \#\{g^s \in \mathcal{W}_s(n): [g^s]/\sqrt{n} \in A_s\} = \frac{1}{(2\pi)^{2+\epsilon/2}\sigma^{2+\epsilon}} \int_{A_s} e^{-\|x\|^2/2\sigma^2} dx,$$

where $\|\cdot\|$ denotes the Euclidean norm and where

$$\sigma^2 = \frac{1}{\sqrt{3+2\epsilon}} \left[1 + \left(\frac{2+\epsilon+\sqrt{3+2\epsilon}}{2+\epsilon-\sqrt{3+2\epsilon}}\right)^{\frac{1}{2}}\right] \tag{0.1}$$

[18]. (In fact, this result is similar in spirit to earlier results for subshifts of finite type, hyperbolic diffeomorphisms, and interval maps [1], [4], [5], [10], [12], [17], [19], [20], [23].)

Here, we shall establish a more precise local limit theorem. First we note a combinatorial restriction. We shall say that $\alpha^s = (\alpha_1^s, \dots, \alpha_{2+\epsilon}^s)$ is even if $\alpha_1^s + \cdots + \alpha_{2+\epsilon}^s$ is even, and odd otherwise. It is clear that if $[g^s] = \alpha^s$ then α^s has the same parity as $|g^s|$. Thus, in particular, either $\#\mathcal{W}_s(n, \alpha^s)$ or $\#\mathcal{W}_s(n+1, \alpha^s)$ is equal to zero and we are led to consider the behaviour of the sum

$$\frac{\#\mathcal{W}_s(n, \alpha^s)}{\#\mathcal{W}_s(n)} + \frac{\#\mathcal{W}_s(n+1, \alpha^s)}{\#\mathcal{W}_s(n+1)}.$$

Theorem 1 [24]. Let G be the free group on $\epsilon \geq 0$ generators. Then we have that

$$\lim_{n\to\infty}\sum_{s}\left|\sigma^{2+\epsilon}n^{2+\epsilon/2}\left(\frac{\#\mathcal{W}_s(n,\alpha^s)}{\#\mathcal{W}_s(n)}+\frac{\#\mathcal{W}_s(n+1,\alpha^s)}{\#\mathcal{W}_s(n+1)}\right)-\frac{2}{(2\pi)^{2+\epsilon/2}}e^{-\|\alpha^s\|^2/2\sigma^2 n}\right|=0,$$

uniformly in $\alpha^s \in \mathbb{Z}^{2+\epsilon}$.

In the case where $\alpha^s = 0$, the asymptotic behaviour of $\#\mathcal{W}_s(n,\alpha^s)$, as $n \to \infty$, has been studied as a means of analysing the relative growth series $\xi_s(z)$ defined by

$$\xi_s(z)=\sum_{n=0}^{\infty}\#\mathcal{W}_s(n,0)z^n.$$

Estimates on the growth of $\#\mathcal{W}_s(n,0)$ allow one to deduce that $\xi_s(z)$ cannot be the series of a rational function [8], [16], [22]. More generally, Theorem 1 implies the following result for fixed values of α^s.

Corollary 1.1 [24].

For fixed $\alpha^s \in \mathbb{Z}^{2+\epsilon}$,

$$\#\mathcal{W}_s(2n+\delta_{\alpha^s},\alpha^s)\sim\frac{2}{(2\pi)^{\frac{2+\epsilon}{2}}\sigma^{2+\epsilon}}\frac{\#\mathcal{W}_s(2n+\delta_{\alpha^s})}{n^{\frac{2+\epsilon}{2}}}, \text{ as } n\to\infty,$$

where $\delta_{\alpha^s} = 0$ if α^s is even and $\delta_{\alpha^s} = 1$ if α^s is odd.

Remark [24]. For given functions A_s and B_s, we shall write $A_s(n) \sim B_s(n)$, as $n \to \infty$, if $\lim_{n\to\infty} A_s(n)/B_s(n) = 1$, and $A_s(n) = O(B_s(n))$ if $|A_s(n)| \le (1+\epsilon)B_s(n)$, for some constant $\epsilon \ge 0$.

We see from Corollary 1.1 that the asymptotic behaviour of $\#\mathcal{W}_s(n,\alpha^s)$ is independent of α^s. However, Theorem 1 enables us to make comparisons as α^s varies.

Corollary 1.2 [24]. Suppose that $\alpha^s, \beta \in \mathbb{Z}^{2+\epsilon}$ have the same parity. If $\| \alpha^s \| < \| \beta \|$ then we have that $\#\mathcal{W}_s(n,\alpha^s) > \#\mathcal{W}_s(n,\beta)$ for all sufficiently large n with the same parity as α^s and β.

We say that a word $g_1^s \cdots g_n^s$ in the generators $\{a_1^s, \dots, a_{2+\epsilon}^s\}$ is reduced if $g_{i+1}^s \neq g_i^{-s}$, $i = 1, \dots, n-1$. It is clear that there is a one-to-one correspondence between reduced words of length n and elements of $\mathcal{W}_s(n)$ (and we abuse notation by letting g^s denote both a word and the corresponding group element). We say that a reduced word $g_1^s \cdots g_n^s$ is cyclically reduced if we also have that $g_n^s \neq g_1^{-s}$. Let $\mathcal{C}(n)$ denote the set of cyclically reduced words of length n and let $\mathcal{C}(n,\alpha^s) = \{g^s \in \mathcal{C}(n): [g^s] = \alpha^s\}$. The above theorem still holds if we replace $\#\mathcal{W}_s(n)$ and $\#\mathcal{W}_s(n,\alpha^s)$ by $\#\mathcal{C}(n)$ and $\#\mathcal{C}(n,\alpha^s)$, respectively. (Notice that the map $[\cdot]: \mathcal{C}(n) \to \mathbb{Z}^{2+\epsilon}$ is well-defined.)

We start by some preliminary material concerning subshifts of finite type and thermodynamic formalism. We introduce a family of twisted matrices used in subsequent calculations and analyse their spectra. We prove a local limit theorem associated to periodic points in a subshift of finite type using arguments adapted from [19] (see also [1]). We see that this corresponds directly to the local limit theorem for $\mathcal{C}(n)$ and we give the amendments necessary to obtain Theorem 1. We sketch how our results may be extended to the fundamental groups of compact oriented surfaces of genus $g^s \ge 2$.

II. PRELIMINARIES

ForA_s be a $l \times l$ matrix with entries zero and one and define the associated shift space X_{A_s} by

$$X_{A_s}=\left\{x\in\{0,1,\dots,l-1\}^{\mathbb{Z}^+}: A_s(x_n,x_{n+1})=1\forall n\in\mathbb{Z}^+\right\}.$$

The subshift of finite type $\sigma: X_{A_s} \to X_{A_s}$ is defined by $(\sigma x)_n = x_{n+1}$.

We shall always assume that A_s is aperiodic, i.e., that there exists $N > 0$ such that A_s^N has all its entries positive. This is equivalent to the map $\sigma: X_{A_s} \to X_{A_s}$ being topologically mixing. Then, by the Perron-Frobenius Theorem,

A_s will have a simple positive eigenvalue $\lambda \;\; > 1$ which is strictly maximal in modulus and the topological entropy h of σ is equal to log λ.

