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Research Paper

Global Stability Analysis of a Vector-Host Epidemic Model with Non-Monotonic Incidence Rate for Host and Bilinear Incidence Rate for Vector

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Abstract
This work develops vector–host epidemic models in which disease transmission among hosts is governed by non-monotonic incidence functions, while transmission in the vector population follows a bilinear incidence structure. The proposed framework is used to examine the mechanisms underlying the spread of vector-borne infections. Stability properties of both the disease-free and endemic equilibrium states are studied using the basic reproduction number. Conditions for local as well as global stability of these equilibria are derived. The analysis demonstrates that the infection persists in the population whenever the associated basic reproduction number R_0*exceeds unity, whereas the disease dies out when* $R_0 < 1$*. Numerical simulations are performed to support and illustrate the analytical findings.*

Keywords: *vector host epidemics, stability, disease-free equilibrium, endemic equilibrium.*

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I. Introduction

In recent decades, outbreaks of vector-borne diseases have been reported across the globe with increasing frequency.Pathogenic agents such as viruses, bacteria, and parasites are transmitted through a variety of vectors, including mosquitoes, flies, ticks, and aquatic snails—between human and animal hosts, giving rise to infections such as malaria, dengue, and yellow fever [1, 4]. These diseases continue to rank among the most serious global health threats, contributing significantly to annual mortality and morbidity in both human and animal populations [9]. Their consequences extend well beyond health, affecting economic sectors such as medical care, agriculture, livestock production, tourism, and international commerce, while also altering ecological balance at regional and global scales.The expanding geographical range and abundance of vectors have been closely linked to increased global mobility [5], intensified trade activities, climatic variability, and rapid, poorly planned urban expansion. In light of these factors, modelling approaches that explicitly integrate host and vector populations are crucial for accurately describing transmission pathways and disease persistence. A substantial body of mathematical literature has been devoted to the formulation and analysis of vector-borne disease models, providing valuable insights into transmission dynamics and control strategies ([1], [2], [3], [4], [5], [6], [7]).

The fundamental definitions and preliminary concepts from dynamical systems theory needed in this chapter are summarized below, based on ([8], [10], [11]).

Second additive compound matrix: Second additive compound matrix $M^{[2]}$ of a matrix $M = (M_{ij})$ of order 3×3 is as follows

$$M^{[2]} = \begin{bmatrix} M_{11} + M_{22} & M_{23} & -M_{13} \\ M_{32} & M_{11} + M_{33} & M_{12} \\ -M_{31} & M_{21} & M_{22} + M_{33} \end{bmatrix}$$

Theorem: Let M be a 3 × 3 real matrix. If tr(M), det(M), and det($M^{[2]}$) are all negative, then all of the eigen values of M have negative real part.

Asymptotically orbitally Stable: An orbit generated by system $\dot{x} = f(x),\ x \in \mathbb{R}^n$, with initial condition on a compact ϕ- invariant subset A of the state space is said to be asymptotically orbitally stable if the invariant state $\Gamma = \{\phi(t, x_0)/x_0 \in A; t \geq 0\}$ is asymptotically stable.

Omega limit point: A point $x_0 \in \mathbb{R}^n$ is called an omega limit point of $x \in \mathbb{R}^n$ if there exists a sequence $\{t_i\}, t_i \to \infty$ such that $\phi(t_i, x) \to x_0$.
Uniform Persistence: It means that strictly positive solutions are eventually bounded uniformly away from the boundary.
Gronwall's Inequality: Let $u: [0, \alpha] \to \mathbb{R}$ be continuous and non-negative. Suppose C ≥ 0, K ≥ 0 are such that

$$u(t) \le C + \int_0^t Ku(s)ds$$

for all $t \in [0, \alpha]$. Then, for all t in this interval,

$$u(t) \le Ce^{Kt}$$

The construction of the models and the corresponding stability analysis are described in this section. For clarity, we first provide a list of the state variables and parameters employed in the chapter:
$S(t)$ denotes the number of susceptible individuals in host population at time *t*
$I(t)$ denotes the number of infected individuals in host population at time *t*
$R(t)$ denotes the number of recovered individuals in host population at time *t*
such that number of host population $N_1(t) = S(t) + I(t) + R(t)$
$M(t)$ denotes the number of susceptible vectors at time *t*
$V(t)$ denotes the number infective vectors population at time *t*
such thatnumber of vector population$N_2(t) = M(t) + V(t)$
The parameter $\alpha_1$is the recruitment rate of host population, λ_1 is the transmission rate from vector to host, β_1 is natural death rate of host population, γ is per capita recovery rate of host, α_2 is the recruitment rate of vector population, $\beta_2$is natural death rate of vector population andλ_2 is the transmission rate from host to vector

we consider the vector host epidemic model where the dynamics of the host is expressed by an SIR model with the non-monotonic incidence rate $\frac{k\lambda_1 S(t)V(t)}{1+a_1V(t)+a_2V^2(t)}$ and the vector population is expressed by a system of susceptible and infected vector with bilinear incidence rate. Here, $a_1$and $a_2$are the parameters which measure effects of social awareness. The parameter $a_1$is chosen to ensure $1 + a_1V + a_2V^2 > 0$ for all $V \geq 0$. The parameter a_2 is assumed to be positive.

