Quest Journals
Journal of Research in Applied Mathematics
Volume 12 ~ Issue 2 (February 2026) pp: 124-127
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

On the completeness of root vectors of completely continuous operators in Hilbert space.

Rakhshanda Dzhabarzadeh
Institute of Mathematics and Mechanics of Azerbaijan

*Received 11 Feb., 2026; **R**evised 20 Feb., 2026; **A**ccepted 23 Feb., 2026 © The author(s) 2026. **P**ublished with open access at **www.questjournals.org***

The spectral theory of operators is the essential direction of functional analysis. Spectral theory of operators develops as the generalization of notions of eigenvalues and eigenvectors to the wider classes of linear spaces, for example, Hilbert space. Completely continuous operators occupy an extremely important place in spectral theory of operators.It is known that completely continuous self-adjoint operator with the zero kernel in Hilbert space has a countable set of eigenvectors, orthogonal to each other, and forms the basis in space. All eigenvalues of these operators are real numbers.Many works of famous mathematicians are devoted to this topic [1],[2],[3],[4]and so on. For example, spectral decomposition of the completely continuous self-adjoint operators was given in all books on functional analysis. Spectral decomposition of the normal operators is given in[1], and spectral decomposition of completely continuous operator is given in [5]. In [6] it is presented the multiple spectral decompositions of completely continuous operator bundle in Hilbert space.

Let A be a linear operator, acting in separabel Hilbert space Necessary definitions and notions for understanding of presenting below material**.**

1. λ is an eigenvalue of operator A ifthere is nonzero element x such that

$$Ax - \lambda x = 0$$

Element x is called an eigenvector, corresponding to this eigenvalue λ.

2.Element x is called a root vector of height k if the following equalities

$$(A - \lambda E)^k x = 0$$

$$(A - \lambda E)^{k-1} x \neq 0$$ satisfy.

If $k = 1$, then x is an eigenvector of operator A with the eigenvalue λ.

3.Completely continuous operator A has a finite order ifseries of eigenvalues of operator $(A^* A)^{\frac{1}{2}}$ (eigenvalues of operator $(A^* A)^{\frac{1}{2}}$ are called $S-$numbers of operator A) in the some positive degrees converge. Lower boundary of such numbers, for which these series converge, is called an order of operator A.

4.In [7] it is introduced the class of operators with negative finite order.Operator A has a negative order $-p$ if $KerA = 0$ and operator A^{-1} is a completely continuous with the finite order p .For the product

AB of two operators A and B with the orders p and s ,correspondingly, (p and s may be positive or negative numbers) has an order , $\frac{ps}{s+p}$ which could be positive or negative number.

5.Each completely continuous with finite order p operator A may be presented as the product $A = CB$ of completely continuous self-adjoint operator B with the finite order s and operator C with the order $\frac{ps}{s-p}$.

In the case ,when completely continuous operator is not self-adjoint, there are some results for completeness of root vectors of completely continuous operators in Hilbert space.

Below we present the famous results sufficient for completeness of root vectors of completely continuous operators, acting in Hilbert space.

1. In V.B.Lidsky's work [8] it is proved that the completely continuous operator A has the complete system of root vectors .

If there is the some system of concentric circles $S_t 9 j = 1,2,...)$ centered at the origin on which resolvent of operator A satisfies condition $\left\|(A-\lambda E)^{-1}\right\|(\leq c$ .Parameter C does not depend on the circles $S_t 9 j = 1,2,...)$.

2.In V.B.Lidsky's work [9] it is proved the summation by generalized Abel method on root vectors of completely continuous operator B . For this it is enough that the quadratic form (Bh, h) lies in sufficient small angle bisector of which is the real axis.

3.[10] If a completely continuous operator A may be presented as the sum of self-adjoint completely continuous operator with finite order p and the completely continuous operator with finite order s , when $s < p$,then the completeness of root vectors of operator A in Hilbert space takes place. Now we introduce the class of completely continuous operators of finite order when the norms of their imaginary parts are sufficient small.

The operator A can be presented in the form

$$A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i} \text{ , where } T = \frac{A+A^*}{2}$$

and $S = \frac{A-A^*}{2i}$ are completely continuous self-adjoint operators.Operator S is called an imaginary part of operator A . It is famousthat each ofthe operators T and S has a countable set of eigenvectors, orthogonal to each other corresponding to real eigenvalues.

Theorem. Let A be a completely continuous operator with the finite order $p > 1$. ,satisfying the following conditions

a) $KerA = 0$

$$Ker(A + A^*) = 0$$

b) $\left\|\left(A-A^{*}\right)\left(A+A^{*}\right)^{-1}\right\| \leq \dfrac{\sin\frac{\pi}{p}}{1+\sin\frac{\pi}{p}} \quad p>1.$

Then operator A has a complete system of root vectors in Hilbert space .

Proof.

We consider the equation $(E-\lambda A)x=0$

where $A=T+iS$.