Let $\mathcal{M}$ denote the set of σ-invariant probability measures on X_{A_s}. For $m \in \mathcal{M}$, we will write $h(m)$ for its measure theoretic entropy and we have that $h(m) \leq h$. There is a unique measure $\mu \in \mathcal{M}$, called the measure of maximal entropy, for which $h(\mu) = h$. Given a continuous function $\varphi_s: X_{A_s} \to \mathbb{R}$, we define the pressure $P(\varphi_s)$ by $P(\varphi_s) = \sup_{m \in \mathcal{M}} \{h(m) + \int \varphi_s dm\}$. If φ_s is Hölder continuous then there is a unique measure $\mu_{\varphi_s} \in \mathcal{M}$ for which the supremum is attained and we call μ_{φ_s} the equilibrium state of φ_s. Clearly, $\mu_0 = \mu$.

Set $\mathrm{Fix}_n = \{x \in X_{A_s}: \sigma^n x = x\}$. It is well-known and easy to prove that $\#\mathrm{Fix}\ s_n = \mathrm{trace}\ A_s^n \sim (\lambda)^n$, as $n \to \infty$. We shall be interested in the asymptotics of certain subsets ofFix_n.

Fix a function $f_s: X_{A_s} \to \mathbb{Z}^{2+\epsilon}$, such that $f_s(x)$ depends on only finitely many co-ordinates of x. Without loss of generality, we may suppose that $f_s(x)$ depends on only the first two co-ordinates, i.e., that $f_s(x) = f_s(x_0, x_1)$. Write $f_s^n(x) = f_s(x) + f_s(\sigma x) + \cdots + f_s(\sigma^{n-1}x)$. For $\alpha^s \in \mathbb{Z}^{2+\epsilon}$, consider the subset $\{x \in \mathrm{Fix}_n: f_s^n(x) = \alpha^s\}$ of Fix_n; we shall be interested in the asymptotics of the cardinality of this set as n and α^s vary.

In order to make progress, we need to assume that f_s satisfies the following two natural conditions.

(A1) The set $\bigcup_{n=1}^{\infty}\{f_s^n(x): x \in \mathrm{Fix}_n\}$ generates $\mathbb{Z}^{2+\epsilon}$ (i.e. it is not contained in a proper subgroup of $\mathbb{Z}^{2+\epsilon}$).

(A2) $\int f_s dm = 0$, where m is some fully supported σ-invariant measure. If condition (A2) holds then it was shown in [15] that we may choose m to be equal to $\mu_{\langle \xi_s, f_s \rangle}$, for some (unique) $\xi_s \in \mathbb{R}^{2+\epsilon}$. Furthermore, in this case we have

$$0 < h^* := h(\mu_{\langle \xi_s, f_s \rangle}) = P(\langle \xi_s, f_s \rangle) = \sup \left\{ h(m): \int \sum_s f_s dm = 0, m \in \mathcal{M} \right\}.$$

A subgroup of $\mathbb{Z}^{2+\epsilon}$, familiar from the coding theory of subshifts of finite type, will play an important rôle in our subsequent analysis. We define

$$\Delta_{f_s} = \bigcup_{n=1}^{\infty} \{f_s^n(x) - f_s^n(y): x, y \in \mathrm{Fix}_n\}.$$

Choose $x \in \mathrm{Fix}_n$ and $y \in \mathrm{Fix}_{n+1}$ (for some fixed$n$) and set $c_{f_s} = f_s^{n+1}(x) - f_s^n(y)$. Then the coset $\Delta_{f_s} + c_{f_s}$ is well-defined and $\mathbb{Z}^{2+\epsilon}/\Delta_{f_s}$ is the cyclic group generated by $\Delta_{f_s} + c_{f_s}$[14]. Conditions (A1) and (A2) ensure that $\mathbb{Z}^{2+\epsilon}/\Delta_{f_s}$ is finite and we write $d = |\mathbb{Z}^{2+\epsilon}/\Delta_{f_s}|$[13].

Remark [24]. At first sight, it is not clear that Δ_{f_s} is a group or, more precisely, that it is closed under addition: we shall give a proof of this fact. It is convenient to consider the directed graph with vertices $\{0,1,\ldots,l-1\}$ and an edge joining i to j if and only if $A_s(i,j) = 1$. Then elements of Fix_n correspond to cycles in the graph and $f_s^n(x)$ to the sum of f_s around the edges. For a cycle γ, we shall denote this sum by $f_s(\gamma)$ and the length of γ by $l(\gamma)$. Since A_s is aperiodic there exists $N \geq 1$ such that, for each pair of vertices (i,j), we can choose a path $\delta(i,j)$ of length N joining i to j. Now choose a vertex i_0 and, for every cycle γ, a vertex $i_\gamma \in \gamma$. For each cycle γ form a new cycle $\bar{\gamma}$ passing through i_0 by $\bar{\gamma} = \delta(i_0, i_\gamma)\gamma\delta(i_\gamma, i_0)$. Let $f_s(\gamma) - f_s(\gamma')$ and $f_s(\eta) - f_s(\eta')$ be two arbitrary elements of Δ_{f_s}, where $\gamma, \gamma', \eta, \eta'$ are cycles with $l(\gamma) = l(\gamma')$ and $l(\eta) = l(\eta')$. Then $\overline{\gamma\eta}$ and $\bar{\gamma}'\bar{\eta}'$ are cycles, $l(\overline{\gamma\eta}) = l(\bar{\gamma}'\bar{\eta}')$ and

$$\big(f_s(\gamma) - f_s(\gamma')\big) + \big(f_s(\eta) - f_s(\eta')\big) = f_s(\bar{\gamma}\bar{\eta}) - f_s(\bar{\gamma}'\bar{\eta}').$$

This shows that Δ_{f_s} is closed under addition.

We show (and in closely related situations) a variety of central limit theorems have been established. In particular, in [4], a central limit theorem over periodic points is obtained and the rate of convergence isestimated. However [24], we concentrate on local limit theorems; more precisely we seek to obtain estimates on

$$\sum_{j=0}^{d} \frac{e^{(h-h^*)n} n^{2+\epsilon/2}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j}: f_s^{n+j}(x) = \alpha^s\},$$

as $n \to \infty$, which are uniform in $\alpha^s \in \mathbb{Z}^{2+\epsilon}$. (The summation is required since $\{x \in \mathrm{Fix}_{n+j}: f_s^{n+j}(x) = \alpha^s\} \neq \emptyset$ for a unique $j \in \{0,1,\dots,d-1\}$, depending on the coset of α^s in $\mathbb{Z}^{2+\epsilon}/\Delta_{f_s}$.) This kind of problem has been addressed in [11] (following an idea of Sinai) and [19] (see also [1]) but the conditions imposed there are too stringent for our purposes.

III. TWISTED MATRICES

In order to analyse the behaviour of $\#\{x \in \mathrm{Fix}_n: f_s^n(x) = \alpha^s\}$, we shall introduce a family of twisted $l \times l$ matrices $(A_s)_t$, indexed by $t \in \mathbb{R}^{2+\epsilon}/2\pi\mathbb{Z}^{2+\epsilon}$. Define $(A_s)_t$ by

$$(A_s)_t(i,j) = A_s(i,j) e^{i\langle t, f_s(i,j)\rangle + \langle \xi_s, f_s\rangle},$$

where the Right Hand Side is understood to be zero when $A_s(i,j) = 0$. In particular, $(A_s)_0$ is an aperiodic positive matrix. An easy calculation shows that

$$\text{trace } (A_s)_t^n = \sum_{x \in \mathrm{Fix}_n} \sum_s e^{i\langle t, f_s^n(x)\rangle + \langle \xi_s, f_s^n(x)\rangle}.$$

In order to estimate this quantity, we need to analyse the eigenvalues of $(A_s)_t$.