II. MODEL FORMULATION

We present a vector host epidemiological model with non-monotonic incidence rate for hosts and the bilinear incidence rate for vectors which is represented by a system (2.1) of differential equations as follows:

$$\frac{dS(t)}{dt} = \alpha_1 - \frac{k\lambda_1 S(t)V(t)}{1 + a_1V(t) + a_2V^2(t)} - \beta_1 S(t)$$
$$\frac{dI(t)}{dt} = \frac{k\lambda_1 S(t)V(t)}{1 + a_1V(t) + a_2V^2(t)} - \gamma I(t) - \beta_1 I(t)$$
$$\frac{dR(t)}{dt} = \gamma I(t) - \beta_1 R(t) \quad (2.1)$$
$$\frac{dM(t)}{dt} = \alpha_2 - \lambda_2 M(t)I(t) - \beta_2 M(t)$$
$$\frac{dV(t)}{dt} = \lambda_2 M(t)I(t) - \beta_2 V(t)$$

From system (2.1)we can easily obtain

$$\frac{d}{dt}N_1(t) = \alpha_1 + \beta_1 N_1(t)$$
$$\frac{d}{dt}N_2(t) = \alpha_2 + \beta_2 N_2(t)$$

The total population sizes of host and vector are asymptotically constant.

i.e. $\lim_{t\to\infty} N_1(t) = \alpha_1/\beta_1, \quad \lim_{t\to\infty} N_2(t) = \alpha_2/\beta_2$

Without loss of generality, we assume that

$$N_1(t) = \alpha_1/\beta_1, N_2(t) = \alpha_2/\beta_2 \text{ for all} t \ge 0.$$

Therefore, the dynamical system (2.1) is qualitatively equivalent to the following dynamical system:

$$\frac{dS(t)}{dt} = \alpha_1 - \frac{k\lambda_1 S(t)V(t)}{1 + a_1V(t) + a_2V^2(t)} - \beta_1 S(t)$$
$$\frac{dI(t)}{dt} = \frac{k\lambda_1 S(t)V(t)}{1 + a_1V(t) + a_2V^2(t)} - \gamma I(t) - \beta_1 I(t) \quad (2.2)$$
$$\frac{dV(t)}{dt} = \lambda_2 \left(\frac{\alpha_2}{\beta_2} - V(t)\right) I(t) - \beta_2 V(t)$$

The values of R and M can be determined from $R = \frac{\alpha_1}{\beta_1} - S - I$ and $M = \frac{\alpha_2}{\beta_2} - V$, respectively.

Biological interpretation requires that all state variables remain non-negative. Taking into account the qualitative properties of the solutions, we confine our investigation of system (2.2) to the following closed set

$$\Gamma = \{(S, I, V) \in \mathbb{R}_+^3 / 0 \leq S + I \leq \alpha_1/\beta_1, 0 \leq V \leq \alpha_2/\beta_2, S \geq 0, I \geq 0\}$$

System (2.2) has disease free equilibrium $E_0(\alpha_1/\beta_1, 0, 0)$.

To find the endemic equilibrium of system (2.2), we set

$$\alpha_1 - \frac{k\lambda_1 S(t)V(t)}{1 + a_1 V(t) + a_2 V^2(t)} + \beta_1 S(t) = 0$$

$$\frac{k\lambda_1 S(t)V(t)}{1 + a_1 V(t) + a_2 V^2(t)} - \gamma I(t) - \beta_1 I(t) = 0$$

$$\lambda_2 \left(\frac{\alpha_2}{\beta_2} - V(t)\right) I(t) - \beta_2 V(t) = 0$$

This gives

$$a_2\beta_1\beta_2^2(\gamma + \beta_1)V^2 + [k\alpha_1\beta_2\lambda_1\lambda_2 + k\lambda_1\beta_2^2(\gamma + \beta_1) + a_1\beta_1\beta_2^2(\gamma + \beta_1)]V + \beta_1\beta_2^2(\gamma + \beta_1) - k\alpha_1\alpha_2\lambda_1\lambda_2 = 0$$

The basic reproduction number is derived as

$$R_0 = \frac{k\alpha_1\alpha_2\lambda_1\lambda_2}{\beta_1\beta_2^2(\gamma + \beta_1)} \tag{2.3}$$

The endemic equilibrium $E^*(S^*, I^*, V^*)$ is given by the following equations

$$S^* = \frac{(\gamma + \mu_1)\beta_2^2(1 + a_1 V^* + a_2 V^{*2})}{k\lambda_1\lambda_2[\alpha_2 - \beta_2 V^*]} \tag{2.4}$$

$$I^* = \frac{\beta_2^2 V^*}{\lambda_2[\alpha_2 - \beta_2 V^*]} \tag{2.5}$$

$$V^* = \frac{-\Delta + \sqrt{\Delta^2 - 4a_2\beta_1^2\beta_2^4(\gamma + \beta_1)^2(1 - R_0)}}{2a_2\beta_1\beta_2^2(\gamma + \beta_1)} \tag{2.6}$$

where $\Delta = k\alpha_1\beta_2\lambda_1\lambda_2 + k\lambda_1\beta_2^2(\gamma + \mu_1) + a_1\beta_1\beta_2^2(\gamma + \beta_1)$

III. STABILITY ANALYSIS

We derive the stability conditions for the disease-free and endemic equilibriums of model (2.2)in this section.

