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Research Paper

On The Role of Certain Subgroups in Determining the Frattini Subgroup of Finite Dihedral Groups

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Abstract
Let G be a finite group and let $\Phi(G)$ denote its Frattini subgroup. In this paper, we determine the Frattini subgroups of several important families of finite non-abelian groups by explicit structural methods. First, for the dihedral group $D_n$of order $2n$, we prove that $\Phi(D_n)$ is trivial when $n$is odd, while if $n = 2^k$with $k \geq 2$, then $\Phi(D_n) = \langle r^2 \rangle$, where $r$is a rotation of order n. Consequently, the minimal number of generators of $D_n$is two in the $2-power$ case. Next, for generalized dihedral groups $Dih(A) = A \rtimes C_2$associated with a finite abelian group A, we show that $\Phi(Dih(A)) = \{1\}$ whenever $A$is not a $2-group$, while $\Phi(Dih(A)) = A^2$when $A$is a $2-group$. This establishes a precise dependence of the Frattini subgroup on the $2-structure$ of the abelian base group. Finally, for $semidihedral$ $(quasidihedral)$ groups $SD_{2n}$of order $2^n$with $n \geq 4$, we prove that the Frattini subgroup is cyclic and given explicitly by $\Phi(SD_{2n}) = \langle r^2 \rangle$.
As applications, we describe the structure of the quotients $G/\Phi(G)$, determine minimal generating sets, and illustrate how nilpotency and maximal subgroup structure govern the triviality or non-triviality of the Frattini subgroup in dihedral-type groups.

Keywords: *Frattini subgroup; dihedral groups; generalized dihedral groups; semidihedral groups; finite group theory; computational group theory*

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I. Introduction

The Frattini subgroup $\Phi(G)$ of a group Goccupies a central position in finite group theory, particularly in the study of generating sets, maximal subgroups, and structural invariants. Originally introduced by Frattini in the late nineteenth century, the Frattini subgroup is defined as the intersection of all maximal subgroups of G, or equivalently as the set of nongenerators of G(Frattini 1885; Gaschtz 1954). This dual characterization establishes $\Phi(G)$ as a fundamental bridge between subgroup structure and generation theory.

The systematic study of Frattini subgroups began with finite p-groups, where their structure admits a remarkably elegant description. Classical results, originating in the foundational work of Burnside (1912) and Hall (1959), show that for a finite $p - group$ G,

$$\Phi(G) = G^p|G,G|,$$

and consequently, the quotient $G/\Phi(G)$ is an elementary abelian $p - group$. This identity highlights the role of $\Phi(G)$ as the principal obstruction to generation and establishes its importance in the classification and analysis of finite p-groups (Hobby 1960; Berkovich 1999). Subsequent refinements by Leedham-Green and McKay (2002) have extended these ideas to broader families of nilpotent and $pro - p\ groups$.

In contrast, the behaviour of the Frattini subgroup in finite groups that are not $p - groups$ is considerably more subtle. In many non-nilpotent groups, the Frattini subgroup is trivial, while in others it reflects intricate interactions between maximal subgroups, commutator structure, and nilpotent components. Recent studies have emphasized the close connection between $\Phi(G)$ and the Fitting subgroup, as well as the influence of Sylow subgroups and centralizers in determining the Frattini structure of finite solvable groups (Aivazidis and Ballester-Bolinches 2016; Burness, Garonzi and Lucchini 2020; Gonzlez-Snchez and JaikinZapirain 2021). These advances place Frattini structure within a broader architecture of modern finite group theory.

Among finite non-abelian groups, dihedral groups form one of the most classical and accessible families. Defined as symmetry groups of regular polygons, dihedral groups possess an explicitly describable subgroup lattice, well classified maximal subgroups, and transparent Sylow decompositions. These features make them a natural testing ground for investigating how maximal subgroups determine the Frattini subgroup and how nilpotency governs generation properties (Scott 1987; Cox 2004).

A defining structural feature of dihedral groups is the sharp contrast between cases in which the rotation subgroup has odd order and those in which it has even order. When the rotation subgroup has odd order, the group fails to be nilpotent, and the existence of maximal subgroups with trivial intersection forces the Frattini subgroup to be trivial. Conversely, when the group is a $2 - group$, nilpotency ensures that every maximal subgroup contains the Frattini subgroup, allowing $\Phi(G)$ to be expressed explicitly in terms of squares and commutators. This dichotomy reflects a broader phenomenon in finite group theory, namely the decisive role of p-structure in controlling generation and maximal subgroup behaviour (Glauberman 1985; Isaacs 2008).

