Quest Journals
Journal of Research in Applied Mathematics
Volume 12 ~ Issue 2 (February 2026) pp: 32-53
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

A survey the spectral analysis of three families of exceptional Laguerre polynomials

Fathi Omran[(1)] and Shawgy Hussein[(2)]
[(1)] *Sudan University of Science and Technology.*
[(2)] *Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.*

Abstract

Following in the same way the systematic methodology and smooth theory of the pioneers authors in their survey paper [31] showing all exceptional sequences with a non singular weight, that they form a complete orthogonal set in their natural Hilbert space setting. Among the exceptional sequence of sets already known are two types of exceptional Laguerre polynomials, called the Type I and Type II exceptional Laguerre polynomials, each delete and omitting m polynomials. They clearly discuss these polynomials and construct the self-adjoint operators generated by their corresponding second-order differential expressions in the complete Hilbert spaces. They present a novel derivation of the Type III family of exceptional Laguerre polynomials along with a detailed wide of its properties. They include several basic representations of these polynomials, orthogonality, norms, completeness, the location of their local extrema and roots, root asymptotics, as well as a complete spectral study of the second-order Type III exceptional Laguerre differential expression. An application and an abbreviation for competents are verified and valid.

Keywords: Orthogonal polynomials; Spectral theory; Root asymptotics

Received 23 Jan., 2026; Revised 04 Feb., 2026; Accepted 06 Feb., 2026 *© The author(s) 2026.*
Published with open access at ***www.questjournas.org***

I. Introduction

An exceptional orthogonal polynomial system is a sequence $\{p_n\}_{n\in\mathbb{N}_0\setminus A}$ with the following characteristic properties:

(a) $deg(p_n) = n$ for $n \in \mathbb{N}_0\setminus A$, where A is a finite subset of $\mathbb{N}_0$;

(b) there exists an interval $I = (a, b)$ and a Lebesgue measurable weight $w > 0$ on I such that

$$\int_I p_n\, p_m\, w = k_n \delta_{n,m} \ \ (n, m \in \mathbb{N}_0\setminus A)$$

for some $k_n > 0$; here $\delta_{n,m}$ denotes the Kronecker delta symbol;

(c) there exists a second-order differential expression

$$\ell[y](x_s) = a_2(x_s)y''(x_s) + a_1(x_s)y'(x_s) + a_0(x_s)y(x_s)$$

and, for each $n \in \mathbb{N}_0\setminus A$, there exists a $\lambda_n^s \in \mathbb{C}$ such that $y = p_n(x_s)$ is a solution of

$$\ell[y](x_s) = \lambda_n^s\, y(x_s)\ (x_s \in I);$$

(d) for $n \in A$, there does not exist a polynomial p of degree n such that $y = p(x_s)$ satisfies $\ell[y] = \lambda^s y$ for any choice of $\lambda^s \in \mathbb{C}$;

(e) all of the moments

$$\int_I x_s^n w(x_s)dx_s \qquad (n \in \mathbb{N}_0)$$

of w exist and are finite.

We call a sequence $\{p_n\}_{n\in\mathbb{N}_0\setminus A}$ satisfying conditions (a).(e) above an exceptional polynomial sequence of codimension $|A|$. Where $|A|$ is the cardinality of A for $A = \{0\}$ see [12], where the authors classified exceptional orthogonal polynomials with one missing degree, and introduced the X_1-Laguerre and the X_1-Jacobi polynomials. The fact that these sequences omit a constant polynomial distinguishes their characterization from the Bochner classification [2] characterizing the Jacobi, Laguerre, and Hermite polynomials; of course, the Bochner classification corresponds to $A = \phi$. See [12] by finding other sequences of exceptional polynomials $\{p_n\}_{n\in\mathbb{N}_0\setminus A}$, satisfying each of the conditions in (a).(e).

We show (see [31]) the three families of exceptional Laguerre polynomials, each spanning a flag of codimension m. So, we deal with two such exceptional Laguerre sequences associated with

$$A = \{0, 1, \ldots, m.1\} \tag{1.1}$$

and another where

$$A = \{1, 2, \ldots, m\}. \tag{1.2}$$

The two exceptional Laguerre sequences are known as the Type I and Type II exceptional Laguerre polynomials with (1.1), obtained by [25,24]; for properties see [14,15,18]. We develop (see [31]) the spectral theory of the two second-order exceptional Laguerre differential equations having the Type I and Type II sequences as eigenfunctions. We name the sequences of polynomials associated with (1.2), type III exceptional Laguerre polynomials, see [10].

Recently, the exceptional orthogonal polynomials is one of the most interesting, and intensive studies see [13.15,11,17,16,20,21,25,24,26,27], [22], [7,9], [29], [19]. The lowest ($m = 1$) such examples, the exceptional X_1-Laguerre and X_1-Jacobi polynomials, are equivalent to those introduced by [12]. Subsequently, (see [26,27]) who first related these orthogonal polynomials to the Darboux transformation and classical orthogonal polynomials. [25] first introduced higher order codimension exceptional orthogonal polynomials for arbitrary positive integers m.

Now, there are several implications of generalizing the Bochner classification theorem. The new exceptional orthogonal polynomials see [23] of constructing self-adjoint operators from Lagrangian symmetric secondorder differential expressions. The exceptional orthogonal polynomial sequences found to date are complete in Hilbert space setting. Then the three exceptional Laguerre sequences that we show (see [31]) are complete since each of them are missing m polynomials. Their completeness suggests that interesting Müntz-type theorems (see, for example, [3] and [5, Theorem 7.6]) in weighted L^2-spaces.

Hence in the following, we review some essential facts shown [31] about the classical Laguerre expression and its solutions, give a rational factorizations and the Darboux transformation as it relates to the Laguerre case, the main properties of the Type I exceptional Laguerre polynomials and then develop the spectral theory of their associated second-order differential expression, construct a self-adjoint operator, generated from the second-order Type I exceptional Laguerre differential expression, which has the Type I exceptional Laguerre polynomials as eigenfunctions, discuss another interesting self-adjoint operator, generated by the Type I exceptional Laguerre differential expression, which has a complete set of eigenfunctions involving the Type III exceptional Laguerre polynomials, treat Type II exceptional Laguerre polynomials in a similar fashion and introduce the Type III exceptional Laguerre polynomials and develop many of their properties. They also compute the norms of these polynomials in Hilbert space H and show that the sequence of Type III exceptional Laguerre polynomials, despite missing polynomials p of degrees $1 \le deg(p) \le m$, forms a complete orthogonal set of polynomials in H, determine the location of the roots of these polynomials, and show that the Type III exceptional Laguerre polynomial of degree $(m + 1 + \epsilon)$ has $(1 + \epsilon)$ positive roots and m negative roots and we give properties of these roots with the roots of the two classical Laguerre polynomials $L_\epsilon^{2+\epsilon}(x_s)$ and $L_\epsilon^{-(2+\epsilon)}(-x_s)$. They also discuss the asymptotic behavior of the roots of these Type III exceptional Laguerre polynomials, and develop spectral properties of the second-order Type III exceptional Laguerre differential expression $\ell_m^{\mathrm{III},1+\epsilon}[.]$and, in particular, determine the self-adjoint operator in H, generated by $\ell_m^{\mathrm{III},1+\epsilon}[.]$, which has the Type III exceptional Laguerre polynomials as eigenfunctions. Lastly, in the Appendix, list some examples of Type III exceptional Laguerre polynomials (see [31]).

All exceptional differential expressions considered are related to the classical Laguerre differential expression by a 1-step Darboux transformation. So, the class of exceptional Laguerre polynomials was generalized to include families that are related to the classical Laguerre polynomials by a multi-step Darboux transformation [6]. Indeed, it can be shown that all three families considered here are particular cases of a general scheme introduced. For the type III class the equivalence takes the form of some interesting identities that involve Wronskian-like determinants of Laguerre polynomials. Thus, Theorems 5.3 and 5.4 are particular cases of Corollary 6.4 and Theorem 6.3 in [6], respectively (see [31]).

For $\mathbb{N}_0 := \mathbb{N}_0 \cup \{0\}$. The set $\mathcal{P}$ will denote the vector space of all complex-valued polynomials $p(x_s)$ in the real variable x_s. For $n \in \mathbb{N}_0$, let $\mathcal{P}_n$ denote the $(n + 1)$-dimensional vector space of all polynomials of degree$\le m + \epsilon$.

2. The classical Laguerre differential expression and rational factorizations

Let

$$\ell^{1+\epsilon}[y] = -x_s\, y'' + (-(2+\epsilon) + x_s)y' \tag{2.1}$$

denote the classical Laguerre differential expression. Then, for each $m \in \mathbb{N}_0$, $y = L_m^{1+\epsilon}(x_s)$ is a solution of

$$\ell^{1+\epsilon}[y](x_s) = my(x_s).$$

Remark 2.1 [31]. In the contributions [12.15], the authors define the Laguerre expression as

$$\tilde{\ell}^{1+\epsilon}[y] = x_s y'' + (2 + \epsilon - x_s)y';$$

for operator-theoretic and spectral-analytic reasons, we elect to define the Laguerre expression as in (2.1).