We linearize the system (2.2) and get the Jacobian matrix J as follows.

$$J(S, I, V) = \begin{bmatrix} -\frac{k\lambda_1 V}{1 + a_1 V + a_2 V^2} - \beta_1 & 0 & -\frac{k\lambda_1 S(1 - a_2 V^2)}{(1 + a_1 V + a_2 V^2)^2} \\ \frac{k\lambda_1 V}{1 + a_1 V + a_2 V^2} & -\gamma - \beta_1 & \frac{k\lambda_1 S(1 - a_2 V^2)}{(1 + a_1 V + a_2 V^2)^2} \\ 0 & \lambda_2\left(\frac{\alpha_2}{\beta_2} - V\right) & -\lambda_2 I - \beta_2 \end{bmatrix} \tag{2.7}$$

We apply Routh-Hurwitz criterion to investigate the local stability of disease-free equilibrium and LaSalle's invariance principle to derive global stability of disease-free equilibrium.

Theorem 1: The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable for $R_0 > 1$.

Proof: At the disease-free equilibrium E_0, equation(2.7) becomes

$$J(E_0) = \begin{bmatrix} -\beta_1 & 0 & -k\lambda_1\frac{\alpha_1}{\beta_1} \\ 0 & -\gamma - \beta_1 & k\lambda_1\frac{\alpha_1}{\beta_1} \\ 0 & \lambda_2\frac{\alpha_2}{\beta_2} & -\beta_2 \end{bmatrix} \tag{2.8}$$

Characteristic equation is given by $J(E_0) - lI = 0$

This gives

$$(-\beta_1 - l)\left[(-\gamma - \beta_1 - l)(-\beta_2 - l) - \frac{k\alpha_1\alpha_2\lambda_1\lambda_2}{\beta_1\beta_2}\right] = 0$$

$$-(\beta_1 + l)\left[l^2 + (\gamma + \beta_1 + \beta_2)l + (\gamma + \beta_1)\beta_2 - \frac{k\alpha_1\alpha_2\lambda_1\lambda_2}{\beta_1\beta_2}\right] = 0$$

$$(\beta_1 + l)\left[l^2 + (\gamma + \beta_1 + \beta_2)l + (\gamma + \beta_1)\beta_2\left(1 - \frac{k\alpha_1\alpha_2\lambda_1\lambda_2}{\beta_1\beta_2^2(\gamma + \beta_1)}\right)\right] = 0$$

$$(\beta_1 + l)[l^2 + (\gamma + \beta_1 + \beta_2)l + (\gamma + \beta_1)\beta_2(1 - R_0)] = 0$$

When $R_0 < 1$, application of the Routh–Hurwitz stability criterion [31] shows that all eigenvalues of the Jacobian matrix have negative real parts, implying that the disease-free equilibrium $E_0$is locally asymptotically stable. In contrast, if $R_0 > 1$, the spectrum contains one eigenvalue with positive real part while the remaining two are negative, and consequently the equilibrium E_0loses stability and becomes unstable.

Theorem 2: If $R_0 \leq 1$then the disease-free equilibrium E_0 is globally asymptotically stable and it is unstable for $R_0 > 1$.
Proof: Consider the function

$$L_1 = \frac{k\lambda_1\alpha_1}{\beta_1\beta_2}V + I$$

$$\frac{dL_1}{dt} = \frac{k\lambda_1\alpha_1}{\beta_1\beta_2}\frac{dV}{dt} + \frac{dI}{dt}$$

$$\frac{dL_1}{dt} = \frac{k\lambda_1\alpha_1}{\beta_1\beta_2}\left[\lambda_2\left(\frac{\alpha_2}{\beta_2} - V\right)I - \mu_2 V\right] + \frac{k\lambda_1 SV}{1 + a_1V + a_2V^2} - (\gamma + \beta_1)I$$

$$= -(\gamma + \beta_1)\left[1 - \frac{k\alpha_1\alpha_2\lambda_1\lambda_2}{\beta_1\beta_2^2(\gamma + \beta_1)}\right]I - \frac{k\alpha_1\lambda_1\lambda_2}{\beta_1\beta_2}VI - \frac{k\alpha_1\lambda_1}{\beta_1}V + \frac{k\lambda_1 SV}{1 + a_1V + a_2\,V^2}$$

$$= -(\gamma + \beta_1)[1 - R_0]I - \frac{k\alpha_1\lambda_1\lambda_2}{\beta_1\beta_2}VI - \frac{k\alpha_1\lambda_1}{\beta_1}V + \frac{k\lambda_1 SV}{1 + a_1V + a_2\,V^2}$$

$$\leq -(\gamma + \beta_1)[1 - R_0]I - \frac{k\alpha_1\lambda_1\lambda_2}{\beta_1\beta_2}VI - \frac{k\alpha_1\lambda_1}{\beta_1}V + k\lambda_1 SV$$

$$\leq -(\gamma + \beta_1)[1 - R_0]I - \frac{k\alpha_1\lambda_1\lambda_2}{\beta_1\beta_2}VI, \qquad \text{as } S \leq \frac{\alpha_1}{\beta_1}$$

We get $\frac{dL_1}{dt} \leq 0$ for $R_0 \leq 1$ and all $t \geq 0$ hence the function L_1 is a Lyapunov function. The equality $\frac{dL_1}{dt} = 0$ holds at the disease-free equilibriumE_0 $(\alpha_1/\beta_1,\ 0,\ 0)$. Thus, $\{E_0\}$ is the largest invariant set in the closed set Γ. Therefore, E_0 is globally stable using LaSalle's invariance principle .