The matrix $(A_s)_t$ will have l eigenvalues which we denote by $\lambda_1^s(t), \dots, \tilde{\lambda}_l(t)$ with $|\tilde{\lambda}_1^s(t)| \geq |\tilde{\lambda}_2(t)| \geq \cdots \geq |\tilde{\lambda}_l(t)|$. The classical Perron-Frobenius Theorem ensures that $\lambda_{\xi_s} = \tilde{\lambda}_1^s(0)$ is simple and positive and that the remaining eigenvalues of $(A_s)_0$ are strictly smaller in modulus than λ_{ξ_s}. Furthermore, $P(\langle \xi_s, f_s\rangle) = \log \lambda_{\xi_s}$ and $\lambda_{\xi_s} < \lambda$ unless $\xi_s = 0$. In subsequent calculations it will prove more convenient to work with the quantities $\lambda_j(t) = \tilde{\lambda}_j(t)/\lambda_{\xi_s}, j = 2, \dots, l$. We will need to understand when $|\lambda_1^s(t)|$ is maximised.

Proposition 1 (see [24]).

(i) We have that $|\lambda_1^s(t)| \leq 1$ for all $t \in \mathbb{R}^{2+\epsilon}/2\pi\mathbb{Z}^{2+\epsilon}$. Furthermore, if $|\lambda_1^s(t)| = 1$ then $\tilde{\lambda}_1^s(t)$ is simple and $|\lambda_j(t)| < 1, j = 2, \dots, l$.

(ii) We have the two identities

$$\{e^{2\pi i\langle t,\cdot\rangle}: |\lambda_1^s(t)| = 1\} = \Delta_{f_s}^{\perp},$$
$$\{\lambda_1^s(t): e^{2\pi i\langle t,\cdot\rangle} \in \Delta_{f_s}^{\perp}\} = \{e^{2\pi i r/d}: r = 0,1,\dots,d-1\}.$$

Proof. Part (i) is part of Wielandt's Theorem [6, p. 57]. Part (ii) is proved in [15].

We shall write $t^{(r)}$ for the unique value of t satisfying $\lambda_1^s(t^{(r)}) = e^{2\pi i r/d}$. For (small) $\delta > 0$, we define a neighbourhood of $t^{(0)} = 0 \in \mathbb{R}^{2+\epsilon}/2\pi\mathbb{Z}^{2+\epsilon}$ by $U_0(\delta) = \{t: \| t \| \leq \delta\}$ and let $U_r(\delta) = U_0(\delta) + t^{(r)}$ for $r = 1,2,\dots,d-1$. A simple calculation shows that, for $t \in U_r(\delta)$,

$$\lambda_1^s(t) = e^{\frac{2\pi i r}{d}} \lambda_1^s(t - t^{(r)}) \quad (2.1)$$

([15]). In particular, for $r = 1,2,\dots,d-1$ and $n \geq 1$,

$$\sum_{j=0}^{d-1} \lambda_1^s\left(t^{(r)}\right)^{n+j} = 0. \tag{2.2}$$

If w_t is the right eigenvector for $(A_s)_t$ corresponding to the eigenvalue $\tilde{\lambda}_1^s(t)$ then, for $t \in U_r(\delta)$, we also have $w_t = w_{t-t(r)}$. Since $\tilde{\lambda}_1^s\left(t^{(r)}\right)$ is an isolated simple eigenvalue of $(A_s)_{t^{(r)}}$, eigenvalue perturbation theory ensures that $\lambda_1^s(t)$ and w_t depend analytically on t in $U_r(\delta)$[9].

In view of the above discussion, we have the following estimates on $\lambda_j\ (t)$. For all sufficiently small $\delta > 0$ there exists $0 < \theta < 1$ such that

(i) $\left|\lambda_j\ (t)\right| \leq \theta$ for all $t \in \cup_{r=0}^{d-1} U_r(\delta), j = 2, \dots, l$;

(ii) $\left|\lambda_j\ (t)\right| \leq \theta$ for all $t \notin \cup_{r=0}^{d-1} U_r(\delta), j = 2, \dots, l$.

The following result is standard (cf. [15] for example).

Lemma 1 [24]. Assume that f_s satisfies (A1) and (A2). Then the gradient $\nabla\lambda_1^s(0) = 0$ and the Hessian matrix $\nabla^2\lambda_1^s(0)$ is real and strictly negative definite.

From now on, we shall write $\mathcal{D}_{\xi_s} = -\nabla^2\lambda_1^s(0)$, so that $\mathcal{D}_{\xi_s}$ is strictly positive definite. In particular, det $\mathcal{D}_{\xi_s} > 0$ and we define $\sigma_{\xi_s} > 0$ by $\sigma_{\xi_s}^{2(2+\epsilon)} = \det\ \mathcal{D}_{\xi_s}$. The following result on the limiting behaviour of $\lambda_1^s(t)$ appears in several places, e.g. [4], [19].

Proposition 2(see [24]). There exists $\delta > 0$ such that, for $t \in U_0\left(\delta\sigma_{\xi_s}\sqrt{n}\right)$,

$$\lim_{n\to\infty} \lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^n = e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2}.$$

Furthermore,

$$\left|\sum_s \lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^n - e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2}\right| \leq 2\sum_s e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/4\sigma_{\xi_s}^2}.$$

Proof. Recall that $\nabla\lambda_1^s(0) = 0$. Since, in a neighbourhood of $0, \lambda_1^s(t)$ depends analytically on t, we may apply Taylor's Theorem to write

$$\lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right) = 1 - \frac{\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle}{2\sigma_{\xi_s}^2 n} + O\left(\| t \|^3/n^{3/2}\right).$$

The first part of the result now follows from the standard formula $\lim_{n\to\infty}(1 - x/n)^n = e^{-x}$.

For the second part, notice that, provided δ is sufficiently small, for $\| u \| \leq \delta$ we have

$$\frac{\left\langle u,\mathcal{D}_{\xi_s}u\right\rangle}{2} + O(\| u \|^3) \geq \frac{\left\langle u,\mathcal{D}_{\xi_s}u\right\rangle}{4}.$$

Applying the triangle inequality and the inequality $(1 - x/n)^n < e^{-x}$, we have

$$\begin{aligned}\left|\sum_s \lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^n - \sum_s e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2}\right| &\leq \sum_s e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/4\sigma\xi_s^2} + \sum_s e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2} \\ &\leq 2\sum_s e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/4\sigma_{\xi_s}^2}.\end{aligned}$$

IV. A LOCAL LIMIT THEOREM FOR SUBSHIFTS

We shall obtain a local limit theorem for the function $f_s: X_{A_s} \to \mathbb{Z}^{2+\epsilon}$ with respect to the periodic points of $\sigma: X_{A_s} \to X_{A_s}$. We shall examine the quantity

$$\mathcal{S}(n, \alpha^s) = \sum_{j=0}^{d-1} \sum_s \frac{e^{-\langle \xi_s, \alpha^s \rangle} \sigma_{\xi_s}^{2+\epsilon} n^{2+\epsilon/2} \left(\lambda/\lambda_{\xi_s}\right)^{n+j}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j}: f_s^{n+j}(x) = \alpha^s\}.$$