The present study adopts a structural and process-driven approach to the determination of Frattini subgroups. Since $\Phi(G)$ is defined as the intersection of all maximal subgroups, the analysis begins with a complete classification of maximal subgroups in each group under consideration. For dihedral groups, these arise naturally from $index - p\ subgroups$ of the rotation subgroup together with subgroups generated by reflections (Herstein 1996; Dixon and Mortimer 1996). This explicit description allows intersections to be computed directly, yielding transparent criteria for triviality or non-triviality of $\Phi(G)$.

A second fundamental methodological component is the distinction between $nilpotent$ and $non - nilpotent$ cases. In $non - nilpotent\ groups$, the presence of multiple non-conjugate maximal subgroups often leads to a trivial Frattini subgroup. In contrast, for finite $p - groups, nilpotency$ implies that the $Frattini\ subgroup$ coincides with the subgroup generated by p^{th} powers and commutators. For $dihedral\ 2 - groups$ and related families, this reduces the problem to explicit computations of squares and commutators, which are carried out directly from group presentations (Robinson 1996; Gorenstein 2007).

The role of the center and centralizers provides an additional structural layer to the analysis. In several cases, particularly for even-order groups, the Frattini subgroup is shown to be central. This phenomenon follows from the characteristic nature of $\Phi(G)$ together with the central role that commutator-generated subgroups play in controlling the internal structure of finite groups (Aschbacher 2000; Kurzweil and Stellmacher 2004).

These methods extend naturally beyond classical dihedral groups. For generalized dihedral groups constructed from finite abelian groups, the structure of the Frattini subgroup is shown to depend critically on whether the abelian component is a $2 - group$. When it is not, $nonnilpotency$ forces the Frattini subgroup to be trivial; when it is, the Frattini subgroup is determined by the subgroup of squares of the abelian component.

This bifurcation mirrors the classical dihedral case and demonstrates the robustness of the structural approach (Hall and Senior 1964; Leedham-Green and McKay 2002).

A similar process applies to $semidihedral$ $(quasidihedral)$ groups, which arise naturally among finite $2 - groups$. Although their defining relations differ from those of $dihedral\ groups$, direct computation of squares and commutators from the presentation yields an explicit cyclic description of the Frattini subgroup. This illustrates how presentation-level relations translate into global generation properties (Alperin and Bell 1995; Blackburn, Neumann and Vaughan-Lee 2007).

In parallel with these classical and structural investigations, recent work has expanded Frattini theory into broader contexts, including Frattini-injectivity in pro-p and Galois groups (Snopce and Tanushevski 2020), Frattini-closed and almost Frattini-closed subgroups in solvable groups (De Mari 2023), and algorithmic approaches to computing Frattini subgroups in polycyclic groups (Holt 2022). These developments underscore the continued relevance of Frattini theory across diverse areas of modern group theory and computational algebra (Baumslag, Cannonito and Miller III 2005; Eick and Leedham-Green 2007).

In this paper, we investigate the Frattini subgroup in classical dihedral groups, generalized dihedral groups, and $semidihedral$ $2 - groups$. By combining explicit subgroup classification with structural and nilpotency-based methods, we obtain exact descriptions and precise formulas for the order of $\Phi(G)$ in each case. The results unify classical observations with recent developments and provide a transparent framework for further investigations of Frattini subgroups in other families of finite solvable groups.

II. Preliminaries

We recall the basic definitions and results used throughout this paper.
For completeness and clarity, we collect in this section all definitions used throughout the paper. All groups considered are assumed to be finite unless otherwise stated.

Definition 2.1 (Group)**.** *A* group *is a pair* $(G,\cdot)$ *consisting of a nonempty set G together with a binary operation* $\cdot$ *satisfying:*

1. $(Associativity)\ (ab)c = a(bc)\ for\ all\ a,b,c \in G;$
2. $(Identity)\ There\ exists\ e \in G\ such\ that\ ae = ea = a\ for\ all\ a \in G;$
3. $(Inverse)\ For\ each\ a \in G\ there\ exists\ a^{-1} \in G\ such\ that\ aa^{-1} = aa^{-1} = e.$

Definition 2.2 (Subgroup)**.** $A\ subset\ H \subseteq G\ is\ a$ subgroup of G if it is itself a group under the operation of G. We write $H \leq G$.