In the following theorems, we investigate the local stability of endemic equilibrium and we derive the global stability of endemic equilibrium in the feasible region Γ by proving uniform persistence of the system (2.2). The asymptotic orbital stability of periodic orbits is further demonstrated using the framework of second compound equations.

Theorem 3: The endemic equilibrium$E^*$of the system (2.2) is locally asymptotically stable if $R_0 > 1$.
Proof: The Jacobian matrix $J(E^*)$ of equation (2.7) becomes:

$$J(E^*) = \begin{bmatrix} -\frac{k\lambda_1V^*}{1 + a_1V^* + a_2V^{*2}} - \beta_1 & 0 & -\frac{k\lambda_1S^*(1 - a_2V^{*2})}{(1 + a_1V^* + a_2V^{*2})^2} \\ \frac{k\lambda_1V^*}{1 + a_1V^* + a_2V^{*2}} & -(\gamma + \beta_1) & \frac{k\lambda_1S^*(1 - a_2V^{*2})}{(1 + a_1V^* + a_2V^{*2})^2} \\ 0 & \lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right) & -\lambda_2 I^* - \beta_2 \end{bmatrix}$$

$$trJ(E^*) = -\frac{k\lambda_1V^*}{1 + a_1V^* + a_2V^{*2}} - 2\beta_1 - \gamma - \lambda_2I^* - \beta_2 < 0 \qquad (2.9)$$

Using the value of S^*, we have

$$\mu_2(\gamma + \beta_1) = \frac{k\lambda_1\lambda_2S^*}{(1 + a_1V^* + a_2V^{*2})}\left(\frac{\alpha_2}{\beta_2} - V^*\right)$$

Also

$$detJ(E^*) = \left(-\frac{k\lambda_1V^*}{1 + a_1V^* + a_2V^{*2}} - \beta_1\right)\left[(-\gamma - \beta_1)(-\lambda_2I^* - \beta_2) - \frac{k\lambda_1S^*(1 - a_2V^{*2})}{(1 + a_1V^* + a_2V^{*2})^2}\lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right)\right]$$
$$- \frac{k\lambda_1S^*(1 - a_2V^{*2})}{(1 + a_1V^* + a_2V^{*2})^2}\frac{k\lambda_1V^*}{(1 + a_1V^* + a_2V^{*2})}\lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right)$$
$$= -\left(\frac{k\lambda_1V^*}{1 + a_1V^* + a_2V^{*2}} + \beta_1\right)(\gamma + \beta_1)(\lambda_2I^* + \beta_2) - \frac{k\lambda_1\beta_1S^*(a_2V^{*2} - 1)}{(1 + a_1V^* + a_2V^{*2})^2}\lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right)$$

$$\therefore detJ(E^*) < 0, \qquad \text{provided} V^{*2} > {}^{1}\!/_{a_2} \qquad (2.10)$$

Second additive compound matrix is given by

$$J^{[2]}(E^*) = \begin{bmatrix} -\frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} - 2\beta_1 - \gamma & \frac{k\lambda_1 S^*(1-a_2V^{*2})}{(1+a_1V^*+a_2V^{*2})^2} & \frac{k\lambda_1 S^*(1-a_2V^{*2})}{(1+a_1V^*+a_2V^{*2})^2} \\ \lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right) & -\frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} - \beta_1 - \lambda_2 I^* - \beta_2 & 0 \\ 0 & \frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} & -\gamma - \beta_1 - \lambda_2 I^* - \beta_2 \end{bmatrix}$$

$$\begin{aligned} detJ^{[2]}(E^*) = &\left(-\frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} - 2\beta_1 - \gamma\right)\left[\left(-\frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} - \beta_1 - \lambda_2 I^* - \beta_2\right)(-\gamma - \beta_1 - \lambda_2 I^* - \beta_2)\right] \\ &- \frac{k\lambda_1 S^*(1-a_2V^{*2})}{(1+a_1V^*+a_2V^{*2})^2}\left[\left(\lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right)\right)(-\gamma - \beta_1 - \lambda_2 I^* - \beta_2)\right] \\ &+ \frac{k\lambda_1 S^*(1-a_2V^{*2})}{(1+a_1V^*+a_2V^{*2})^2}\lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right)\frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} \\ = &-\left(\frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} + 2\beta_1 + \gamma\right)\left[\left(\frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}} + \beta_1 + \lambda_2 I^* + \beta_2\right)(\gamma + \beta_1 + \lambda_2 I^* + \beta_2)\right] \\ &- \frac{k\lambda_1 S^*(a_2V^{*2}-1)}{(1+a_1V^*+a_2V^{*2})^2}\left[\lambda_2\left(\frac{\alpha_2}{\beta_2} - V^*\right)\left(\gamma + \beta_1 + \lambda_2 I^* + \beta_2 + \frac{k\lambda_1 V^*}{1+a_1V^*+a_2V^{*2}}\right)\right] \\ &< 0, \quad \text{provided} V^{*2} > 1/a_2 \qquad (2.11) \end{aligned}$$

We have, tr J(E*), det J(E*) and det $J^{[2]}$ (E*) all negative. Hence, all eigen values of J(E*) have negative real part . Hence the theorem follows.