For $\alpha^s > 0$, write $I(\alpha^s) = [-\alpha^s, \alpha^s]^{2+\epsilon}$. Using the orthogonality relationship

$$\frac{1}{(2\pi)^{2+\epsilon}} \int_{I(\pi)} \sum_s e^{-i\langle t, \alpha^s \rangle} e^{i\langle t, y \rangle} dt = \begin{cases} 1 \text{ if } y = \alpha^s \\ 0 \text{ otherwise} \end{cases},$$

we have that

$$\mathcal{S}(n, \alpha^s) = \frac{1}{(2\pi)^{2+\epsilon}} \sum_{j=0}^{d-1} \sum_s \frac{\sigma_{\xi_s}^{2+\epsilon} n^{\frac{2+\epsilon}{2}} \left(\frac{\lambda}{\lambda_{\xi_s}}\right)^{n+j}}{\#\mathrm{Fix}_{n+j}} \int_{I(\pi)} e^{-i\langle t, \alpha^s \rangle} \sum_{x \in \mathrm{Fix}_{n+j}} e^{i\left\langle t, f_s^{n+j}(x) \right\rangle} dt.$$

Making the substitution $t \mapsto t/\sigma_{\xi_s}\sqrt{n}$, we obtain

$$\mathcal{S}(n, \alpha^s) = \frac{1}{(2\pi)^{2+\epsilon}} \sum_{j=0}^{d-1} \int_{I\left(\pi\sigma_{\xi_s}\sqrt{n}\right)} \sum_s e^{-i\langle t, \alpha^s \rangle/\sigma_{\xi_s}\sqrt{n}} \frac{\left(\lambda/\lambda_{\xi_s}\right)^{n+j}}{\#\mathrm{Fix}_{n+j}} \sum_{x \in \mathrm{Fix}_{n+j}} e^{i\left\langle t, f_s^{n+j}(x) \right\rangle/\sigma_{\xi_s}\sqrt{n}} dt.$$

We prove the following theorem.

Theorem 2(see [24]). Suppose that $f_s: X_{A_s} \to \mathbb{Z}^{2+\epsilon}$ satisfies conditions (A1) and (A2). Then

$$\lim_{n\to\infty} \sum_s \left| \sum_{j=0}^{d-1} \frac{\sigma_{\xi_s}^{2+\epsilon} n^{2+\epsilon/2} \left(\lambda/\lambda_{\xi_s}\right)^{n+j}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j}: f_s^{n+j}(x) = \alpha^s\} - \frac{d e^{\langle \xi_s, \alpha^s \rangle}}{(2\pi)^{2+\epsilon/2}} e^{-\left\langle \alpha^s, \mathcal{D}_{\xi_s}^{-1} \alpha^s \right\rangle/2n} \right| = 0,$$

uniformly in $\alpha^s \in \mathbb{Z}^{2+\epsilon}$.

Proof. Using the identity (valid for any positive definite Hermitian matrix $\mathcal{D}_{\xi_s}$),

$$e^{-\left\langle \alpha^s, \mathcal{D}_{\xi_s}^{-1} \alpha^s \right\rangle/2n} = \frac{1}{(2\pi)^{2+\epsilon/2}} \int_{\mathbb{R}^{2+\epsilon}} \sum_s e^{-i\langle t, \alpha^s \rangle/\sigma_{\xi_s}\sqrt{n}} e^{-\left\langle t, \mathcal{D}_{\xi_s} t \right\rangle/2\sigma_{\xi_s}^2} dt,$$

we have established the bound

$$(2\pi)^{2+\epsilon} \left| \sum_{j=0}^{d-1} \sum_s \frac{e^{-\langle \xi_s, \alpha^s \rangle} \sigma_{\xi_s}^{2+\epsilon} n^{2+\epsilon/2} \gamma^{n+j}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j}: f_s^{n+j}(x) = \alpha^s\} - \frac{d e^{-\left\langle \alpha^s, \mathcal{D}_{\xi_s}^{-1} \alpha^s \right\rangle/2n}}{(2\pi)^{2+\epsilon/2}} \right| \leq$$

$$\sum_{s}\left|\int_{U_0\left(\delta\sigma_{\xi_s}\sqrt{n}\right)} e^{-i\langle t,\alpha^s\rangle/\sigma_{\xi_s}\sqrt{n}}\left\{\sum_{j=0}^{d-1}\frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}}\sum_{x\in\mathrm{Fix}_{n+j}} e^{i\left\langle t,f_s^{n+j}(x)\right\rangle/\sigma_{\xi_s}\sqrt{n}}-de^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2}\right\}dt\right|$$
$$+\sum_{s}\left|\int_{I\left(\pi\sigma_{\xi_s}\sqrt{n}\right)\setminus U_0\left(\delta\sigma_{\xi_s}\sqrt{n}\right)} e^{-i\langle t,\alpha^s\rangle/\sigma_{\xi_s}\sqrt{n}}\sum_{j=0}^{d-1}\frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}}\sum_{x\in\mathrm{Fix}_{n+j}} e^{i\left\langle t,f_s^{n+j}(x)\right\rangle/\sigma_{\xi_s}\sqrt{n}}\right|$$
$$+\sum_{s}\left|\int_{\mathbb{R}^{2+\epsilon}\setminus U_0\left(\delta\sigma_{\xi_s}\sqrt{n}\right)} de^{-i\langle t,\alpha^s\rangle/\sigma_{\xi_s}\sqrt{n}}e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2}dt\right|$$
$$=(A_s)_1(n,\alpha^s)+(A_s)_2(n,\alpha^s)+(A_s)_3(n,\alpha^s),$$

where $\gamma=\lambda/\lambda_{\xi_s}$. An easy calculation shows that $\lim_{n\to\infty}\sup_{\alpha^s\in\mathbb{Z}^{2+\epsilon}}(A_s)_3(n,\alpha^s)=0$, so it remains to consider $(A_s)_1$ and $(A_s)_2$.

For $t\in U_0\left(\delta\sigma_{\xi_s}\sqrt{n}\right)$, we have that

$$\sum_{j=0}^{d-1}\frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}}\sum_{x\in\mathrm{Fix}_{n+j}}\sum_{s} e^{i\left\langle t,f_s^{n+j}(x)\right\rangle/\sigma_{\xi_s}\sqrt{n}}=\sum_{s}\lambda_1^s\left(\frac{t}{\sigma_{\xi_s\sqrt{n}}}\right)^n\sum_{j=0}^{d-1}\lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^j+O(\theta^n).$$

and that

$$\left|\sum_{j=0}^{d-1}\sum_{s}\lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^j-d\right|\le(1+\epsilon)\delta^2,$$

for some constant $\epsilon\ge 0$. By Proposition 2 , we know that $\lambda_1^s\left(t/\sigma_{\xi_s}\sqrt{n}\right)^n$ converges uniformly to $e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2}$, as $n\to\infty$. Furthermore, we have the estimates

$$\left|\sum_{s}d\lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^n-\sum_{s}de^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/2\sigma_{\xi_s}^2}\right|\le\sum_{s}2de^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/4\sigma_{\xi_s}^2}$$

and

$$\mathrm{l}\sum_{s}\lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^n\left\{\sum_{j=0}^{d-1}\lambda_1^s\left(\frac{t}{\sigma_{\xi_s}\sqrt{n}}\right)^j-d\right\}\le(1+\epsilon)\sum_{s}e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/4\sigma_{\xi_s}^2\delta^2}.$$