A rational factorization of $-\ell^{1+\epsilon}[.]$ is an identity of the form

$$-\ell^{1+\epsilon} = BA - \lambda^s, \tag{2.2}$$

where A and B are first-order linear differential expressions with rational coefficients. We call

$$\hat{\ell}^{1+\epsilon} := AB - \lambda^s \tag{2.3}$$

the partner operator corresponding to the above rational factorization. Suppose $\phi(x_s)$ is a quasirational solution (that is, $\frac{\phi'(x_s)}{\phi(x_s)}$ is a rational function) of $A[y] = 0$. Notice, from (2.2), that

$$\ell^{1+\epsilon}[\phi](x_s) = \lambda^s \phi(x_s).$$

The operators A and B are given by

$$A[y](x_s) = b(x_s)\left(y'(x_s) - \frac{\phi'(x_s)}{\phi(x_s)}\; y(x_s)\right)$$

and

$$B[y](x_s) = \hat{b}(x_s)\big(y'(x_s) - \hat{w}(x_s)y(x_s)\big),$$

where $b(x_s)$ is a rational function, called the factorization gauge, and

$$\hat{b}(x_s) = \frac{x_s}{b(x_s)}, \qquad \hat{w}(x_s) = \frac{\phi'(x_s)}{\phi(x_s)} + \frac{b'(x_s)}{b(x_s)} - \frac{2+\epsilon-x_s}{x_s}.$$

In this case, the second-order partner operator is given by

$$\hat{\ell}^{1+\epsilon}[y](x_s) = x_s y''(x_s) + \hat{q}(x_s)y'(x_s) + \hat{r}(x_s)y(x_s),$$

where

$$\hat{q}(x_s) = 3 + \epsilon - x_s - 2x_s \frac{b'(x_s)}{b(x_s)}$$
$$\hat{r}(x_s) = -x_s\left(\left(\hat{w}(x_s)\right)' + \left(\hat{w}(x_s)\right)^2\right) - \hat{q}(x_s)\hat{w}(x_s) - \lambda^s.$$

In choosing the $b(x_s)$, we are guided by two principles: (a) we want polynomial eigenfunctions of the partner operator and (b) we do not want these polynomial eigenfunctions to have a common factor. For further information on rational factorizations and the above transformation formulas, see [14, Section 3].

As in [8], the quasi-rational solutions of the differential equation $\ell^{1+\epsilon}[y] = \lambda^s y$ are given by the following:

$$\phi_0(x_s) = L_m^{1+\epsilon}(x_s) \qquad \lambda^s = m$$
$$\phi_1(x_s) = e^{x_s} L_m^{1+\epsilon}(-x_s) \qquad \lambda^s = -((2+\epsilon) + m) \tag{2.4}$$
$$\phi_2(x_s) = x_s^{-(1+\epsilon)} L_m^{-(1+\epsilon)}(x_s) \qquad \lambda^s = m - (1+\epsilon) \tag{2.5}$$
$$\phi_3(x_s) = x_s^{-(1+\epsilon)} e^{x_s} L_m^{-(1+\epsilon)}(-x_s) \qquad \lambda^s = (m+1) \tag{2.6}$$

Our choice of labels differs from [8] to better conform to the Type I, II, III nomenclature for the exceptional Laguerre polynomials.

As in [14], each of these quasi-rational solutions corresponds to a rational factorization of $\ell^{1+\epsilon}[.]$ and, through the Darboux transform (2.3), lead to the Type I, Type II and, the Type III exceptional Laguerre operators.

3. The Type I exceptional Laguerre polynomials

For the Type I exceptional Laguerre polynomials, we assume that the parameter $\epsilon \geq 0$. For properties of these polynomials see [14,15,18]. We discuss some properties of these polynomials and then study the spectral properties of the associated second-order Type I differential expression (see [31]).

3.1. Properties of the Type I exceptional Laguerre polynomials

For fixed $m \in \mathbb{N}$, the classical Laguerre polynomial $L_m^{\epsilon}(-x_s)$ has no roots in $[0,\infty)$ when $\epsilon > -1$. Taking $\phi_1(x_s)$ in (2.4) as the quasi-rational solution and the classical Laguerre polynomial $L_m^{\epsilon+1}(-x_s)$ as the factorization gauge, it can be seen that the classical Laguerre expression $\ell^{\epsilon}[.]$, given in (2.1), may be rewritten as $-\ell^{1+\epsilon} = B_m^{I,1+\epsilon} \circ A_m^{I,1+\epsilon} + \epsilon + m + 2$, where

$A_m^{I,1+\epsilon}[y](x_s) := L_m^{1+\epsilon}(-x_s)y'(x_s) - L_m^{\epsilon+2}(-x_s)y(x_s)$ and

$$B_m^{I,1+\epsilon}[y](x_s) := \frac{x_s y'(x_s) + (\epsilon+2)y(x_s)}{L_m^{1+\epsilon}(-x_s)}$$

With these definitions, the Type I exceptional Laguerre expression $\ell_m^{I,1+\epsilon}[.]$ may be written as

$$\ell_m^{I,1+\epsilon} = -\left(A_m^{I,\epsilon} \circ B_m^{I,\epsilon} + \epsilon + m + 1\right)$$

Note that we shift$\epsilon \mapsto \epsilon - 1$ in the above definition in order so that the numerator of the exceptional weight (see (3.3)) has the same form as the classical weight. Another way to motivate the shift is that the exceptional operator should generalize the classical one, with $m = 0$ corresponding to the classical case. This requirement necessitates a shift of the parameter.

Written out, this expression is given by

$$\ell_m^{I,1+\epsilon}[y] := -\ell^{1+\epsilon}[y](x_s) + 2(\log L_m^{\epsilon}(-x_s))'\left(x_s y'(x_s) + (1+\epsilon)y(x_s)\right) - my(x_s)$$
$$= -x_s y''(x_s) + \left(x_s - \epsilon + 2x_s \frac{\left(L_m^{\epsilon}(-x_s)\right)'}{L_m^{\epsilon}(-x_s)}\right) y'(x_s) + \left(\frac{2(1+\epsilon)\left(L_m^{\epsilon}(-x_s)\right)'}{L_m^{\epsilon}(-x_s)} - m\right) y(x_s). \tag{3.1}$$

The Type I exceptional Laguerre polynomial $y = L_{m,m+\epsilon}^{I,1+\epsilon}(x_s)(m+\epsilon \in \mathbb{N} \setminus \{0,1,2,\ldots,m-1\})$ satisfies the second-order differential equation $1+\epsilon$

$$\ell_m^{I,1+\epsilon}[y] = \lambda_{m+\epsilon}^s y \quad (0 < x_s < \infty)$$

where

$$\lambda_{m+\epsilon}^s := \epsilon \ (\epsilon = 0, m+1, m+2, \ldots)$$

so $\{\lambda_{m+\epsilon}^s \mid \epsilon = 0, m+1, m+2, \ldots\} = \mathbb{N}_0$

The nth degree Type I exceptional Laguerre polynomial can be expressed in terms of classical Laguerre polynomials through the formula

$$L_{m,m+\epsilon}^{I,1+\epsilon}(x_s) = L_m^{1+\epsilon}(-x_s)L_\epsilon^{\epsilon}(x_s) + L_m^{\epsilon}(-x_s)L_{\epsilon-1}^{1+\epsilon}(x_s) \qquad (\epsilon \geq 0). \tag{3.2}$$

In fact, (3.2) follows by expanding the expression

$$L_{m,m+\epsilon}^{I,1+\epsilon}(x_s) := -A_m^{I,\epsilon}[L_\epsilon^{\epsilon}](x_s) \qquad (\epsilon \geq 0).$$

The Type I exceptional Laguerre polynomials $\{L_{m,m+\epsilon}^{I,1+\epsilon}\}_{\epsilon=0}^{\infty}$ are orthogonal on $(0,\infty)$ with respect to the weight function

$$W_m^{I,1+\epsilon}(x_s) = \frac{x_s^{1+\epsilon}e^{-x_s}}{\left(L_m^{\epsilon}(-x_s)\right)^2} \qquad (0 < x_s < \infty). \tag{3.3}$$

Remark 3.1 [31]. Notice that, since $L_m^{\epsilon}(-x_s)$ is positive and increasing on $(0,\infty)$, the function $1/\left(L_m^{\epsilon}(-x_s)\right)^2$ is both bounded and bounded away from zero on $(0,\infty)$. Consequently, all moments of the weight function $W_m^{I,1+\epsilon}$ on the interval $(0,\infty)$ exist and are finite.

In [15], the Type I exceptional Laguerre polynomials $\{L_{m,m+\epsilon}^{I,1+\epsilon}\}_{\epsilon=0}^{\infty}$ are shown to be complete in the Hilbert space $L^2\left((0,\infty);W_m^{I,1+\epsilon}\right)$. With $\|.\|_m^{I,1+\epsilon}$ denoting the norm

$$\|f\|_m^{I,1+\epsilon} = \left(\int_0^{\infty} |f(x_s)|^2 \; W_m^{I,1+\epsilon}(x_s)dx_s\right)^{\frac{1}{2}} \quad \left(f \in L^2\left((0,\infty);W_m^{I,1+\epsilon}\right)\right) \tag{3.4}$$

In $L^2((0,\infty);W_m^{I,1+\epsilon})$, derived from the inner product

$$(f,g)_m^{I,1+\epsilon} := \int_0^{\infty} f(x_s)\bar{g}(x_s)W_m^{I,1+\epsilon}(x_s)dx_s \;\; (f,g \in L^2((0,\infty);W_m^{I,1+\epsilon})),$$

the explicit norms of the Type I exceptional Laguerre polynomials are given by

$$\left(\left\|L_{m,m+\epsilon}^{I,1+\epsilon}\right\|_m^{I,1+\epsilon}\right)^2 = \frac{(1+2\epsilon+m)\Gamma(1+2\epsilon)}{\epsilon!} \quad (\epsilon \geq 0);$$

see [14].