Theorem 4: If $R_0 > 1$, then system (2.12) is uniformly persistent, that is, there exists $\epsilon > 0$ (independent of initial conditions), such that $\lim\inf_{t\to\infty} S(t) > \epsilon$, $\lim\inf_{t\to\infty} I(t) > \epsilon$ and $\lim\inf_{t\to\infty} V(t) > \epsilon$.

Proof: To prove this, we show the following results:

(i) For system (4.23), E_0 is only omega-limit point on the boundary of Γ.

(ii) For $R_0 > 1$, E_0 cannot be the omega-limit point of any orbit in Int Γ.

(i) The vector field is transversal to the boundary of Γ, except in the S-axis, which is invariant for the system (2.2).

On the S-axis, we have

$$\frac{dS}{dt} = \alpha_1 - \beta_1 S$$

It follows from the above expression that $S \to \alpha_1/\beta_1$ as $t \to \infty$. So, the first part is proved.

(ii) Now we define the following function in Γ.

$$L_2(t) = \frac{\beta_1\beta_2(1+R_0)}{2k\lambda_1\alpha_1}I(t) + V(t)$$

$$\frac{dL_2(t)}{dt} = \frac{\beta_1\beta_2(1+R_0)}{2k\lambda_1\alpha_1}\frac{dI(t)}{dt} + \frac{dV(t)}{dt}$$

$$\begin{aligned} &= \frac{\beta_1\beta_2(1+R_0)}{2k\lambda_1\alpha_1}\left(\frac{k\lambda_1 S(t)V(t)}{1+a_1V(t)+a_2V^2(t)} - \gamma I(t) - \beta_1 I(t)\right) + \lambda_2\left(\frac{\alpha_2}{\beta_2} - V(t)\right)I(t) - \beta_2 V(t) \\ &= \frac{\beta_1\beta_2(1+R_0)}{2\alpha_1}\frac{S(t)V(t)}{(1+a_1V(t)+a_2V^2(t))} - \beta_2 V(t) + \lambda_2\left(\frac{\alpha_2}{\beta_2} - V(t)\right)I(t) - \frac{\beta_1\beta_2(1+R_0)}{2k\lambda_1\alpha_1}(\gamma+\beta_1)I(t) \\ &= \frac{\beta_1\beta_2(1+R_0)}{2\alpha_1}\frac{S(t)V(t)}{(1+a_1V(t)+a_2V^2(t))} - \beta_2 V(t) + \lambda_2\left(\frac{\alpha_2}{\beta_2} - V(t)\right)I(t) - \frac{(1+R_0)}{2R_0}\frac{\lambda_2\alpha_2}{\beta_2}I(t) \\ &= \frac{\beta_1\beta_2(1+R_0)}{2\alpha_1}\frac{S(t)V(t)}{(1+a_1V(t)+a_2V^2(t))} - \beta_2 V(t) + \lambda_2\left(\frac{\alpha_2}{\beta_2} - V(t) - \frac{1}{2}\left(\frac{1}{R_0}+1\right)\frac{\alpha_2}{\beta_2}\right)I(t) \\ &\geq \frac{\beta_1\beta_2(1+R_0)}{2\alpha_1}\left[S(t) - \frac{2\alpha_1}{\beta_1(1+R_0)}\right]V(t) + \lambda_2\left[\frac{\alpha_2}{\beta_2} - V(t) - \frac{1}{2}\left(\frac{1}{R_0}+1\right)\frac{\alpha_2}{\beta_2}\right]I(t) \end{aligned}$$

Since $R_0 > 1$, then $\frac{1}{2}\left(\frac{1}{R_0}+1\right) < 1$ and $\frac{2}{(1+R_0)} < 1$. Hence, there exists a neighborhood U of E_0 such that for $(S,I,V)\epsilon U \cap Int\Gamma$, the expressions $S(t) - \frac{2\alpha_1}{\beta_1(1+R_0)}$ and $\frac{\alpha_2}{\beta_2} - V(t) - \frac{1}{2}\left(\frac{1}{R_0}+1\right)\frac{\alpha_2}{\beta_2}$ are positive. In this neighborhood $U(E_0)$, we have that $\frac{dL_2(t)}{dt} > 0$ in $U(E_0) - \{E_0\}$. Now the level sets of L are the plane

$$\frac{\beta_1\beta_2(1+R_0)}{2k\lambda_1\alpha_1}I(t)+V(t)=C$$

which move away from the S-axis when C increases. Since L_2 is increasing along the orbits starting in $U \cap Int\Gamma$, all solutions of system (2.2) move away from E_0 .

Theorem 5: When $R_0 > 1$, the endemic equilibrium E^* is globally asymptotically stable.

Proof: The system (4.23) is uniformly persistent, and E^* is locally asymptotically stable for $R_0 > 1$. The following proposition shows that the system (4.23) has the property of stability of periodic orbits [46].