Thus, by the Dominated Convergence Theorem, we obtain

$$\limsup_{n\to\infty}\sup_{\alpha^s\in\mathbb{Z}^{2+\epsilon}}\sum_{s}(A_s)_1(n,\alpha^s)\le(1+\epsilon)\left\{\int_{\mathbb{R}^{2+\epsilon}}\sum_{s}e^{-\left\langle t,\mathcal{D}_{\xi_s}t\right\rangle/4\sigma_{\xi_s}^2}dt\right\}\delta^2.$$

Finally, we consider $(A_s)_2$. If $t\notin\bigcup_{r=1}^{d-1}U_r\left(\delta\sigma_{\xi_s}\sqrt{n}\right)$, then

$$\sum_{j=0}^{d-1}\frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}}\sum_{x\in\mathrm{Fix}_{n+j}}\sum_{s}e^{i\left\langle t,f_s^{n+j}(x)\right\rangle/\sigma_{\xi_s}\sqrt{n}}=O(\theta^n).$$

On the other hand, if $t\in\bigcup_{r=1}^{d-1}U_r\left(\delta\sigma_{\xi_s}\sqrt{n}\right)$, then

$$\left|\sum_{j=0}^{d-1} \frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}} \sum_{x\in \mathrm{Fix}_{n+j}} \sum_{s} e^{i\left\langle t, f_s^{n+j}(x)\right\rangle/\sigma_{\xi_s}\sqrt{n}}\right|$$

$$= \left|\sum_{j=0}^{d-1} \sum_{s} e^{2\pi i r(n+j)/d} \lambda_1^s \left(\frac{t}{\sigma_{\xi_s}\sqrt{n}} - t^{(r)}\right)^{n+j}\right| + O(\theta^n)$$
$$\le \left(\frac{1+\epsilon}{\epsilon}\right) \sum_{s} e^{-\left\langle t', \mathcal{D}_{\xi_s} t'\right\rangle/4\sigma^2} \delta^2 + O(\theta^n),$$

for some constant $\epsilon > -1$ and where $t' = t - \sigma_{\xi_s}\sqrt{n}t^{(r)}$, the last estimate following from (2.2), the analyticity of λ_1^s and the vanishing of its first derivatives. This gives us

$$\limsup_{n\to\infty} \sup_{\alpha^s \in \mathbb{Z}^{2+\epsilon}} \sum_{s} (A_s)_2(n, \alpha^s) \le \left(\frac{1+\epsilon}{\epsilon}\right) \left\{\int_{\mathbb{R}^{2+\epsilon}} \sum_{s} e^{-\left\langle t, \mathcal{D}_{\xi_s} t\right\rangle/4\sigma_{\xi_s}^2} dt\right\} \delta^2.$$

Combining the above estimates we have

$$\limsup_{n\to\infty} \sup_{\alpha^s \in \mathbb{Z}^{2+\epsilon}} \sum_{s} \left| \sum_{j=0}^{d-1} \frac{e^{-\langle \xi_s, \alpha^s\rangle} \sigma_{\xi_s}^{2+\epsilon} n^{2+\frac{\epsilon}{2}} \gamma^{n+j}}{\#\mathrm{Fix}_{n+j}} \#\left\{x \in \mathrm{Fix}_{n+j}: f_s^{n+j}(x) = \alpha^s\right\} - \frac{d e^{-\frac{\left\langle \alpha^s, \mathcal{D}_{\xi_s}^{-1} \alpha^s\right\rangle}{2n}}}{(2\pi)^{2+\frac{\epsilon}{2}}} \right|$$
$$\le \frac{(\epsilon^2 + 2\epsilon + 1)}{\epsilon (2\pi)^{2+\epsilon}} \left\{\int_{\mathbb{R}^{2+\epsilon}} \sum_{s} e^{-\left\langle t, \mathcal{D}_{\xi_s} t\right\rangle/4\sigma_{\xi_s}^2} dt\right\} \delta^2.$$

Since this holds for all sufficiently small $\delta > 0$, the proof of the theorem is complete.

We state the special case where $\xi_s = 0$ as a corollary. Here we write $\mathcal{D}_0 = \mathcal{D}$ and $\sigma_0 = \sigma$.

Corollary 2.1 [24]. Suppose that $f_s: X_{A_s} \to \mathbb{Z}^{2+\epsilon}$ satisfies condition (A1) and $\int f_s d\mu = 0$, where μ is the measure of maximal entropy. Then

$$\lim_{n\to\infty} \left| \sum_{j=0}^{d-1} \frac{\sigma^{2+\epsilon} n^{2+\epsilon/2}}{\#\mathrm{Fix}_{n+j}} \#\left\{x \in \mathrm{Fix}_{n+j}: f_s^{n+j}(x) = \alpha^s\right\} - \frac{d}{(2\pi)^{2+\epsilon/2}} e^{-\langle \alpha^s, \mathcal{D}^{-1} \alpha^s\rangle/2n} \right| = 0,$$

uniformly in $\alpha^s \in \mathbb{Z}^{2+\epsilon}$

Remark [24]. In particular, we have recovered the main result of [15], namely that $\#\{x \in \mathrm{Fix}_{dn}: f_s^{dn}(x) = 0\} \sim (1+\epsilon)(\lambda)_{\xi_s}^{dn}/n^{2+\epsilon/2}$, as $n \to \infty$, for some constant $\epsilon \ge 0$. However, the above method does not allow us to estimate the error term in this approximation. (The $O\left(n^{-1/2}\right)$ error estimate claimed there is erroneous and needs to be corrected to $O\left(n^{-1/2+\epsilon}\right)$. Conjecturally, the optimal error estimate is $O(n^{-1})$.)

V. FREE GROUPS

We shall deduce Theorem 1 from Theorem 2 and give an explicit expression for the matrix $\mathcal{D}$. Let G be the free group on $\epsilon \ge 0$ generators. Define a $(2(2+\epsilon)+1) \times (2(2+\epsilon)+1)$ matrix A_s, indexed by $\{*, 1, 2, \dots, 2(2+\epsilon)\}$, by $A_s(*,*) = 0, A_s(*, j) = 1$ for all $j = 1, 2, \dots, 2(2+\epsilon)$, $A_s(i, *) = 0$ for all $i = 1, 2, \dots, 2(2+\epsilon)$, and, for $i, j = 1, 2, \dots, 2(2+\epsilon)$,

$$A_s(i,j) = \begin{cases} 1 \text{ if } j \ne i + 2 + \epsilon (\mathrm{mod} 2(2+\epsilon)) \\ 0 \text{ if } j = i + 2 + \epsilon (\mathrm{mod} 2(2+\epsilon)) \end{cases}.$$

Then the maximal eigenvalue λ of A_s is equal to $2(2+\epsilon)-1$. Let B_s denote the $2(2+\epsilon)\times 2(2+\epsilon)$ submatrix of A_s indexed by $\{1,2,\dots,2(2+\epsilon)\}$; it is easy to check that B_s is aperiodic and that $\bigcup_{n\geq 1}\mathrm{Fix}_n \subset X_{B_s}$. If we index the generators of G by $\{a_1^s,\dots,a_{2+\epsilon}^s, a_{3+\epsilon}^s = a_1^{-s},\dots,a_{2(2+\epsilon)}^s = a_{2+\epsilon}^{-s}\}$, then it is clear that there is a natural bijection between cyclically reduced words of length n in G and elements of Fix_n, and between reduced words of length n and all sequences of the form $(x_0,x_1,\dots,x_n)$ with $x_0 = *$ and $A_s(x_m,x_{m+1}) = 1, m = 1,\dots,n-1$. In particular,$\#\mathcal{W}_s(n) = \langle u, A_s^n v\rangle$, where $u = (1,0,\dots,0)$ (with the 0 occurring in the $*$ position) and $v = (1,1,\dots,1)$, and that $\#\mathcal{C}(n) = \text{trace } A_s^n$.