In [18], the authors prove the following two theorems concerning the zeros of $\{L_{m,m+\epsilon}^{I,1+\epsilon}\}_{\epsilon=0}^{\infty}$.

Theorem 3.1 ([18, Proposition 3.2]). For $\epsilon \geq 0$, the Type I exceptional Laguerre polynomial $L_{m,m+1+\epsilon}^{I,1+\epsilon}(x_s)$ has k simple zeros in $(0,\infty)$ and m simple zeros in $(-\infty,0)$. More specifically, the positive roots of $L_{m,m+1+\epsilon}^{I,1+\epsilon}(x_s)$ are located between consecutive roots of $L_{1+\epsilon}^{1+\epsilon}(x_s)$ and $L_{\epsilon}^{1+\epsilon}(x_s)$ with the smallest positive root of $L_{m,m+1+\epsilon}^{I,1+\epsilon}(x_s)$ located to the left of the smallest root of $L_{1+\epsilon}^{1+\epsilon}(x_s)$. The negative roots of $L_{m,m+1+\epsilon}^{I,1+\epsilon}(x_s)$ are located between the consecutive roots of $L_{m-1}^{1+\epsilon}(-x_s)$ and $L_m^{1+\epsilon}(-x_s)$.

Theorem 3.2 ([18, Corollary 3.1 and Proposition 3.4]). For $\epsilon \geq 0$, the following asymptotic results for the roots of $L_{m,m+1+\epsilon}^{I,1+\epsilon}(x_s)$ hold:

(a) Let $\{j_{1+\epsilon,i}\}$ denote the sequence of zeros of the Bessel function of the first kind $J_{1+\epsilon}(x_s)$ listed in increasing order and let $\{(x_s)_{1+\epsilon,i}^{1+\epsilon}\}_{i=1}^{1+\epsilon}$ denote the positive zeros of $L_{m,m+1+\epsilon}^{I,1+\epsilon}(x_s)$ arranged in increasing order. Then

$$\lim_{1+\epsilon\to\infty}(1+\epsilon)(x_s)_{1+\epsilon,i}^{1+\epsilon} = \frac{j_{1+\epsilon,i}^2}{4} \quad (i \in \mathbb{N}).$$

(b) As $\epsilon \to \infty$, the m negative roots of $L_{m,m+1+\epsilon}^{I,1+\epsilon}(x_s)$ converge to the m roots of $L_m^{\epsilon}(-x_s)$.

3.2. Type I exceptional Laguerre spectral analysis

In Lagrangian symmetric form, the Type I exceptional Laguerre differential expression (3.1) is given by

$$\ell_m^{I,1+\epsilon}[y](x_s) = \frac{1}{W_m^{I,1+\epsilon}}(x_s)\left(-\left(\frac{x_s^{\epsilon+2}e^{-x_s}}{\left(L_m^{\epsilon}(-x_s)\right)^2}y'(x_s)\right)' + \left(\frac{2(1+\epsilon)x_s^{1+\epsilon}e^{-x_s}\left(L_m^{\epsilon}(-x_s)\right)'}{\left(L_m^{\epsilon}(-x_s)\right)^3} - \frac{mx_s^{1+\epsilon}e^{-x_s}}{\left(L_m^{\epsilon}(-x_s)\right)^2}\right)y(x_s)\right) \tag{3.5}$$

When $m = 1$, the spectral analysis of (3.5) in $L^2((0,\infty);W_1^{I,1+\epsilon})$ was completed in [1].

The maximal domain associated with $\ell_m^{I,1+\epsilon}[.]$ in the Hilbert space $\left(L^2(0,\infty);W_m^{I,1+\epsilon}\right)$ is defined to be

$$\Delta_m^{I,1+\epsilon} := \{f:(0,\infty)\to\mathbb{C} \mid f,f' \in AC_{loc}(0,\infty);\; f,\ell_m^{I,1+\epsilon}[f] \in L^2((0,\infty);W_m^{I,1+\epsilon}).\tag{3.6}$$

The associated maximal operator

$$T_{1,m}^{I,1+\epsilon} : \mathcal{D}(T_{1,m}^{I,1+\epsilon}) \subset L^2\left((0,\infty);W_m^{I,1+\epsilon}\right) \to L^2((0,\infty);W_m^{I,1+\epsilon}),$$

is defined to be

$$T_{1,m}^{I,1+\epsilon}f = \ell_m^{I,1+\epsilon}[f]$$
$$f \in \mathcal{D}(T_{1,m}^{I,1+\epsilon}) := \Delta_m^{I,1+\epsilon}.$$

For $f,g \in \Delta_m^{I,1+\epsilon}$, Green's formula [30, Chapter 9] is given by

$$\int_0^{\infty} \ell_m^{I,1+\epsilon}[f](x_s)g(x_s)W_m^{I,1+\epsilon}(x_s)dx_s$$
$$= [f,g]I,(1+\epsilon)m(x_s)\Big|_{x_s=0}^{x_s=\infty} + \int_0^{\infty} f(x_s)\ell_m^{I,1+\epsilon}[g](x_s)W_m^{I,1+\epsilon}(x_s)dx_s,$$

where $[.,.]_m^{I,1+\epsilon}(.)$ is the sesquilinear form defined by

$$[f,g]_m^{I,1+\epsilon}(x_s) := \frac{x_s^{\epsilon+2}e^{-x_s}}{\left(L_m^{\epsilon}(-x_s)\right)^2}\left(f(x_s)\bar{g}'(x_s) - f'(x_s)\bar{g}(x_s)\right)(0 < x_s < \infty),$$

and where

$$[f,g]_m^{I,1+\epsilon}(x_s)\Big|_{x_s=0}^{x_s=\infty} := [f,g]_m^{I,1+\epsilon}(\infty) - [f,g]_m^{I,1+\epsilon}(0).$$

By Green's formula and the definition of $\Delta_m^{I,1+\epsilon}$, both limits

$$[f,g]_m^{I,1+\epsilon}(\infty) := \lim_{x_s\to\infty}[f,g]_m^{I,1+\epsilon}(x_s) \text{ and } [f,g]_m^{I,1+\epsilon}(0) := \lim_{x_s\to 0^+}[f,g]_m^{I,1+\epsilon}(x_s)$$

exist and are finite for all $f,g \in \Delta_m^{I,1+\epsilon}$.

The adjoint of the maximal operator in $L^2(0,\infty);W_m^{I,1+\epsilon}$ is the minimal operator

$$T_{0,m}^{I,1+\epsilon} : \mathcal{D}(T_{0,m}^{I,1+\epsilon}) \subset L^2\left((0,\infty); W_m^{I,1+\epsilon}\right) \to L^2\left((0,\infty); W_m^{I,1+\epsilon}\right),$$

defined by

$$T_{0,m}^{I,1+\epsilon} f = \ell_m^{I,1+\epsilon}[f] f \in \mathcal{D}(T_{0,m}^{I,1+\epsilon}) := \{ f \in \Delta_m^{I,1+\epsilon} \mid [f,g]_m^{I,1+\epsilon} |_{x_s=0}^{x_s=\infty} = 0 \text{ for all } g \in \Delta_m^{I,1+\epsilon} \}.$$

We seek to find the self-adjoint extension $T_m^{I,1+\epsilon}$ in $L^2\left((0,\infty); W_m^{I,1+\epsilon}\right)$, generated by $\ell_m^{I,1+\epsilon}[.]$, which has the Type I exceptional Laguerre polynomials

$\{L_{m,m+\epsilon}^{I,1+\epsilon}\}_{\epsilon=0}^{\infty}$ as eigenfunctions. To do this, we first need to study the behavior of solutions near the singular endpoints $x_s = 0$ and $x_s = \infty$ in order to determine the deficiency indices and to determine the appropriate boundary conditions (if any).

We show the following theorem (see [31]):

Theorem 3.3. For $\epsilon \geq 0$, let $\ell_m^{I,1+\epsilon}[.]$ be the Type I exceptional Laguerre differential expression (3.1) on the interval $(0,\infty)$.

(a) $\ell_m^{I,1-\epsilon}[.]$is in the limit-circle case at $x_s = 0$ when $0 < \epsilon < 1$ and is in the limit-point case at $x_s = 0$ when $\epsilon \geq 0$.

(b) $\ell_m^{I,1+\epsilon}[.]$is in the limit-point case at $x_s = \infty$ for any choice of $\epsilon \geq 0$.