Let $P(t)=(S(t),I(t),V(t))$ *be a periodic solution of system* (2.2). To prove the stability of periodic orbits, it is sufficient to prove that the following linear non-autonomous system,

$$W'(t)=\left(J^{[2]}P(t)\right)W(t)$$

is asymptotically stable.

The second additive compound matrix is given by

$$J^{[2]}(S,I,V)=\begin{bmatrix} -\frac{k\lambda_1 V}{1+a_1V+a_2V^2}-2\beta_1-\gamma & \frac{k\lambda_1 S(1-a_2V^2)}{(1+a_1V+a_2V^2)^2} & \frac{k\lambda_1 S(1-a_2V^2)}{(1+a_1V+a_2V^2)^2} \\ \lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right) & -\frac{k\lambda_1 V}{1+a_1V+a_2V^2}-\beta_1-\lambda_2 I-\beta_2 & 0 \\ 0 & \frac{k\lambda_1 V}{1+a_1V+a_2V^2} & -(\gamma+\beta_1)-\lambda_2 I-\beta_2 \end{bmatrix}$$

For the solution $P(t)$, equation (2.11) becomes,

$$\begin{aligned} W_1'(t) &= -\left(\frac{k\lambda_1 V}{1+a_1V+a_2V^2}+2\beta_1+\gamma\right)W_1(t)+\frac{k\lambda_1 S(1-a_2V^2)}{(1+a_1V+a_2V^2)^2}W_2(t)+\frac{k\lambda_1 S(1-a_2V^2)}{(1+a_1V+a_2V^2)^2}W_3(t) \\ W_2'(t) &= \lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)W_1(t)-\left(\frac{k\lambda_1 V}{1+a_1V+a_2V^2}+\beta_1+\lambda_2 I+\beta_2\right)W_2(t) \quad (2.12) \\ W_3'(t) &= \frac{k\lambda_1 V}{1+a_1V+a_2V^2}W_2(t)-(\gamma+\beta_1+\lambda_2 I+\beta_2)W_3(t) \end{aligned}$$

To prove that system (2.12) is asymptotically stable, we consider the following function

$$L_3\big(W_1(t),W_2(t),W_3(t),S(t),I(t),V(t)\big)=\left\|W_1(t),\frac{I(t)}{V(t)}W_2(t),\frac{I(t)}{V(t)}W_3(t)\right\|$$

where $\|\cdot\|$ is the norm in $\mathbb{R}^3$ defined by

$$\|W_1(t),W_2(t),W_3(t)\|=\sup\{|W_1|,|W_2+W_3|\}$$

From theorem (2.10), we obtain that the orbit of $P(t)$ remains at a positive distance from the boundary of Γ. There exists constant c > 0 such that

$$L_3\big(W_1(t),W_2(t),W_3(t),S(t),I(t),V(t)\big)\geq c\|W_1(t),W_2(t),W_3(t)\| \quad (2.13)$$

Let $(W_1(t),W_2(t),W_3(t))$ be a solution of the system (3.6) and

$$L_3(t)=\sup\left\{|W_1(t)|,\frac{I(t)}{V(t)}|W_2(t)+W_3(t)|\right\} \quad (2.14)$$

Thus, we obtain the following inequalities

$$\begin{aligned} D_+|W_1(t)| &\leq -\left(\frac{k\lambda_1 V}{1+a_1V+a_2V^2}+2\beta_1+\gamma\right)|W_1(t)|+\frac{k\lambda_1 S(1-a_2V^2)}{(1+a_1V+a_2V^2)^2}(|W_2(t)+W_3(t)|) \\ &\leq -\left(\frac{k\lambda_1 V}{1+a_1V+a_2V^2}+2\beta_1+\gamma\right)|W_1(t)|+\frac{k\lambda_1 S(1-a_2V^2)}{(1+a_1V+a_2V^2)^2}\frac{V}{I}\left(\frac{I}{V}|W_2(t)+W_3(t)|\right) \\ D_+|W_2(t)| &\leq \lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)|W_1(t)|-\left(\frac{k\lambda_1 V}{1+a_1V+a_2V^2}+\beta_1+\lambda_2 I+\beta_2\right)|W_2(t)| \quad (2.15) \\ D_+|W_3(t)| &\leq \frac{k\lambda_1 V}{1+a_1V+a_2V^2}|W_2(t)|-(\gamma+\beta_1+\lambda_2 I+\beta_2)|W_3(t)| \end{aligned}$$

From second and third inequality of system (2.15), we have

$$\begin{aligned} D_+(|W_2(t)+W_3(t)|) &\leq \lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)|W_1(t)|-(\beta_1+\lambda_2 I+\beta_2)|W_2(t)|-(\gamma+\beta_1+\lambda_2 I+\beta_2)|W_3(t)| \\ &\leq \lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)|W_1(t)|-(\beta_1+\lambda_2 I+\beta_2)|W_2(t)+W_3(t)|-\gamma|W_3(t)| \\ &\leq \lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)|W_1(t)|-(\beta_1+\lambda_2 I+\beta_2)|W_2(t)+W_3(t)| \end{aligned}$$