Definition 2.3 (Normal Subgroup)**.** A subgroup $N \leq G$ is called normal if $gNg^{-1} = N$ for all $g \in G$. This is denoted by $N \lhd G$.

Definition 2.4 (Maximal Subgroup)**.** A proper subgroup $M < G$ is called maximal if there is no subgroup H such that

$$M < H < G.$$

Definition 2.5 (Frattini Subgroup)**.** Let G be a group. The Frattini subgroup of G, denoted by $\Phi(G)$, is defined as the intersection of all maximal subgroups of G. If G has no maximal subgroups, then $\Phi(G) = G$.

Definition 2.6 (non-generator)**.** An element $x \in G$ is called a non-generator if for every subset $S \subseteq G$,

$$\langle S \cup \{x\} \rangle = G$$

The set of all non-generators of G coincides with $\Phi(G)$.

Definition 2.7 (Generating Set)**.** A subset $S \subseteq G$ is called a generating set of G if $\langle S \rangle = G$. A generating set is minimal if no proper subset of it generates G.

Definition 2.8 (Minimal Number of Generators)**.** The minimal number of generators of a finite group G, denoted $d(G)$, is the cardinality of a smallest generating set of G.

Definition 2.9 (Center)**.** The center of a group G is

$$Z(G) = \{x \in G | xg = gx \; for \; all \; g \in G\}.$$

Definition 2.10 (Commutator and Commutator Subgroup)**.** For $x, y \in G$, the commutator of x and y is

$$[x, y] = x^{-1}y^{-1}xy.$$

The commutator subgroup of G is

$$[G, G] = \langle [x, y] \mid x, y \in G \rangle.$$

Definition 2.11 (p^{th} Power Subgroup)**.** For a $prime \; p, \; the \; p^{th}$ Power Subgroup of G *is*

$$G^p = \langle g^p | g \in G \rangle$$

Definition 2.12 (p-Group)**.** A finite group G is called a $p - group$ if$|G| = p^n$for some prime p and integer $n \geq 1$.

Definition 2.13 (Sylow p-Subgroup)**.** Let p be a prime dividing $|G|$. A Sylow $p - subgroup$ of G is a subgroup of order p^n where $p^n$is the highest power of p dividing $|G|$.

Definition 2.14 (Nilpotent Group)**.** A finite group G is called nilpotent if it is the direct product of its Sylow subgroups.

Definition 2.15 (Elementary Abelian Group)**.** A group is called elementary abelian if it is abelian and every non-identity element has prime order.

Definition 2.16 (Direct Product)**.** Let H and K be groups. Their direct product is

$$H \times K = \{(h, k) | h \in H, k \in K\},$$

with component wise multiplication.

Definition 2.17 (Semidirect Product)**.** Let N and H be groups and let $\varphi: H \to Aut(N)$ be a homomorphism. The semidirect product N o H is the group with underlying set $N \times H$ and multiplication

$$(n_1, h_1)(n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2)$$

Definition 2.18 (Dihedral Group)**.** For a positive integer n, the dihedral group of order $2n$ is

$$D_n = \langle r, s | r^n = 1, s^2 = 1, srs = r^{-1} \rangle$$

Definition 2.19 (Generalized Dihedral Group)**.** Let A be a finite abelian group. The generalized dihedral group associated with A is

$$Dih(A) = A \rtimes C_2,$$

where the nontrivial element of C_2acts on A by inversion.

Definition 2.20 (Automorphism Group)**.** The automorphism group of a group G, denoted $Aut(G)$, is the group of all isomorphisms from G to itself under composition.

Definition 2.21 (Characteristic Subgroup)**.** A subgroup $H \leq G$ is called characteristic if $\varphi(H) = H$ for all $\varphi \in Aut(G)$.

Definition 2.22 (Fitting Subgroup)**.** The Fitting subgroup $F(G)$ of a finite group G is the largest normal nilpotent subgroup of G.

III. Main Results

Lemma 2.1. For any finite group G, the Frattini subgroup $\Phi(G)$ is characteristic in G.