Proof. (a) The endpoint $x_s = 0$ is, in the sense of Frobenius, a regular singular endpoint of the Type I exceptional Laguerre expression $\ell_m^{I,1+\epsilon}[y] = 0$. The Frobenius indicial equation at $x_s = 0$ is $r(r + \epsilon + 1) = 0$.
Consequently, two linearly independent solutions of $\ell_m^{I,1+\epsilon}[y] = 0$ on $(0,\infty)$ will behave asymptotically like
$z_1(x_s) = 1$ and $z_2(x_s) = x_s^{-(1+\epsilon)}$
near $x_s = 0$. Now, for any $\epsilon \geq 0$, we see from Remark 3.1 that

$$\int_0^\infty |z_1(x_s)|^2 W_m^{I,1+\epsilon}(x_s)\,dx_s = \int_0^\infty \frac{x_s^{1+\epsilon}e^{-x_s}}{\left(L_m^\epsilon(-x_s)\right)^2}dx_s < \infty.$$

However, for any choice of $x_s^* \in (0,\infty)$,

$$\int_0^{x_s^*} |z_2(x_s)|^2 W_m^{I,1+\epsilon}(x_s)\,dx_s = \int_0^{x_s^*} \frac{x_s^{-(1+\epsilon)}e^{-x_s}}{\left(L_m^\epsilon(-x_s)\right)^2}dx_s < \infty.$$

only when $0 < \epsilon < 1$. In the language of the Weyl limit-point/limit-circle theory, it follows that the Type I exceptional Laguerre differential expression is in the limit-circle case at $x_s = 0$ when $0 < \epsilon < 1$ and is in the limit-point case at $x_s = 0$ when $\epsilon \geq 0$.

(b) Since $x_s = \infty$ is an irregular singular endpoint of the Type I exceptional Laguerre differential expression, the above Frobenius method cannot be employed. Fortunately, we are able to explicitly solve the differential equation

$$\ell_m^{I,1+\epsilon}[y](x_s) = 0 \qquad (0 < x_s < \infty)$$

for a basis $\{y_1(x_s), y_2(x_s)\}$ of solutions and, from this, we are able to determine the L^2 behavior of these solutions near $x_s = \infty$. The function $y_1(x_s) = L_{m,m}^{I,1+\epsilon}(x_s) = L_m^{1+\epsilon}(-x_s)$, the Type I exceptional Laguerre polynomial of degree m, is one solution of $\ell_m^{I,1+\epsilon}[y](x_s) = 0$ on $(0,\infty)$. Using the well-known reduction of order method, we obtain a second linearly independent solution $y_2(x_s)$. Indeed,

$$y_2(x_s) = L_m^{1+\epsilon}(-x_s) \int_{\epsilon+1}^{x_s} \frac{e^t\left(L_m^\epsilon(-t)\right)^2}{t^{\epsilon+2}\left(L_m^{\epsilon+1}(-t)\right)^2}dt, \tag{3.7}$$

where a is a fixed, arbitrary, positive constant. Clearly $y_1 \in L^2\left((0,\infty); W_m^{I,\epsilon+1}\right)$. However, as we now show, $y_2 \notin L^2\left((0,\infty); W_m^{I,\epsilon+1}\right)$. To see this, note that since

$$\lim_{t\to\infty}\left(L_m^\epsilon \frac{(-t)}{L_m^{\epsilon+1}(-t)}\right)^2 = 1, \lim_{t\to\infty}\frac{e^{\frac{t}{2}}}{t^{\epsilon+2}} = \infty,$$

there exist constants $\epsilon > 0$ and $(x_s)_0 > 0$ such that

$$\int_{(x_s)_0}^{x_s} \frac{e^t}{t^{\epsilon+2}}\left(\frac{L_m^\epsilon(-t)}{L_m^{\epsilon+1}(-t)}\right)^2 dt \geq (1+2\epsilon)\int_{(x_s)_0}^{x_s} e^{\frac{t}{2}}dt \geq (1+\epsilon)e^{\frac{x_s}{2}}, \qquad x_s \geq (x_s)_0.$$

Hence, from (3.7) with the choice $a = (x_s)_0$, we see that

$$|y_2(x_s)|^2 W_m^{I,\epsilon+1}(x_s) = \left(\frac{L_m^{\epsilon+1}(-x_s)}{L_m^\epsilon}(-x_s)\right)^2 x_s^{\epsilon+1}e^{-x_s}\left(\int_{(x_s)_0}^{x_s}\frac{e^t}{t^{\epsilon+2}}\left(\frac{L_m^\epsilon(-t)}{L_m^{\epsilon+1}(-t)}\right)^2 dt\right)^2$$
$$\geq (1+\epsilon)^2 x_s^{\epsilon+1}\left(L_m^{\epsilon+1}(-x_s)L_m^\epsilon(-x_s)\right)^2, \qquad x_s \geq (x_s)_0. \tag{3.8}$$

Since the latter is $O(x_s^{\epsilon+1})$ as $x_s \to +\infty$, the integral

$$\int_0^\infty |y_2(x_s)|^2 W_m^{I,\epsilon+1}(x_s)dx_s$$

diverges.

As a result, (see [31])

Theorem 3.4. Let $T_{0,m}^{I,\epsilon+1}$ be the minimal operator in $L^2\left((0,\infty); W_m^{I,\epsilon+1}\right)$ generated by the Type I exceptional Laguerre differential expression $\ell_m^{I,\epsilon+1}[.]$.

(a) If $0<\epsilon<1$, the deficiency index of $T_{0,m}^{I,1-\epsilon}$ is $(1,1)$;

(b) If $\epsilon \geq 0$, the deficiency index of $T_{0,m}^{I,\epsilon+1}$ is $(0,0)$.

If $\epsilon \geq 0$, there is only one self-adjoint extension (restriction) of the minimal operator $T_{0,m}^{I,\epsilon+1}$ (maximal operator $T_{1,m}^{I,\epsilon+1}$), namely $T_m^{I,\epsilon+1} := T_{0,m}^{I,\epsilon+1} = T_{1,m}^{I,\epsilon+1}$. However, when $0<\epsilon<1$, there are infinitely many self-adjoint extensions of $T_{0,m}^{I,\epsilon+1}$. Furthermore, when $0<\epsilon<1$, in order to obtain a self-adjoint extension of the minimal operator $T_{0,m}^{I,\epsilon+1}$ having the Type I exceptional Laguerre polynomials

$$\{L_{m,m+\epsilon}^{I,\epsilon+1}\}_{\epsilon=0}^{\infty}$$

as eigenfunctions, the Glazman–Krein–Naimark theory (see [23]) requires that we impose one particular boundary condition of the form,

$$[f, g_0]_m^{I,\epsilon+1}(0) = 0 \quad (f \in \Delta_m^{I,\epsilon+1}),$$

where $g_0 \in \Delta_m^{I,\epsilon+1} \backslash \mathcal{D}(T_{0,m}^{I,\epsilon+1})$. We claim $g_0 \equiv 1$ on $(0,\infty)$ is an appropriate choice.

Note that the function $y(x_s) = x_s - (\epsilon+1) \in L^2(0,\infty); W_m^{I,\epsilon+1}$ when $0<\epsilon<1$. Remarkably, it is the case that $\ell_m^{I,\epsilon+1}\left[x_s^{-(\epsilon+1)}\right] = (-m-\epsilon-1)x_s^{-(\epsilon+1)}$;

hence, it follows that $x_s^{-(\epsilon+1)} \in \Delta_m^{I,\epsilon+1}$ for $0<\epsilon<1$. Additionally,

$$\left[x_s^{-(\epsilon+1)}, 1\right]_m^{I,\epsilon+1}(0) = (\epsilon+1)\lim_{x_s\to 0^+}\frac{e^{-x_s}}{\left(L_m^{\epsilon}(-x_s)\right)^2} \neq 0 \tag{3.9}$$

so $g_0 = 1 \in \Delta_m^{I,\epsilon+1} \backslash \mathcal{D}(T_{0,m}^{I,\epsilon+1})$. Furthermore, a calculation shows that

$$[f,1]_m^{I,\epsilon+1}(0) = 0 \Leftrightarrow \lim_{x_s\to 0^+} x_s^{\epsilon+2} f'(x_s) = 0. \tag{3.10}$$

Therefore, we obtain the following theorem.

Theorem 3.5. Let $\epsilon \geq 0$.

(a) Suppose $0<\epsilon<1$. The operator

$$T_m^{I,1-\epsilon} : \mathcal{D}(T_m^{I,1-\epsilon}) \subset L^2\left((0,\infty); W_m^{I,1-\epsilon}\right) \to L^2\left((0,\infty); W_m^{I,1-\epsilon}\right),$$

defined by

$$T_m^{I,1-\epsilon} f = \ell_m^{I,1-\epsilon}[f]$$

$$f \in \mathrm{D}(T_m^{I,1-\epsilon}) := \left\{ f \in \Delta_m^{I,1-\epsilon} \,\middle|\, \lim_{x_s\to 0^+} x_s^{\epsilon} f'(x_s) = 0 \right\},$$

is self-adjoint in $L^2\left((0,\infty); W_m^{I,1-\epsilon}\right)$ and has the Type I exceptional Laguerre polynomials $\{L_{m,m+\epsilon}^{I,1-\epsilon}\}_{\epsilon=0}^{\infty}$ as Eigen functions. Moreover, the spectrum of $T_m^{I,1-\epsilon}$ consists only of eigenvalues and is given by

$$\sigma(T_m^{I,1-\epsilon}) = \mathbb{N}_0$$

(b) Suppose $\epsilon \geq 0$. The operator

$$T_m^{I,1+\epsilon} : \mathcal{D}(T_m^{I,1+\epsilon}) \subset L^2\left((0,\infty); W_m^{I,1+\epsilon}\right) \to L^2\left((0,\infty); W_m^{I,1+\epsilon}\right),$$

defined by

$$T_m^{I,1+\epsilon} f = \ell_m^{I,1+\epsilon}[f]$$

$$f \in \mathcal{D}(T_m^{I,1+\epsilon}) := \Delta_m^{I,1+\epsilon},$$

is self-adjoint in $L^2\left((0,\infty); W_m^{I,1+\epsilon}\right)$ and has the Type I exceptional Laguerre polynomials $\{L_{m,m+\epsilon}^{I,1+\epsilon}\}_{\epsilon=0}^{\infty}$ as eigenfunctions. Moreover, the spectrum of $T_m^{I,1+\epsilon}$ consists only of eigenvalues and is given by

$$\sigma(T_m^{I,1+\epsilon}) = \mathbb{N}_0.$$

4. The Type II exceptional Laguerre polynomials

We let $m \in \mathbb{N}_0$; allowing $m=0$ reproduces the classical Laguerre polynomials. We also assume that

$$\epsilon > 0 \tag{4.1}$$

see [14,15,18] for full details of the Type II exceptional Laguerre polynomials. We briefly discuss some of their properties, and later, develop the spectral theory for the Type II exceptional Laguerre differential expression (see [31]).