Thus, we obtain

$$D_+\left(\frac{I}{V}|W_2(t)+W_3(t)|\right)=\left(\frac{I'}{I}-\frac{V'}{V}\right)\frac{I}{V}|W_2(t)+W_3(t)|+\frac{I}{V}D_+|W_2(t)+W_3(t)|$$

$$\leq \left(\frac{I'}{I}-\frac{V'}{V}\right)\frac{I}{V}|W_2(t)+W_3(t)|+\lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)\frac{I}{V}|W_1(t)|-\frac{I}{V}(\beta_1+\lambda_2 I+\beta_2)|W_2(t)+W_3(t)|$$

$$\leq \lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)\frac{I}{V}|W_1(t)|+\left[\frac{I'}{I}-\frac{V'}{V}-\beta_1-\lambda_2 I-\beta_2\right]\frac{I}{V}|W_2(t)+W_3(t)| \quad (4.37)$$

From the first equation of system (4.36) and equation (2.16), we get

$$D_+L_3(t)\leq sup\{g_1(t),g_2(t)\}W(t) \quad (2.17)$$

where

$$g_1(t)=-\left(\frac{k\lambda_1 V}{1+a_1V+a_2V^2}+2\beta_1+\gamma\right)+\frac{k\lambda_1 S(1-a_2V^2)}{(1+a_1V+a_2V^2)^2}\frac{V}{I}$$

$$g_2(t)=\lambda_2\left(\frac{\alpha_2}{\beta_2}-V\right)\frac{I}{V}+\frac{I'}{I}-\frac{V'}{V}-\beta_1-\lambda_2 I-\beta_2$$

Rewriting the second and third equation of system (4.23) as follows

$$\frac{I'}{I}=\frac{k\lambda_1 SV}{(1+a_1V+a_2V^2)I}-(\gamma+\beta_1) \quad (2.18)$$

$$\frac{V'}{V}=\lambda_2\left(\frac{\alpha_2}{\beta_2}-V(t)\right)\frac{I}{V}-\beta_2 \quad (2.19)$$

We have

$$g_1(t)\leq-\left(\frac{k\lambda_1 V}{1+a_1V+a_2V^2}+2\beta_1+\gamma\right)+\frac{k\lambda_1 S}{(1+a_1V+a_2V^2)^2}\frac{V}{I}$$

$$=-\frac{k\lambda_1 V}{1+a_1V+a_2V^2}-\beta_1-\beta_1-\gamma+\frac{k\lambda_1 S}{(1+a_1V+a_2V^2)^2}\frac{V}{I}$$

$$=-\frac{k\lambda_1 V}{1+a_1V+a_2V^2}-\beta_1+\frac{I'}{I}\text{ , using(4.39)}$$

$$g_1(t)<-\beta_1+\frac{I'}{I} \quad (2.20)$$

$$g_2(t)=\frac{V'}{V}+\frac{I'}{I}-\frac{V'}{V}-\beta_1-\lambda_2 I\text{ , using(2.19)}$$

$$g_2(t)=\frac{I'}{I}-\beta_1-\lambda_2 I$$

$$g_2(t)\leq-\beta_1+\frac{I'}{I} \quad (2.21)$$

Hence

$$sup\{g_1(t),g_2(t)\}\leq-\beta_1+\frac{I'}{I}$$

From equation (2.17) and Gronwall's inequality , we obtain

$$L_3(t)\leq L_3(0)I(t)e^{-\beta_1 t}<L_3(0)\frac{\alpha_1}{\mu_1}e^{-\beta_1 t}$$

which implies that L_3 (t)→0 as t→∞. By (4.34), we obtain

$$\big(W_1(t),W_2(t),W_3(t)\big)\to 0 \text{ as } t\to\infty.$$

Hence the linear system (2.12) is asymptotically stable and hence the periodic solution is asymptotically orbitally stable. Therefore, the endemic equilibrium E^* is globally asymptotically stable.

IV. NUMERICAL SIMULATION

We present the numerical simulation to validate the theoretical results using MATLAB. Fig. 3.1 shows that the disease-free equilibrium exists for $R_0 < 1$. Fig. 3.2 indicates that disease becomes endemic for $R_0 > 1$. As the parameters a_1 and a_2 increase, number of infective individuals decreases. This is shown in fig .3.3 and fig.3.4, respectively.

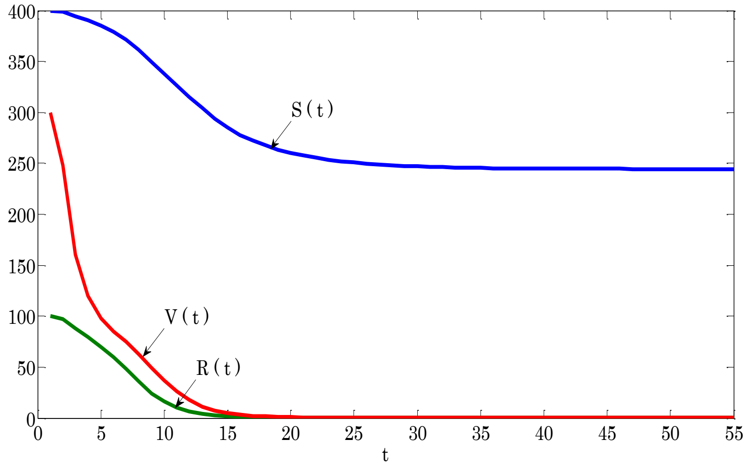

Fig.3.1 Here, $S(0) = 400, I(0) = 100, V(0) = 300, k = 0.016, \alpha_1 = 11, \alpha_2 = 100, \lambda_1 = 0.007, \lambda_2 = 0.009, \beta_1 = 0.045, \ \beta_2 = 0.61, \gamma = 0.12, a_1 = 0.02, \ a_2 = 0.01, R_0 = 0.40.$