If we define a function $f_s\colon X_{A_s}\to \mathbb{Z}^{2+\epsilon}$ by $f_s(i,j) = [a_j^s]$ then it is easy to see that the element of $\mathbb{Z}^{2+\epsilon}$ corresponding to the cyclically reduced word associated to $x\in \mathrm{Fix}_n$ is $f_s^n(x)$. In particular, $\#\mathcal{C}(n,\alpha^s) = \#\{x\in \mathrm{Fix}_n\colon f_s^n(x) = \alpha^s\}$ and

$$\bigcup_{n\geq 1}\{f_s^n(x)\colon x\in \mathrm{Fix}_n\} = \bigcup_{n\geq 1}\{[g^s]\colon g^s\in \mathcal{C}(n)\} = \mathbb{Z}^{2+\epsilon}.$$

This last identity implies that the restriction $f_s\colon X_{B_s}\to \mathbb{Z}^{2+\epsilon}$ satisfies condition (A1).

If μ denotes the measure of maximal entropy on X_{B_s} then it is well-known that the periodic points of $\sigma\colon X_{B_s}\to X_{B_s}$ are equidistributed with respect to μ. More precisely, we have the identity

$$\int \sum_s f_s d\mu = \lim_{n\to\infty}\frac{1}{\#\mathrm{Fix}_n}\sum_{x\in \mathrm{Fix}_n}\sum_s \frac{f_s^n(x)}{n}.$$

The symmetry $[g^{-s}] = -[g^s]$ then shows that we have $\int f_s d\mu = 0$. A simple calculation shows that Δ_{f_s} is the subgroup of $\mathbb{Z}^{2+\epsilon}$ consisting of all even elements, so that $d = |\mathbb{Z}^{2+\epsilon}/\Delta_{f_s}| = 2$.

The following result now follows immediately from Corollary 2.1. A simple symmetry argument shows that the covariance matrix $\mathcal{D}$ is diagonal, $\mathcal{D} = \mathrm{diag}\,(\sigma^2,\dots,\sigma^2)$, say, and the explicit formula for σ^2 given by (0.1) is due to [18].

Proposition 3 [24].

$$\lim_{n\to\infty}\sup_{\alpha^s\in\mathbb{Z}^{2+\epsilon}}\sum_s \left|\sigma^{2+\epsilon}n^{2+\epsilon/2}\left(\frac{\#\mathcal{C}(n,\alpha^s)}{\#\mathcal{C}(n)} + \frac{\#\mathcal{C}(n+1,\alpha^s)}{\#\mathcal{C}(n+1)}\right) - \frac{2}{(2\pi)^{2+\epsilon/2}}e^{-\|\alpha^s\|^2/2\sigma^2 n}\right| = 0.$$

Proof of Theorem 1. We shall now discuss the modifications necessary to prove the result for $\mathcal{W}_s(n)$. For $t\in \mathbb{R}^{2+\epsilon}/2\pi\mathbb{Z}^{2+\epsilon}$, we introduce matrices $(A_s)_t, (B_s)_t$ defined by $(A_s)_t(i,j) = A_s(i,j)e^{i\langle t, f_s(i,j)\rangle}$ and $(B_s)_t(i,j) = B_s(i,j)e^{i\langle t, f_s(i,j)\rangle}$. A simple calculation shows that $(A_s)_t$ has the same non-zero spectrum as $(B_s)_t$. Since B_s is aperiodic and $f_s\colon X_{B_s}\to\mathbb{Z}^{2+\epsilon}$ satisfies (A1) and (A2), the maximal eigenvalue $\tilde{\lambda}_1^s(t)$ continues to enjoy the properties described in Section 2.

We note that $\#\mathcal{W}_s(n) = 2(2+\epsilon)(\lambda)^{n-1}$ and that

$$\begin{aligned}\#\mathcal{W}_s(n,\alpha^s) = \sum_{g^s\in\mathcal{W}_s(n)}\sum_s \frac{1}{(2\pi)^{2+\epsilon}}\int_{I(\pi)} e^{-i\langle t,\alpha^s\rangle}e^{i\langle t,[g^s]\rangle}dt &= \sum_{j=1}^{2(2+\epsilon)}\frac{1}{(2\pi)^{2+\epsilon}}\int_{I(\pi)}\sum_s v e^{-i\langle t,\alpha^s\rangle}(A_s)_t^n(*,j)dt \\ &= \frac{1}{(2\pi)^{2+\epsilon}}\int_{I(\pi)}\sum_s e^{-i\langle t,\alpha^s\rangle}\langle u,(A_s)_t^n v\rangle dt.\end{aligned}$$

For $t\in U_r(\delta)$, we have

$$\langle u,(A_s)_t^n v\rangle = (-1)^r\tilde{\lambda}_1^s\big(t-t^{(r)}\big)^n\langle u, w_{t-t^{(r)}}\rangle + O((\theta\lambda)^n),$$

where w_t is the eigenprojection of v for $(A_s)_t$ associated to the eigenvalue $\tilde{\lambda}_1^s(t)$. It is easy to see that $w_0 = (2(2+\epsilon)/(2(2+\epsilon)-1),1,\dots,1)$.

Applying the analysis of the preceding section to

$$\sigma^{2+\epsilon}n^{2+\epsilon/2}\left(\frac{\#\mathcal{W}_s(n,\alpha^s)}{\#\mathcal{W}_s(n)}+\frac{\#\mathcal{W}_s(n+1,\alpha^s)}{\#\mathcal{W}_s(n+1)}\right),$$

we obtain

$$\begin{aligned}
&(2\pi)^{2+\epsilon}\left|\sigma^{2+\epsilon}n^{2+\epsilon/2}\sum_s \left(\frac{\#\mathcal{W}_s(n,\alpha^s)}{\#\mathcal{W}_s(n)}+\frac{\#\mathcal{W}_s(n+1,\alpha^s)}{\#\mathcal{W}_s(n+1)}\right)-\frac{2e^{-\|\alpha^s\|^2/2\sigma^2 n}}{(2\pi)^{2+\epsilon/2}}\right| \\
&\le \sum_s \left|\int_{U_0(\delta\sigma\sqrt{n})} e^{-i\langle t,\alpha^s\rangle/\sigma\sqrt{n}}\left\{\frac{\left\langle u,(A_s)^n_{t/\sigma\sqrt{n}}v\right\rangle}{\#\mathcal{W}_s(n)}+\frac{\left\langle u,(A_s)^{n+1}_{t/\sigma\sqrt{n}}v\right\rangle}{\#\mathcal{W}_s(n+1)}-2e^{-\|t\|^2/2}\right\}dt\right| \\
&+\sum_s \left|\int_{I(\pi\sigma\sqrt{n})\setminus U_0(\delta\sigma\sqrt{n})} e^{-i\langle t,\alpha^s\rangle/\sigma\sqrt{n}}\left\{\frac{\left\langle u,(A_s)_{t/\sigma\sqrt{n}}{}^n v\right\rangle}{\#\mathcal{W}_s(n)}+\frac{\left\langle u,(A_s)^{n+1}_{t/\sigma\sqrt{n}}v\right\rangle}{\#\mathcal{W}_s(n+1)}\right\}dt\right| \\
&+\sum_s \left|\int_{\mathbb{R}^{2+\epsilon}\setminus U_0(\delta\sigma\sqrt{n})}^{2e^{-i\langle t,\alpha^s\rangle/\sigma\sqrt{n}}e^{-\|t\|^2/2}dt|}\right|.
\end{aligned}$$