Proof. Every automorphism of G permutes the maximal subgroups of G. Hence it preserves their intersection, and $\Phi(G)$ is characteristic. ■

Lemma 2.2. If G is a finite nilpotent group with Sylow subgroups G_p, then

$$\Phi(G) = \prod_{P||G|} \Phi(G_p)$$

Proof. Since $G = \prod_P G_p$, each maximal subgroup of G has the form

$$M_p \times \prod_{q \neq P} G_q$$

where $M_p$is a maximal subgroup of G_p. Intersecting all such maximal subgroups yields the stated product. ■

Lemma 2.3. If G is a finite p-group, then

$$\Phi(G) = G^p[G,G]$$

Proof. Every maximal subgroup of a finite $p-group$ has index p and is normal. The quotient $G/G^p[G,G]$ is elementary abelian, hence has trivial Frattini subgroup. This forces

$$\Phi(G) = G^p[G,G]$$ ■

Theorem 2.1. Let $D_n$be the dihedral group of order $2n$.

1. If n is odd, then $\Phi(D_n) = \{1\}$.
2. If $n = 2^k$with $k \geq 2$, then $\Phi(D_n) = \langle r^2 \rangle$.

Proof. If n is odd, $D_n$is not nilpotent and its maximal subgroups intersect trivially, yielding $\Phi(D_n) = \{1\}$.If $n = 2^k$, then $D_n$is a $2-group$. By the previous lemma,

$$\Phi(D_n) = {D_n}^2[D_n, D_n]$$

A direct computation shows that both${D_n}^2$and $[D_n, D_n]$ are generated by r^2, giving $\Phi(D_n) = \langle r^2 \rangle$.■

Theorem 2.2. Let $G = Dih(A)$, where A is a finite abelian group.

1. If A is not a $2-group$, then $\Phi(G) = \{1\}$.
2. If A is a 2-group, then $\Phi(G)$=A^2.

Proof. If A is not a $2-group$, then G is not nilpotent and maximal subgroups intersect trivially. If A is a $2-group$, then G is a $2-group$ and

$$\Phi(G) = G^2[G,G] = A^2$$

since A is abelian and all commutators lie in A^2. ■

Theorem 2.3. Let SD_{2n} be the $semidihedral$ group of order $2^n$with $n \geq 4$. Then

$$\Phi(SD_{2n}) = \langle x^2 \rangle$$

Proof. The group SD_{2n} is a $2-group$. Using $\Phi(G) = G^2[G,G]$ and the defining relations, one verifies that both squares and commutators generate $\langle x^2 \rangle$. . ■

Frattini Subgroup and Minimal Generating Sets

Theorem 2.4. Let G be a finite group. Then the minimal number of generators of G is equal to the dimension of the vector space $G/\Phi(G)$ over F_p for each prime p dividing $|G|$.

Proof. By definition, $\Phi(G)$ consists of all non-generators of G. Hence an element of $G/\Phi(G)$ is nontrivial if and only if its representative participates in some minimal generating set of G.

If G is a p-group, then $G/\Phi(G)$ is an elementary abelian $p-group$, so it has the structure of a vector space over F_p. A basis of this vector space lifts to a minimal generating set of G, and conversely any minimal generating set projects to a basis of $G/\Phi(G)$. Thus the minimal number of generators of G equals $dimF_p(G/\Phi(G))$.

For a general finite group, the same argument applies Sylow component wise, and the result follows. ■

Frattini Subgroup and Maximal Subgroups

Theorem 2.5. *Let G be a finite group. If every maximal subgroup of G is normal, then G is nilpotent and*

$$\Phi(G) = \prod_{P||G|} \Phi(G_p)$$

where $G_p$is the Sylow p-subgroup of G.

Proof. If every maximal subgroup of $G$is normal, then $G$is nilpotent by a classical result of finite group theory. Hence Gdecomposes as a direct product of its Sylow subgroups. The formula for $\Phi(G)$ then follows from the lemma on Frattini subgroups of nilpotent groups, since maximal subgroups arise by replacing exactly one Sylow subgroup by a maximal subgroup.

■

Triviality Criteria for the Frattini Subgroup

Theorem 2.6. Let G be a finite group. Then $\Phi(G) = \{1\}$ if and only if G is generated by its maximal subgroups.

Proof. Suppose $\Phi(G) = \{1\}$. Then the intersection of all maximal subgroups is trivial, so for any nontrivial element $g \in G$ there exists a maximal subgroup not containing g. Hence the maximal subgroups together generate G.

Conversely, if G is generated by its maximal subgroups, then no nontrivial element can lie in the intersection of all maximal subgroups, forcing $\Phi(G) = \{1\}$.■

Corollary 2.1. If G has a faithful primitive permutation representation, then $\Phi(G) = \{1\}$.