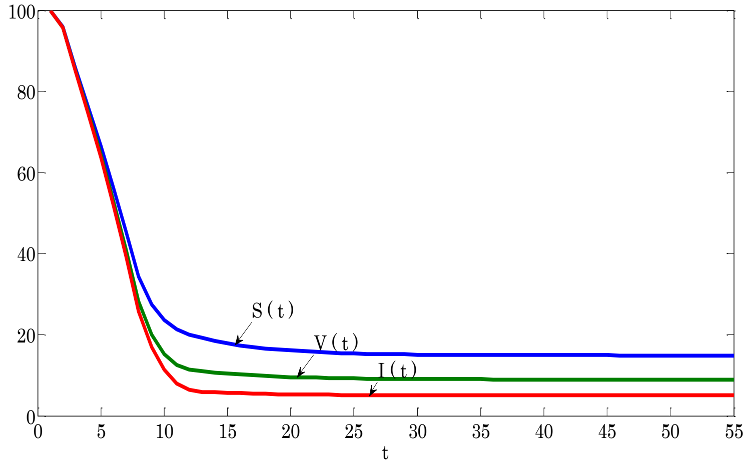

Fig.3.2 Here, $S(0) = 400, I(0) = 100, V(0) = 300, k = 1.1, \alpha_1 = 11, \alpha_2 = 100, \lambda_1 = 0.007, \lambda_2 = 0.009, \beta_1 = 0.045, \ \beta_2 = 0.61, \gamma = 0.12, a_1 = 0.02, \ a_2 = 0.01, R_0 = 27.59.$

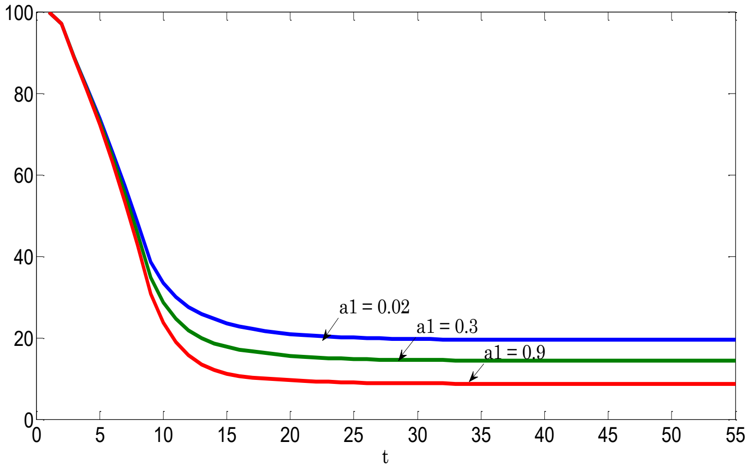

Fig. 3.3 Dependence of I^* on a_1

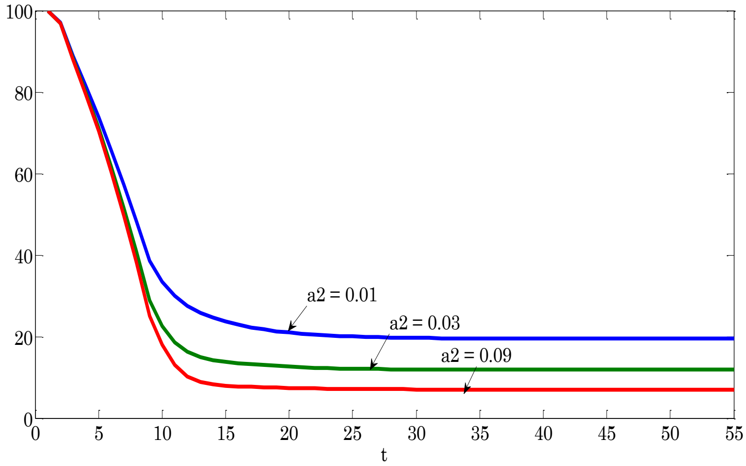

Fig. 3.4 Dependence of I^* on a_2

V. DISCUSSION AND CONCLUSION

This study examined the transmission dynamics and evolutionary aspects of vector-borne diseases by extending classical vector–host models to incorporate parameters representing hygienic practices and the application of insecticides or repellents for vector management. Analysis of the basic reproduction numbers, together with numerical simulations of the four models considered in this chapter, reveals that the implementation of stronger preventive interventions, such as quarantine, isolation, and enhanced vector control in endemic regions, can substantially reduce the infected population, lower transmission intensity, and decrease the environmental carrying capacity for mosquitoes, thereby contributing to disease elimination.The estimated basic reproduction numbers $R_0$for these models are considerably larger than those reported in other models,suggesting that the current levels of host-based preventive behavior and vector control efforts are insufficient to suppress disease persistence. If such measures remain inadequate, the risk of recurrent or future outbreaks remains significant. These findings underscore the need for public health authorities to design and implement more rigorous and effective intervention strategies for controlling vector-borne epidemics.

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