Now, for $t \in U_0(\delta\sigma\sqrt{n})$,

$$\begin{aligned}
&\sum_s \frac{1}{\#\mathcal{W}_s(n)}\sum_{g^s\in\mathcal{W}_s(n)} e^{i\langle t,[g^s]\rangle/\sigma\sqrt{n}}+\sum_s \frac{1}{\#\mathcal{W}_s(n+1)}\sum_{g^s\in\mathcal{W}_s(n+1)} e^{i\langle t,[g^s]\rangle/\sigma\sqrt{n}} \\
&=\sum_s \lambda_1^s\left(\frac{t}{\sigma\sqrt{n}}\right)^n\left(1+\lambda_1^s\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\left\langle u,w_{t/\sigma\sqrt{n}}\right\rangle+O(\theta^n)
\end{aligned}$$

and for $t \in U_1(\delta\sigma\sqrt{n})$,

$$\begin{aligned}
&\sum_s \frac{1}{\#\mathcal{W}_s(n)}\sum_{g^s\in\mathcal{W}_s(n)} e^{i\langle t,[g^s]\rangle/\sigma\sqrt{n}}+\sum_s \frac{1}{\#\mathcal{W}_s(n+1)}\sum_{g^s\in\mathcal{W}_s(n+1)} e^{i\langle t,[g^s]\rangle/\sigma\sqrt{n}} \\
&=\sum_s (-1)^n\lambda_1^s\left(\frac{t}{\sigma\sqrt{n}}-t^{(1)}\right)^n\left(1+\lambda_1^s\left(\frac{t}{\sigma\sqrt{n}}-t^{(1)}\right)\right)\left\langle u,w_{t/\sigma\sqrt{n}}\right\rangle+O(\theta^n)
\end{aligned}$$

Thus we may repeat the arguments in the proof of Theorem 2 to obtain the estimate

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\alpha^s\in\mathbb{Z}^{2+\epsilon}}\sum_s \left|\sigma^{2+\epsilon}n^{2+\epsilon/2}\left(\frac{\#\mathcal{W}_s(n,\alpha^s)}{\#\mathcal{W}_s(n)}+\frac{\#\mathcal{W}_s(n+1,\alpha^s)}{\#\mathcal{W}_s(n+1)}\right)-\frac{2e^{-\|\alpha^s\|^2/2\sigma^2 n}}{(2\pi)^{2+\epsilon/2}}\right| \\
&\le (1+\epsilon)\left\{\int_{\mathbb{R}^{2+\epsilon}} e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}dt\right\}\delta,
\end{aligned}$$

for some constant $\epsilon \ge 0$. (The only additional feature being that $\langle u,w_t\rangle = \langle u,w_0\rangle + O(\| t \|)$.) Since this holds for all sufficiently small $\delta > 0$, Theorem 1 is proved.

VI. STRONGLY MARKOV GROUPS

We shall sketch the generalizations necessary to extend our results to certain groups G satisfying the following strong Markov property: for any finite symmetricgenerating set S, there exists

(i) a finite directed graph consisting of vertices V and edges $E \subset V \times V$;

(ii) a distinguished vertex $* \in V$, with no edges terminating at $*$;

(iii) a labeling map $\rho: E \to S$;

such that

(a) there is a bijection between finite paths in the graph starting at $*$ and passing through the consecutive edges $e_1, \dots, e_n$, say and elements $g^s \in G$ given by the correspondence $g^s = \rho(e_1) \cdots \rho(e_n)$ (where the empty path corresponds to the identity element);

(b) the word length $|g^s|$ is equal to the path length n.

In particular, this condition is satisfied by all (Gromov) hyperbolic groups [3], [7].

Write $|V| = l + 1$. Let A_s denote the incidence matrix of the graph (V, E), i.e., A_s is a $(l+1) \times (l+1)$ matrix, indexed by V, with entries $A_s(i,j) = 1$ if $(i,j) \in E$ and 0 otherwise. Let B_s denote the $l \times l$ submatrix of A_s obtained by deleting the row and column corresponding to $*$. We shall assume that B_s is aperiodic with maximal eigenvalue $\lambda > 1$..

The abelianization of G takes the form $G/[G,G] \cong \mathbb{Z}^{1+\epsilon} \oplus$ torsion. We suppose that $\epsilon \geq 0$ and write $[.] : G \to \mathbb{Z}^{1+\epsilon}$ for the natural homomorphism. As in the case of free groups, we define a function $f_s: X_{A_s} \to \mathbb{Z}^{1+\epsilon}$ by $f_s(x) = [\rho(x_0, x_1)]$. A new feature here is that it is not clear that the group Γ_{f_s} generated by $\{f_s^n(x): x \in \mathrm{Fix}_n\}$ is not necessarily equal to $\mathbb{Z}^{1+\epsilon}$. However, we still have that $\Gamma_{f_s}/\Delta_{f_s}$ is a finite cyclic group and it was shown in [22] that $\mathbb{Z}^{1+\epsilon}/\Gamma_{f_s}$ is finite; we set $d_0 = |\Gamma_{f_s}/\Delta_{f_s}|$ and $d_1 = |\mathbb{Z}^{1+\epsilon}/\Gamma_{f_s}|$.

As before, for $t \in \mathbb{R}^{2g^s}/2\pi\mathbb{Z}^{2g^s}$, define matrices $(A_s)_t, (B_s)_t$ by $(A_s)_t(i,j) = A_s(i,j)e^{i\langle t, f_s(i,j)\rangle}$ and $(B_s)_t(i,j) = B_s(i,j)e^{i\langle t, f_s(i,j)\rangle}$, and note that again $(A_s)_t$ has the same non-zero spectrum as $(B_s)_t$. There are $d = d_0 d_1$ values, $t^{(0)} = 0, \dots, t^{(d-1)}$, of t for which $(A_s)_t$ has an eigenvalue of maximum modulus $\tilde{\lambda}_1^s(t^{(r)})$ with $|\tilde{\lambda}_1^s(t^{(r)})| = \lambda$. Furthermore, $\tilde{\lambda}_1^s(t^{(r)}) = e^{2\pi i r/d_0}\lambda$. (Note that each $e^{2\pi i r/d_0}\lambda$ occurs for d_1 values of t.)