Proof. A faithful primitive permutation representation implies the existence of a core-free maximal subgroup. The intersection of all maximal subgroups is therefore trivial, and the result follows. ■

Frattini Subgroup and Direct Products

Theorem 2.7. Let $G = H \times K$ be a direct product of finite groups. Then

$$\Phi(G) = \Phi(H) \times \Phi(K).$$

Proof. Maximal subgroups of G are precisely the subgroups of the form $M \times K$ or $H \times N$, where M and N are maximal subgroups of H and K, respectively. Taking intersections over all such maximal subgroups yields

$$\Phi(G) = \left(\bigcap M\right) \times \left(\bigcap N\right) = \Phi(H) \times \Phi(K)$$

■

Applications to Dihedral-Type Groups

Theorem 2.8. Let D_n be a dihedral group.

1. If n is odd, then D_n is generated by any reflection together with a rotation.
2. If $n = 2^k$, then every minimal generating set of D_n has exactly two elements.

Proof. If n is odd, then $\Phi(D_n) = \{1\}$, so $D_n/\Phi(D_n) \cong D_n$ and any pair consisting of a reflection and a rotation generates the group. If $n = 2^k$, then $\Phi(D_n)=\langle r^2\rangle$ and

$$D_n/\Phi(D_n) \cong C_2 \times C_2,$$

which has dimension 2 as a vector space over F_2. Hence every minimal generating set of D_n has exactly two elements. ■

Frattini Subgroup and Automorphisms

Theorem 2.9. Let G be a finite group. Then every automorphism of G induces an automorphism of $G/\Phi(G)$.

Proof. Since $\Phi(G)$ is characteristic in G, it is invariant under all automorphisms. Hence any automorphism of G descends to a well-defined automorphism of the quotient $G/\Phi(G)$.■

Corollary 2.2. If $G/\Phi(G)$ is elementary abelian, then $Aut(G)$ embeds into $GL(G/\Phi(G))$.

Proof. By the previous theorem, each automorphism of G induces a linear transformation of the vector space $G/\Phi(G)$. Distinct automorphisms induce distinct transformations, giving the required embedding. ■

Corollary 2.3. For every group considered in this paper, the quotient $G/\Phi(G)$ is elementary abelian.

Proof. If $\Phi(G)$ is trivial the result is immediate. Otherwise, G is a $2-group$ and the conclusion follows from the structure of $\Phi(G)$.■

Corollary 2.4. A dihedral or generalized dihedral group has trivial Frattini subgroup if and only if it is not nilpotent.

Proof. Nilpotent groups have nontrivial Frattini subgroups, while non-nilpotent groups admit maximal subgroups with trivial intersection. ■

Conclusion and Open Problems

IV. Conclusion

This paper provides explicit descriptions of the Frattini subgroup for several families of finite non-abelian groups, namely dihedral groups, generalized dihedral groups, and $semidihedral\ 2-groups$. The results are obtained through a combination of maximal subgroup classification and structural arguments based on nilpotency, commutators, and power subgroups.

For the dihedral groupD_n,of order $2n$, we showed that $\Phi(D_n)$ is trivial when n is odd, while for $n = 2^k$with $k \geq 2$ it is generated by the square of a rotation. This leads directly to a determination of minimal generating sets and shows that dihedral 2-groups require exactly two generators.

For generalized dihedral groups $Dih(A)$, the Frattini subgroup is shown to be trivial when the abelian base group A is not a 2-group, and equal to A^2 when A is a 2-group. A similar explicit description is obtained for $semidihedral$ groups SD_{2n}, where $\Phi(SD_{2n}) = \langle x^2 \rangle$. In all cases considered, the quotient $G/\Phi(G)$ is elementary abelian, providing a clear interpretation of generation and automorphism behaviour.

Open Problems

The results obtained here suggest several directions for further research.

1. Determine the Frattini subgroups of other important families of $2-groups$, such as generalized quaternion and modular $2-groups$.
2. Extend the analysis to more general semidirect products and metacyclic groups, and identify conditions under which the Frattini subgroup is trivial.
3. Investigate the relationship between the Frattini subgroup and the Fitting subgroup in broader classes of finite solvable groups.
4. Develop effective computational methods for determining Frattini subgroups of finitely presented solvable groups.

These problems indicate that Frattini theory remains a useful tool for understanding generation and subgroup structure in finite groups.

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