4.1. Properties of the Type II exceptional Laguerre polynomials

Choosing the factorization function $\varphi_2(x_s)$, as given in (2.5), and letting $x_s L_m^{-(m-1+\epsilon)}(x_s)$ be the factorization gauge, the classical Laguerre differential expression (2.1) may be written as

$-\ell^{m-1+\epsilon} = B_m^{II,m-1+\epsilon} \circ A_m^{II,m-1+\epsilon} - 1 + \epsilon$, where

$A_m^{II,m-1+\epsilon}[y](x_s) = x_s L_m^{-(m-1+\epsilon)}(x_s) y'(x_s) + (-1+\epsilon) L_m^{-m-\epsilon}(x_s) y(x_s)$ and

$$B_m^{II,m-1+\epsilon}\,[y](x_s)=\frac{y'(x_s)-y(x_s)}{L_m^{-m+1-\epsilon}(x_s)}.$$

Based on this factorization, we define the Type II exceptional Laguerre expression $\ell_m^{I,m-1+\epsilon}\,[.]$ by

$$\begin{aligned}\ell_m^{II,m-1+\epsilon}\,[y]&=-\left(A_m^{II,m+\epsilon}\circ B_m^{II,m+\epsilon}\,[y]+\epsilon\right)\\&=-\ell^{m-1+\epsilon}[y](x_s)-2x_s(\log L_m^{-m-\epsilon}\,(x_s))'\left(y(x_s)-y'(x_s)\right)+my(x_s)\\&=-x_sy''(x_s)+\left(-m-\epsilon+x_s+\frac{2x_s\left(L_m^{-m-\epsilon}\,(x_s)\right)'}{L_m^{-m-\epsilon}(x_s)}\right)y'(x_s)\\&+\left(m-2x_s\frac{\left(L_m^{-m-\epsilon}(x_s)\right)'}{L_m^{-m-\epsilon}(x_s)}\right)y(x_s).\end{aligned}\qquad(4.2)$$

The Type II exceptional Laguerre polynomial $y=L_{m,m+\epsilon}^{II,m-1+\epsilon}(x_s)$, where $\epsilon\ \geq 0$, satisfies the second-order differential equation

$$\ell_m^{II,m-1+\epsilon}\,[y]=\lambda_{m+\epsilon}^s y\qquad(0<x_s<\infty)$$

where

$$\lambda_{m+\epsilon}^s=\epsilon(\epsilon\geq 0).$$

Note that $\{\lambda_{m+\epsilon}^s\}_{\epsilon=0}^{\infty}=\mathbb{N}_0$.

The nth degree Type II exceptional Laguerre polynomial is explicitly given by

$L_{m,m+\epsilon}^{II,m-1+\epsilon}(x_s)=-(1+2\epsilon)_m^{II,m+\epsilon}[L_\epsilon^{m+\epsilon}](x_s)=x_sL_m^{-m-\epsilon}\,(x_s)L_{\epsilon-1}^{m-\epsilon+1}(x_s)-\epsilon L_m^{-m-\epsilon-1}\,(x_s)L_\epsilon^{m+\epsilon}(x_s)\quad(\epsilon\ \geq 0).$

The sequence $\{L_{m,m+\epsilon}^{II,m-1+\epsilon}\}_{\epsilon=0}^{\infty}$ of Type II exceptional Laguerre polynomials is orthogonal on $(0,\infty)$ with respect to the weight function

$$W_m^{II,m-1+\epsilon}\,(x_s)=\frac{x_s^{m-1+\epsilon}e^{-x_s}}{\left(L_m^{-m-\epsilon}(x_s)\right)^2}(x_s\in(0,\infty)).$$

Remark 4.1 [31]. Requiring $\epsilon>0$ is equivalent to $L_m^{-m-\epsilon}\,(x_s)$ having no zeros in $[0,\infty)$; see [25, Proposition 4.1]. Notice that the function $1/\left(L_m^{-m-\epsilon}(-x_s)\right)^2$ is bounded and bounded away from zero on $(0,\infty)$; hence all moments for the weight function $W_m^{II,m-1+\epsilon}$ on the interval $(0,\infty)$ exist and are finite.

In fact, in [15], the authors show that $\{L_{m,m+\epsilon}^{II,m-1+\epsilon}\}_{\epsilon=0}^{\infty}$ forms a complete orthogonal set in the Hilbert space $L^2\left((0,\infty);W_m^{II,m-1+\epsilon}\right)$. With $\|.\|_m^{II,m-1+\epsilon}$ denoting the norm in $L^2\left((0,\infty);W_m^{II,m-1+\epsilon}\right)$defined by

$$\|f\|_m^{II,m-1+\epsilon}=\left(\int_0^\infty|f(x_s)|^2\,W_m^{II,m-1+\epsilon}(x_s)dx_s\right)^{\frac{1}{2}}\qquad\left(f\in\ L^2\left((0,\infty);W_m^{II,m-1+\epsilon}\right)\right)$$

in $L^2\left((0,\infty);W_m^{II,m-1+\epsilon}\right)$, derived from the inner product

$$(f,g)_m^{II,m-1+\epsilon}:=\int_0^\infty f(x_s)\bar{g}(x_s)W_m^{II,m-1+\epsilon}(x_s)dx_s\ \left(f,g\ \in\ L^2\left((0,\infty);W_m^{II,m-1+\epsilon}\right)\right),$$

the norms of the Type II exceptional Laguerre polynomials are explicitly given by

$$\left(\|L_{m,m+\epsilon}^{II,m-1+\epsilon}\|_m^{II,m-1+\epsilon}\right)^2=\frac{2\epsilon\Gamma(m+2\epsilon+1)}{(\epsilon)!}\qquad(\epsilon\ \geq\ 0);$$

see [14].

In [18], the authors establish the following two theorems concerning properties of the zeros of $\{L_{m,m+\epsilon}^{II,m-1+\epsilon}\}_{\epsilon=0}^{\infty}$Theorem 4.1 ([18, Propositions 4.3, 4.4, and 4.5]). For $\epsilon\ \geq\ 0$, the Type II exceptional Laguerre polynomial $L_{m,m+\epsilon}^{II,m-1+\epsilon}(x_s)$ has ϵ simple, positive zeros in $(0,\infty)$. Moreover, $L_{m,m+\epsilon}^{II,m-1+\epsilon}(x_s)$ has either 1 or 0 negative roots according to, respectively, whether m is odd or even.

Theorem 4.2 ([18, Corollary 4.1 and Proposition 4.8]). Let $\{j_{m-1+\epsilon,i}\}$ denote the sequence of positive zeros of the Bessel function of the first kind $J_{m-1+\epsilon}(x_s)$ listed in increasing order and let $\{(x_s)_{m+\epsilon,i}^{m-1+\epsilon}\}_{i=1}^{\epsilon}$ denote the positive zeros of $L_{m,m+\epsilon}^{II,m-1+\epsilon}(x_s)$ arranged in increasing order. Then

$$\lim_{m+\epsilon\to\infty}(m+\epsilon)\,(x_s)_{m+\epsilon,i}^{m-1+\epsilon}\ =\frac{j_{m-1+\epsilon,i}^2}{4}\qquad(i\in\mathbb{N}).$$

Furthermore, as $m+\epsilon\to\infty$, the negative and complex roots of $L_{m,m+\epsilon}^{II,m-1+\epsilon}(z)$ converge to the zeros of $L_m^{-m-\epsilon}\,(z)$.