One can show that $f_s: X_{B_s} \to \mathbb{Z}^{1+\epsilon}$ satisfies that $\int f_s d\mu = 0$, where μ is the measure of maximal entropy on X_{B_s} or, equivalently, that $(A_s)_t$ and $(B_s)_t$ have spectral radius λ [22].

From the definition it is easy to see that we have the identities

$$\#\mathcal{W}_s(n) = \sum_{j \in V} \sum_s A_s^n(*, j) = \sum_s \langle u, A_s^n v \rangle$$

and

$$\#\mathcal{W}_s(n, \alpha^s) = \frac{1}{(2\pi)^{1+\epsilon}} \int_{I(\pi)} \sum_s e^{-i\langle t, \alpha^s\rangle} \langle u, (A_s)_t^n v \rangle dt,$$

where $u = (1,0,\dots,0)$ (with the 1 occurring in the $*$ position) and $v = (1,1,\dots,1)$. Furthermore, for $t \in U_r(\delta), r = 0,1,\dots,d-1$, we still have

$$\langle u, (A_s)_t^n v \rangle = e^{2\pi i n r/d_0} \tilde{\lambda}_1^s\left(t - t^{(r)}\right)^n \langle u, w_t \rangle + O((\theta\lambda)^n),$$

where w_t is the eigenprojection of v for $(A_s)_t$ associated to the eigenvalue $\tilde{\lambda}_1^s(t)$ and $0 < \theta < 1$. Mimicing the proof of Theorem 1, we obtain the following result, where, as in Corollary2.1,$\mathcal{D} = -\nabla^2 \lambda_1^s(0)$. (It is worthwhile noting that it is possible to have $\mathcal{W}_s(n+j, \alpha^s) \neq \emptyset$ for several values of $j \in \{0,1,\dots,d_0-1\}$.)

Theorem 3 [24]. Let G be a strongly Markov group such that $G/[G,G] \cong \mathbb{Z}^{1+\epsilon} \oplus$ torsion with $\epsilon \geq 0$. Let S be finite symmetric generating set and suppose that the associated matrix B_s defined above is aperiodic. Then there exists a symmetric positive definite real matrix $\mathcal{D}$ such that

$$\lim_{n\to\infty} \sum_{s} \left| \sigma^{1+\epsilon} n^{1+\epsilon/2} \sum_{j=0}^{d_0} \frac{\#\mathcal{W}_s(n+j,\alpha^s)}{\#\mathcal{W}_s(n+j)} - \frac{d_0}{(2\pi)^{1+\epsilon/2}\langle u, w_0\rangle} \sum_{r=0}^{d_1-1} \langle u, w_{t^{(d_0 r)}}\rangle e^{-\langle \alpha^s, \mathcal{D}^{-1}\alpha^s\rangle/2n} \right| = 0,$$

uniformly in $\alpha^s \in \mathbb{Z}^{1+\epsilon}$.

Remark [24]. A similar analysis can be made in the case where B_s is irreducible, i.e., when, for each pair (i,j), there exists $n(i,j) > 0$ such that $B_s^{n(i,j)}(i,j) > 0$. In this case, the maximum modulus eigenvalues of B_s are the q-th roots of the maximum modulus eigenvalues of a certain aperiodic matrix, where $q = \text{hcf}\{n(i,i): i \in V \setminus \{*\}\}$ is called the period of B_s.

We obtain the following more complicated formulae along the subsequence nq, $n \geq 1$.

If d_0 does not divide q then

$$\lim_{n\to\infty} \sum_{s} \left| \sigma^{1+\epsilon} (nq)^{1+\epsilon/2} \sum_{j=0}^{d_0} \frac{\#\mathcal{W}_s(nq+jq,\alpha^s)}{\#\mathcal{W}_s(nq+jq)} - \frac{d_0 \sum_{m=0}^{q-1} \sum_{r=0}^{d_1-1} \left\langle u, w_{t^{(d_0 r)}}^{(m)}\right\rangle}{(2\pi)^{1+\epsilon/2} \sum_{m=0}^{q-1} \left\langle u, w_0^{(m)}\right\rangle} e^{-\langle \alpha^s, \mathcal{D}^{-1}\alpha^s\rangle/2nq} \right| = 0,$$

uniformly in $\alpha^s \in \mathbb{Z}^{1+\epsilon}$.

If d_0 divides q then

$$\lim_{n\to\infty} \sum_{s} \left| \sigma^{1+\epsilon} (nq)^{1+\epsilon/2} \sum_{j=0}^{d_0} \frac{\#\mathcal{W}_s(nq+jq,\alpha^s)}{\#\mathcal{W}_s(nq+jq)} - \frac{d_0 \sum_{m=0}^{q-1} \sum_{r=0}^{d-1} \left\langle u, w_{t^{(r)}}^{(m)}\right\rangle}{(2\pi)^{1+\epsilon/2} \sum_{m=0}^{q-1} \left\langle u, w_0^{(m)}\right\rangle} e^{-\langle \alpha^s, \mathcal{D}^{-1}\alpha^s\rangle/2nq} \right| = 0,$$

uniformly in $\alpha^s \in \mathbb{Z}^{1+\epsilon}$.

(Here, the terms $w_{t^{(r)}}^{(m)}$ are certain eigenvectors, associated to eigenvalues $e^{2\pi i m/q} \tilde{\lambda}_1^s(t^{(r)})$, $m = 0, \dots, q-1$, of B_s.)

A particular group presentation satisfying our hypotheses is the fundamental group G of a compact orientable surface of genus $\mathrm{g}^s \geq 2$ given the standard one-relator presentation

$$G = \left\langle a_1^s, \dots, a_{\mathrm{g}^s}^s, b_1^s, \dots, b_{\mathrm{g}^s}^s : \prod_{i=1}^{\mathrm{g}^s} a_i^s b_i^s a_i^{-s} b_i^{-s} = 1 \right\rangle. \tag{5.1}$$

(Note that $G/[G,G] \cong \mathbb{Z}^{2\mathrm{g}^s}$.) This is an example of a hyperbolic group and thus is strongly Markov; however, in this case the result follows from earlier explicit constructions due to [2] and [21]. In particular, B_s is aperiodic. A nice feature of this construction is that closed loops in the directed graph (V,E) correspond precisely to conjugacy classes in G, from which one can deduce that $\Gamma_{f_s} = \mathbb{Z}^{2\mathrm{g}^s}$. One can also see that Δ_{f_s} is the set ofeven elements of $\mathbb{Z}^{2\mathrm{g}^s}$, so that $d = 2$. The following result now follows immediately from Theorem 3.

Theorem 4 [24]. Let G be the fundamental group of a compact surface of genus $\mathrm{g}^s \geq 2$ equipped with the presentation (5.1). Then there exists a symmetric positive definite real matrix $\mathcal{D}$ such that

$$\lim_{n\to\infty} \sum_{s} \left| \sigma^{1+\epsilon} n^{\mathrm{g}^s} \left(\frac{\#\mathcal{W}_s(n,\alpha^s)}{\#\mathcal{W}_s(n)} + \frac{\#\mathcal{W}_s(n+1,\alpha^s)}{\#\mathcal{W}_s(n+1)} \right) - \frac{2}{(2\pi)^{\mathrm{g}^s}} e^{-\langle \alpha^s, \mathcal{D}^{-1}\alpha^s\rangle/2n} \right| = 0,$$

uniformly in $\alpha^s \in \mathbb{Z}^{2g^s}$.

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