Further,

$$iST^{-1}=\left(A-A^{*}\right)\left(A+A^{*}\right)^{-1}.$$

The equation $(E-\lambda A)x=0$ takes the form $\left[E-\lambda(E+iST)T\right]x=0$

$$[E-\lambda\left(E+(A-A^{*})\left(A+A^{*}\right)^{-1}\right)\left(A+A^{*}\right)]x=0 \quad \textbf{(1)}$$

After simple transformation we get $[\left(E-(A^{*}-A)\left(A+A^{*}\right)^{-1}\right)^{-1}-\lambda\left(A+A^{*}\right)]x=0$

(2) and $\left[\left(E+\sum_{i=1}^{\infty}\left(A^{*}-A\right)\left(A+A^{*}\right)^{-1}\right)^{i}-\lambda\left(A+A^{*}\right)\right]x=0$ **(3)**

Using the condition of the theorem, by the end we have

$$\left\|(\left(E-\left(A^{*}-A\right)\left(A+A^{*}\right)\right)^{-1}\right\|=1+\frac{\left\|\left(A-A^{*}\right)\left(A+A^{*}\right)^{-1}\right\|}{1-\left\|\left(A-A^{*}\right)\left(A+A^{*}\right)^{-1}\right\|}\leq$$

$$\leq 1-+\frac{\dfrac{\sin\frac{\pi}{p}}{1+\sin\frac{\pi}{\rho}}}{1-\dfrac{\sin\frac{\pi}{\rho}}{1+\sin\frac{\pi}{\rho}}}=\sin\frac{\pi}{\rho} \quad \textbf{(4)}$$

All conditions of the Keldysh's theorem[11] and generalized theorem[12]are fulfilled The conditions of Keldysh's theorem in the case when operator bundle, linearly depending on parameter , are following, $A+A^{*}$ is a completely continuous self adjoint operator with the finite order p ,and operator at λ^{0} is also completely continuous.

The generalized theorem [12]allows the existence of bounded part with small norm to the operator ,standing at λ^0 .In our case the operator at parameter λ^0 has a bounded with norm less then $\sin\frac{\pi}{\rho}$.

$$\sum_{i=1}^{\infty}\left\|\left(A-A^*\right)\left(A+A^*\right)^{-1}\right\|^i,$$

Really, $< \dfrac{\dfrac{\sin\frac{\pi}{p}}{1+\sin\frac{\pi}{\rho}}}{1-\dfrac{\sin\frac{\pi}{\rho}}{1+\sin\frac{\pi}{\rho}}} = \sin\frac{\pi}{\rho}$.

Consequently, the completeness of eigen and associated vectors of bundle(3) is true.

Thus, root vectors of operator A form the complete system in Hilbert space. (In the case when operator bundle depends on linearly on parameter from the completeness of eigen and associated vectors of the equation (3) it follows the completeness of root vectors of operator A .

Thus, we have the completeness of root vectors of operator A in Hilbert space H .

Theorem is proven.

References

[1]. N.IAchirzer and I.M.QlazmannThe theory of linear operators in Hilbert space . Publishing “Science”.Editor –in-chief of physical and mathematical literature. Moscow, 1966 (in Russian)

[2]. I.Ts.Gokhberq, M.G. Krein.Introduction to the theory of linear not self-adjoint operators in Hilbert space ,Publishing House “Nauka”,Moskva,1965,pp.448 (in Russian)

[3]. Lusternic L.A. and SobolevV.I.Elements of functional analysis.1965,pp.513 (in Russian)

[4]. F. Riss and B.Sekefalvi -Nad .Elements of function’ s theory and functional analysis Nauka,1976, pp.544

[5]. RakhshandaDzhabarzadehDecompositions of completely continuous operator in Hilbert space. Quest Journal of Research in Applied MathematicsVol.11,issue 3,pp.105-107,2025

[6]. RakhshandaDzhabarzadeh.Spectral decomposition of operator polynomial bundle. Quest Journal of Research in Applied MathematicsVol.11,issue 7,pp.13-18,2025

[7]. DzhabarzadehR.M.To the thery of operator bundles. About class of operator Some questions of theory of nonlinear analysis. Baku,1990, issue 2 ,pp.76-86.

[8]. Lidsky V.B. The condition of completeness of root subspaces of not self-adjoint operators with discrete spectrum. Proceeding of Moscow Mathematical Society ,1950, volume8, pp.84-120 (in Russian)

[9]. V.B.Lidsky. On summation of series on main vectors of not self-adjoint operators. Proceeding of Moscow Mathematical Society, vol.11, pp.3-45,1962.

[10]. R.M.Dzhabarzadeh, A.N. Jabrailova. Spectral theory of operator bundles The scientific heritage(Budapest,Hungary)The journal registered and published in Hungary. Vol. 2,No 86, 2022, pp.30-33

[11]. Keldysh M.V. About completeness of eigen functions of some class of not self-adjointlinear operators. Advancesin Mathematical sciences,1971,vol.27, issue 4,pp.15-41 (in Russian)

[12]. Allakhverdiev J.E. About completeness of system of eigen and associated elements of not self-adjoint operator bundle closed to normal. DAN SSSR,vol.115, 2, 1957, 207-210(in Russian)