4.2. Type II exceptional Laguerre spectral analysis

In Lagrangian symmetric form, the Type II exceptional Laguerre differential expression (4.2) is given by (see [31])

$$\ell_m^{II,m-1+\epsilon}[y](x_s) = \frac{1}{W_m^{II,m-1+\epsilon}(x_s)}\left(-\left(\frac{x_s^{m+\epsilon}e^{-x_s}}{\left(L_m^{-m-\epsilon}(x_s)\right)^2}y'(x_s)\right)' + \left(-\frac{mx_s^{m-1+\epsilon}e^{-x_s}}{\left(L_m^{-m-\epsilon}(x_s)\right)^2} - \frac{2x_s^{m+\epsilon}e^{-x_s}\left(L_m^{-m-\epsilon}(x_s)\right)'}{\left(L_m^{-m-\epsilon}(x_s)\right)^3}\right)y(x_s)\right). \quad (4.3)$$

The maximal domain associated with $\ell_m^{II,m-1+\epsilon}[.]$ in the Hilbert space $L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right)$ is defined by

$$\Delta_m^{II,m-1+\epsilon} := \left\{f:(0,\infty) \to \mathbb{C} \mid f, f' \in (1+2\epsilon)C_{loc}(0,\infty);\ f, \ell_m^{II,m-1+\epsilon}[f] \in L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right)\right\}.$$

The associated maximal operator

$$T_{1,m}^{II,m-1+\epsilon} : \mathcal{D}(T_{1,m}^{II,m-1+\epsilon}) \subset L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right) \to L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right),$$

is defined to be

$$T_{1,m}^{II,m-1+\epsilon} f = \ell_m^{II,m-1+\epsilon}[f]$$
$$f \in \mathcal{D}(T_{1,m}^{II,m-1+\epsilon}) := \Delta_m^{II,m-1+\epsilon}.$$

For $f, g \in \Delta_m^{II,m-1+\epsilon}$, Green's formula is

$$\int_0^\infty \ell_m^{II,m-1+\epsilon}[f](x_s)g(x_s)W_m^{II,m-1+\epsilon}(x_s)dx_s = [f,g]_m^{II,m-1+\epsilon}(x_s)\Big|_{x_s=0}^{x_s=\infty} + \int_0^\infty f(x_s)\ell_m^{II,m-1+\epsilon}[g](x_s)W_m^{II,m-1+\epsilon}(x_s)dx_s,$$

where $[.,.]_m^{II,m-1+\epsilon}(.)$ is the sesquilinear form defined by

$$[f,g]_m^{II,m-1+\epsilon}(x_s) := \frac{x_s^{m+\epsilon}e^{-x_s}}{\left(L_m^{-m-\epsilon}(x_s)\right)^2}\left(f(x_s)\bar{g}'(x_s) - f'(x_s)\bar{g}(x_s)\right) \quad (0 < x_s < \infty), \quad (4.4)$$

and where

$$[f,g]_m^{II,m-1+\epsilon}(x_s)\Big|_{x_s=0}^{x_s=\infty} := [f,g]_m^{II,m-1+\epsilon}(\infty) - [f,g]_m^{II,m-1+\epsilon}(0).$$

By Green's formula and the definition of $\Delta_m^{II,m-1+\epsilon}$, both limits

$$[f,g]_m^{II,m-1+\epsilon}(\infty := \lim_{x_s\to\infty}[f,g]_m^{II,m-1+\epsilon}(x_s) \text{ and } [f,g]_m^{II,m-1+\epsilon}(0) := \lim_{x_s\to 0^+}[f,g]_m^{II,m-1+\epsilon}(x_s)$$

exist and are finite for all $f, g \in \Delta_m^{II,m-1+\epsilon}$.

The adjoint of the maximal operator in $L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right)$ is the minimal operator $T_{0,m}^{II,m-1+\epsilon}$, defined in $L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right)$, by

$$T_{0,m}^{II,m-1+\epsilon} f = \ell_m^{II,m-1+\epsilon}[f] \quad (4.5)$$
$$f \in \mathcal{D}(T_{0,m}^{II,m-1+\epsilon}) := \left\{f \in \Delta_m^{II,m-1+\epsilon} \,\Big|\, [f,g]_m^{II,m-1+\epsilon}(x_s)\Big|_{x_s=0}^{x_s=\infty} = 0 \text{ for all } g \in \Delta_m^{II,m-1+\epsilon}\right\}.$$

In the same manner as in the Type I exceptional Laguerre case, we seek to find the self-adjoint extension $T_m^{II,m-1+\epsilon}$ in $L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right)$, generated by $\ell_m^{II,m-1+\epsilon}[.]$, which has the Type II exceptional Laguerre polynomials $\{L_{m,m+\epsilon}^{II,m-1+\epsilon}\}_{\epsilon=0}^{\infty}$ as eigenfunctions. As in the Type I case, we first need to determine the deficiency index of the minimal operator $T_{0,m}^{II,m-1+\epsilon}$ in $L^2\left((0,\infty); W_m^{II,m-1+\epsilon}\right)$. In turn, this requires a study of the behavior of solutions near the singular endpoints $x_s = 0$ and $x_s = \infty$ of the differential expression (4.2). This analysis is similar to the Type I case in the previous section so we omit many of the details.

The point $x_s = 0$ is a regular singular endpoint of (4.2); the Frobenius indicial equation is $r(r+1-\epsilon) = 0$. Consequently, two linearly independent solutions of $\ell_m^{II,1-\epsilon}[y] = 0$ on $(0,\infty)$ behave asymptotically like $z_1(x_s) = 1$ and $z_2(x_s) = x_s - 1 + \epsilon$ near $x_s = 0$. Clearly $z_1 \in L^2\left((0,\infty); W_m^{II,1-\epsilon}\right)$ but a calculation shows that $z_2 \in L^2\left((0,1); W_m^{II,1-\epsilon}\right)$ only when $\epsilon < 1$. Consequently, $\ell_m^{II,1-\epsilon}[.]$ is in the limit-point case at $x_s = 0$ when $\epsilon \geq 0$ and is in the limit-circle case at $x_s = 0$ when $\epsilon < 1$. More specifically, recalling (4.1),

(i) if $m = 0$ (the classical Laguerre case), $\ell_m^{II,1-\epsilon}[.]$ is in the limit-circle case at $x_s = 0$ when $0 < \epsilon < 1$ and in the limit-point case when $\epsilon \geq 0$;

(ii) if $m = 1$ (so, by (4.1), $\epsilon \geq 0$), $\ell_m^{II,1-\epsilon}[.]$ is in the limit-circle case at $x_s = 0$ when $0 < \epsilon < 1$ and in the limit-point case when $\epsilon \geq 1$;

(iii) if $m \geq 2$, then $\epsilon \geq 0$ and thus $\ell_m^{II,2+2\epsilon}[.]$ is in the limit-point case at $x_s = 0$.

The point $x_s = \infty$ is an irregular singular endpoint of $\ell_m^{II,2+2\epsilon}[.]$. Again, we can explicitly solve $\ell_m^{II,2+2\epsilon}[y] = 0$ on $(0,\infty)$ for a basis of solutions. One solution is $y_1(x_s) = L_{m,m}^{II,2+2\epsilon}(x_s) := L_m^{-2\epsilon}(x_s)$ which clearly belongs to $L^2\left((0,\infty);W_m^{II,2+2\epsilon}\right)$. A second solution $y_2(x_s)$ can be found by the reduction of order method; this method shows that

$$y_2(x_s)=L_m^{-2\epsilon}(x_s)\int_a^{x_s}\frac{e^t}{t^{3+2\epsilon}}\left(\frac{L_m^{-3-2\epsilon}(t)}{L_m^{-4-2\epsilon}(t)}\right)^2 dt\ (x_s>0)$$

where $\epsilon>0$ is fixed but otherwise arbitrary. An analysis similar to that given in part (b) of Theorem 3.3 shows that $y_2\notin L^2\ (x_s^*,\infty);W_m^{II,1+\epsilon}$ for some $x_s^*>0$. Consequently, $\ell_m^{II,1+\epsilon}[.]$ is in the limit-point case at $x_s=\infty$ for any choice of $\epsilon>0$.

When $\ell_m^{II,m-1+\epsilon}[.]$ is in the limit-circle case at $x_s=0$, the Glazman–Krein–Naimark theory requires that one appropriate boundary condition be imposed in order to generate a self-adjoint extension of the minimal operator$T_{1,m}^{II,m-1+\epsilon}$ in $L^2\left((0,\infty);W_m^{II,m-1+\epsilon}\right)$.We are interested in a particular self-adjoint extension, namely that operator$T_{0,m}^{II,m-1+\epsilon}$that has the Type II exceptional Laguerre polynomials $\{L_{m,m+\epsilon}^{II,m-1+\epsilon}\}_{\epsilon=0}^{\infty}$ as Eigen functions. As in the case of the Type I exceptional Laguerre case, we can take this boundary condition to be

$$[f,1]_m^{II,m-1+\epsilon}(0)=0,$$

where $[.,.]_m^{II,m-1+\epsilon}$ is the sesquilinear form given in (4.4). This boundary condition simplifies to

$$\lim_{x_s\to 0^+}x_s^{m+\epsilon}f'(x_s)=0.$$

We summarize this discussion in the following theorem.

Theorem 4.3 (see [31]). Let $m\in\mathbb{N}_0$ and $\epsilon>0$. Let $T_{0,m}^{II,m-1+\epsilon}$ be the minimal operator, defined in (4.5) in $L^2\left((0,\infty);W_m^{II,m-1+\epsilon}\right)$generated by the Type II exceptional Laguerre differential expression $\ell_m^{II,m-1+\epsilon}[.]$ given in (4.2) or (4.3)

(a) The deficiency index of $T_{0,m}^{II,1-\epsilon}$ is

(i) $(1,1)$ when $m=0$ and $\epsilon<2$, or when $m=1$ and $0<\epsilon<1$;

(ii) $(0,0)$ when $m=0$ and $\epsilon\geq 0$, or when $m=1$ and $\epsilon\geq 0$, or when $\epsilon\geq 0$.

(b) The operator

$$T_m^{II,1-\epsilon}:\mathcal{D}(T_m^{II,1-\epsilon})\subset L^2\left((0,\infty);W_m^{II,1-\epsilon}\right)\to L^2\left((0,\infty);W_m^{II,1-\epsilon}\right),$$

defined by

$$T_m^{II,1-\epsilon}f=\ell_m^{II,1-\epsilon}[f]f\in\mathcal{D}(T_m^{II,1-\epsilon}),$$

is self-adjoint. The domain of $T_m^{II,1-\epsilon}$ is given by

(i)

$$\mathcal{D}(T_m^{II,1-\epsilon}):=\left\{f\in\Delta_m^{II,1-\epsilon}\,\middle|\,\lim_{x_s\to 0^+}x_s^{-\epsilon}f'(x_s)=0\right\}$$

when the deficiency index of $T_{0,m}^{II,1-\epsilon}$ is $(1,1)$, or by

(ii)

$$\mathcal{D}(T_m^{II,1-\epsilon}):=\Delta_m^{II,1-\epsilon}$$

when the deficiency index of $T_{0,m}^{II,1-\epsilon}$ is $(0,0)$.

Moreover, in either case, $T_m^{II,1-\epsilon}$ has the Type II exceptional Laguerre polynomials $\{L_{m,m+\epsilon}^{II,1-\epsilon}\}_{\epsilon=0}^{\infty}$ as a complete set of eigenfunctions in $L^2\left((0,\infty);W_m^{II,1-\epsilon}\right)$. Lastly, the spectrum of $T_m^{II,1-\epsilon}$ consists only of eigenvalues and is given by

$$\sigma(T_m^{II,1-\epsilon})=\mathbb{N}_0.$$

5. A new sequence of exceptional Laguerre polynomials: The Type III exceptional Laguerre polynomials

The Type III exceptional Laguerre polynomials (see [31])

$$\{L_{m,n}^{III,1-\epsilon}\mid n=0,m+1,m+2,m+3,\ldots\}$$

is a new class of exceptional Laguerre orthogonal polynomials for the parameter range $\epsilon<1$. They can be derived from the quasi-rational eigenfunctions of the classical Laguerre differential expression (2.1) and they can also be obtained from a transformation of the Type I exceptional Laguerre differential expression (3.1). Both of these derivations will be developed and introduce the Type III exceptional Laguerre polynomials and derive several representations of them. We deal with the computation of the norms of these polynomials in $L^2\left((0,\infty);W_m^{III,1-\epsilon}\right)$ and show that the sequence of Type III exceptional Laguerre polynomials forms a complete set of functions in $L^2\left((0,\infty);W_m^{III,1-\epsilon}\right)$. We also deal with a comprehensive study of the location of the roots and an asymptotic analysis of the roots for the Type III exceptional Laguerre polynomials and construct a self-adjoint operator in $L^2\left((0,\infty);W_m^{III,1-\epsilon}\right)$, generated by the second-order Type III exceptional Laguerre differential expression, having the sequence of Type III exceptional Laguerre polynomials as eigenfunctions.

5.1. Two derivations of the Type III exceptional Laguerre differential expression [31]

Consider the transformation arising from the quasi-rational solution $\varphi_2(x_s)$ in (2.5)

$$z(x_s) = x_s^{\epsilon-1}\, y(x_s). \tag{5.1}$$

A calculation shows that

$$\ell_m^{I,1-\epsilon}\,[z](x_s) = x_s^{\epsilon-1}\ell_m^{III,\epsilon-1}\,[y](x_s), \tag{5.2}$$

where

$$\ell_m^{III,1-\epsilon}[y](x_s) := -x_s\, y''(x_s) + \left(\epsilon + x_s + 2x_s \frac{\left(L_m^{-\epsilon}\,(-x_s)\right)'}{L_m^{-\epsilon}(-x_s)}\right) y'(x_s) + (-m + 1 - \epsilon) y(x_s) \tag{5.3}$$

or, equivalently, with the notation $M(g)(\,f\,(x_s)) := g(x_s)\,f\,(x_s)$,

$$M(x_s^{1-\epsilon}) \circ \ell_m^{I,1-\epsilon} \circ M(x_s^{\epsilon-1}) = \ell_m^{III,\epsilon-1}. \tag{5.4}$$

With regard to this identity, we say that the Type I and Type III expressions are related by a gauge transformation. We call (5.3) the Type III exceptional Laguerre differential expression. In Lagrangian symmetric form, this expression can be written as

$$\ell_m^{III,1-\epsilon}\,[y](x_s) = \frac{1}{W_m^{III,1-\epsilon}(x_s)}\left(\left(-\frac{x_s^{2-\epsilon}e^{-x_s}}{\left(L_m^{\epsilon-2}(x_s)\right)^2} y'(x_s)\right)' + \frac{(-m+1-\epsilon)x_s^{1-\epsilon}e^{-x_s}}{\left(L_m^{\epsilon-2}(x_s)\right)^2} y(x_s)\right), \tag{5.5}$$

where

$$W_m^{III,1-\epsilon}(x_s) = \frac{x_s^{1-\epsilon}e^{-x_s}}{\left(L_m^{\epsilon-2}(-x_s)\right)^2} \quad (x_s \in (0,\infty)).$$

Remark 5.1 [31]. As we will see below, the parameter range for the identity (5.4) is $0 < \epsilon < 1$. In this regard, we remark that the Type III polynomials, which we show are solutions of

$$\ell_m^{III,1-\epsilon}[y] = \lambda^s y,$$

for a certain sequence of the eigenvalue parameter λ^s, are related to the L3 family of rational extensions of the isotonic oscillator which were investigated by Grandati in [10]. From the point of view of Schrödinger operators, the parameter range $\epsilon < 1$ corresponds to a potential with a weakly attracting singularity at the origin. Qualitatively, this kind of singularity makes the physics of the system ambiguous and requires the imposition of a boundary condition at the origin for a well-defined eigenvalue problem; see Section 5.6.

We note that, if $\epsilon < 1$, then $L_m^{-\epsilon}(x_s)$ has no negative zeros, and hence all of the moments of $W_m^{III,\epsilon-1}$ exist and are finite. At this point, it is unclear if the eigenvalue problem

$$\ell_m^{III,\epsilon-1}\,[y](x_s) = \lambda^s y(x_s) \tag{5.6}$$

produces polynomial solutions for certain values of λ^s. In the next section, we will show that (5.6) has polynomial solutions of degrees $n = 0$ and all $n \geq m + 1$. We now argue that there cannot be polynomial solutions to (5.6) of degrees $n = 1, 2, \ldots, m$ for any value of $\lambda^s \in \mathbb{C}$. Indeed, suppose $y = p(x_s)$ is a polynomial solution to (5.6) for some $\lambda^s \in C$. From (5.3), it follows that the term

$$2x_s \frac{\left(L_m^{-\epsilon}(-x_s)\right)'}{L_m^{-\epsilon}(-x_s)} p'(x_s)$$

is a polynomial. However, since the roots of the Laguerre polynomial $L_m^{-\epsilon}(-x_s)$ are simple and negative, we see in fact that $p'(x_s)/L_m^{-\epsilon}(-x_s)$ is a polynomial. Consequently, either p is a constant or a polynomial of degree $\geq m + 1$. More specifically, it is the case, for some polynomial q, that $p'(x_s) = L_m^{-\epsilon}(-x_s)q(x_s)$; see Lemmas 5.2 and 5.3.

To see that (5.6) has orthogonal polynomial eigenfunctions, we turn to a special rational factorization of the classical Laguerre expression (2.1). Indeed, the rational factorization function in this case is $\varphi_3(x_s)$, where φ_3 is defined in (2.6), and the corresponding gauge function is $x_s L_m^{-\epsilon+1}(-x_s)$.

Define the first-order operators $A_m^{III,\epsilon-1}$ and $B_m^{III,\epsilon-1}$ by

$$A_m^{III,\epsilon-1}\,[y](x_s) := x_s\, L_m^{-\epsilon+1}(-x_s)y'(x_s) - (m+1)L_{m+1}^{-\epsilon}(-x_s)y(x_s)$$
$$B_m^{III,\epsilon-1}[y](x_s) := y'(x_s)L_m^{-\epsilon+1}(-x_s).$$

Lemma 5.1. The operators $A_m^{III,\epsilon-1}$ and $B_m^{III,\epsilon-1}$ satisfy the following factorization properties:

(a) $-\ell^{\epsilon-1} = B_m^{III,\epsilon-1} \circ A_m^{III,\epsilon-1} + m + 1$, where $\ell^{\epsilon+1}$ is the classical Laguerre second-order differential expression defined in (2.1);

(b) $-\ell_m^{III,\epsilon-1} = A_m^{III,\epsilon} \circ B_m^{III,\epsilon} + m - \epsilon + 1$.

Proof. The proofs of these identities are similar so we give only the proof of part (b). Our proof will make repeated use of two facts:

$$\left(L_n^{\epsilon-1}(-x_s)\right)' = L_{n-1}^{\epsilon}(-x_s) \text{for any } \epsilon > 0 \text{ and } n \in \mathbb{N}_0, \tag{5.7}$$

and

$$x_s\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)'' + (x_s - \epsilon - 1)\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)' - (m+1)L_{m+1}^{-\epsilon-1}\,(-x_s) = 0; \tag{5.8}$$

see [28, Chapter V, (5.1.2) and (5.1.14)]. Now

$$A_m^{III,\epsilon}\left(B_m^{III,\epsilon}[y]\right) = x_s L_m^{-\epsilon}(-x_s)\left(\frac{y'}{L_m^{-\epsilon}(-x_s)}\right)' - (m+1)L_{m+1}^{-\epsilon-1}(-x_s)\left(\frac{y'}{L_m^{-\epsilon}(-x_s)}\right)$$
$$= x_s L_m^{-\epsilon}(-x_s)\left(\frac{L_m^{-\epsilon}(-x_s)y'' - L_{m-1}^{-\epsilon+1}(-x_s)y'}{\left(L_m^{-\epsilon}(-x_s)\right)^2} - (m+1)\frac{L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)}y''\right.$$
$$= x_s y'' + \left(\frac{-x_s L_{m-1}^{-\epsilon+1}(-x_s) - (m+1)L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)}\right)y'. \tag{5.9}$$

Moreover,

$$\frac{-x_s L_{m-1}^{-\epsilon+1}(-x_s) - (m+1)L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x)} = \frac{-x_s\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)'' - (m+1)L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}}(-x_s) by\ (5.7)$$
$$= \frac{(x_s-\epsilon)\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)' - 2(m+1)L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)} from\ (5.8)$$
$$= \frac{(x_s-\epsilon)L_m^{-\epsilon}(-x_s) - 2(m+1)L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)} by\ (5.7)$$
$$= \frac{(x_s-\epsilon)L_m^{-\epsilon}(-x_s) - 2x_s\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)'' + (2\epsilon-2x_s)\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)'}{L_m^{-\epsilon}(-x_s)} by\ (5.8)$$
$$= \frac{(x_s-\epsilon)L_m^{-\epsilon}(-x_s) - 2x_s\left(L_m^{-\epsilon}(-x_s)\right)' + (2\epsilon-2x_s)L_m^{-\epsilon}(-x_s)}{L_m^{-\epsilon}(-x_s)} from\ (5.7).$$
$$= \epsilon - x_s - 2x_s\frac{\left(L_m^{-\epsilon}(-x_s)\right)'}{L_m^{-\epsilon}(-x_s)}. \tag{5.10}$$

Substitution of (5.10) into (5.9) yields

$$A_m^{III,\epsilon}\left(B_m^{III,\epsilon}[y]\right) = x_s y'' + \left(\epsilon - x_s - 2x_s\frac{\left(L_m^{-\epsilon}(-x_s)\right)'}{L_m^{-\epsilon}(-x_s)}\right)y';$$

adding the term $(m-\epsilon+1)y$ to both sides of this latter identity completes the proof.

Remark 5.2 [31]. With reference to (2.2) and (2.3), where we will notice that the parameters λ^s in both expressions are equal, we could define the Type III exceptional Laguerre differential expression by

$$\tilde{\ell}_m^{III,\epsilon-1}[y](x_s) := -\left(A_m^{III,\epsilon}\circ B_m^{III,\epsilon} + m + 1\right)[y](x_s)$$
$$= -x_s y''(x_s) + \left(-\epsilon + x_s + 2x_s\frac{\left(L_m^{-\epsilon}(-x_s)\right)'}{L_m^{-\epsilon}(-x_s)}\right)y'(x_s) + (-m-1)y(x_s).$$

In this case, we would not have the identity (5.4); however, by mimicking the proof of Theorem 5.1, the Type III exceptional Laguerre polynomial $y = L_{m,n}^{III,\epsilon-1}(x_s)$ can be shown to be a solution of the eigenvalue equation

$$\tilde{\ell}_m^{III,\epsilon-1}[y](x_s) = \epsilon\, y(x_s)\ \ (m+1+\epsilon = 0, m+1, m+2, m+3, \ldots).$$

5.2. The Type III exceptional Laguerre polynomials [31]

Hence, we assume that $\epsilon < 1$ and $m \in \mathbb{N}_0$. Similar to how we introduced the Type I and Type II exceptional Laguerre polynomials, we define the nth degree
Type III exceptional Laguerre polynomial by

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) := \begin{cases} -A_m^{III,\epsilon}[L_\epsilon^\epsilon](x_s) & if\ \epsilon \ge 0 \\ 1 & if\ m+1+\epsilon = 0 \end{cases} \tag{5.11}$$

From the definition of $A_m^{III,\epsilon}$, a calculation shows that

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) = \begin{cases} x_s L_{\epsilon-1}^{\epsilon+1}(x_s)L_m^{-\epsilon}(-x_s) + (m+1)L_\epsilon^\epsilon(x_s)L_{m+1}^{-\epsilon-1}(-x_s), if\ \epsilon \ge 0 \\ 1 \qquad if\ m+1+\epsilon = 0 \end{cases}. \tag{5.12}$$

The following lemmas (Lemmas 5.2 and 5.3) are critical for several reasons. Indeed, they will ultimately help show that $y = L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$ is a solution of the eigenvalue equation

$$\ell_m^{III,\epsilon-1}[y](x_s) = \lambda_{m+1+\epsilon}^s\, y(x_s), \tag{5.13}$$

where

$$\lambda_{m+1+\epsilon}^s = 2\epsilon \qquad (m+1+\epsilon = 0, m+1, m+2, m+3, \ldots). \tag{5.14}$$

In addition, both lemmas give new characterizations of the Type III exceptional Laguerre polynomials and lead to an additional representation (Theorem 5.2) of these polynomials. Lastly, these lemmas will be critically important in our analysis of the location of the roots (Lemma 5.4 and Theorem 5.5) of $\{L_{m,m+1+\epsilon}^{III,\epsilon-1}\}$ and to proving root asymptotic results (Theorem 5.6) of these roots.

Lemma 5.2 (see [31]). For $(1+\epsilon) \in \mathbb{N}$,

$$\frac{\left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)'}{L_m^{-\epsilon}(-x_s)} = -x_s L_{\epsilon-2}^{\epsilon+2}(x_s) + (\epsilon+1-x_s)L_{\epsilon-1}^{\epsilon+1}(x_s) + (m+1)L_\epsilon^\epsilon(x_s). \tag{5.15}$$

Proof. Recall the representation (5.12):

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) = x_s\, L_{\epsilon-1}^{\epsilon+1}(x_s) L_m^{-\epsilon}(-x_s) + (m+1) L_\epsilon^{\epsilon}(x_s) L_{m+1}^{-\epsilon-1}(-x_s).$$

Using the Laguerre identity

$$\left(L_{1+\epsilon}^{\epsilon-1}(x_s)\right)' = -L_\epsilon^{\epsilon}(x_s), \tag{5.16}$$

we see that

$$\left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)' = L_{\epsilon-1}^{\epsilon+1}(x_s) L_m^{-\epsilon}(-x_s) - x_s\, L_{\epsilon-2}^{\epsilon+2}(x_s) L_m^{-\epsilon}(-x_s) + x_s\, L_{\epsilon-1}^{\epsilon+1}(x_s) L_{m-1}^{-\epsilon+1}(-x_s) - (m+1) L_{\epsilon-1}^{\epsilon+1}(x_s) L_{m+1}^{-\epsilon-1}(-x_s) + (m+1) L_\epsilon^{\epsilon}(x_s) L_m^{-\epsilon}(-x_s).$$

Therefore,

$$\frac{\left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)'}{L_m^{-\epsilon}(-x_s)} = -x_s\, L_{\epsilon-2}^{\epsilon+2}(x_s) + (m+1) L_\epsilon^{\epsilon}(x_s) + L_{\epsilon-1}^{\epsilon+1}(x_s)\left(1 + x_s \frac{L_{m-1}^{-\epsilon+1}(-x_s)}{L_m^{-\epsilon}(-x_s)} - (m+1)\frac{L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)}\right). \tag{5.17}$$

Now, using (5.16),

$$1 + x_s \frac{L_{m-1}^{-\epsilon+1}(-x_s)}{L_m^{-\epsilon}(-x_s)} - (m+1)\frac{L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)} = \frac{L_m^{-\epsilon}(-x_s) + x_s\, L_{m-1}^{-\epsilon+1}(-x_s) - (m+1) L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)}$$
$$= \frac{x_s\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)'' + \left(L_{m+1}^{-\epsilon-1}(-x_s)\right)' - (m+1) L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)}. \tag{5.18}$$

Since $y = L_{m+1}^{-\epsilon-1}(x_s)$ satisfies $x_s y'' + (-\epsilon - x_s)y' + (m+1)y = 0$, a simple calculation shows that $y = L_{m+1}^{-\epsilon-1}(-x_s)$ satisfies

$$x_s\, y'' + (-\epsilon + x_s)y' - (m+1)y = 0.$$

Hence (5.18) becomes

$$1 + x_s \frac{L_{m-1}^{-\epsilon+1}(-x_s)}{L_m^{-\epsilon}(-x_s)} - (m+1)\frac{L_{m+1}^{-\epsilon-1}(-x_s)}{L_m^{-\epsilon}(-x_s)} = \frac{(\epsilon + 1 - x_s)\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)'}{L_m^{-\epsilon}(-x_s)} = \epsilon + 1 - x_s \tag{5.19}$$

since $\left(L_{m+1}^{-\epsilon-1}(-x_s)\right)' = L_m^{-\epsilon}(-x_s)$. Substituting (5.19) into (5.17) establishes (5.15).

Lemma 5.3 (see [31]). For $(1+\epsilon) \in \mathbb{N}$,

$$\left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)' = (m+1+\epsilon) L_\epsilon^{\epsilon}(x_s) L_m^{-\epsilon}(-x_s). \tag{5.20}$$

In particular, for all $(1+\epsilon) \in \mathbb{N}$, the Type III exceptional Laguerre polynomial $L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$ has a local extremum at each of the m roots of $L_m^{-\epsilon}(-x_s)$ and at each of the ϵ roots of $L_\epsilon^{\epsilon}(x_s)$.

Proof. The Laguerre polynomial $y = L_\epsilon^{\epsilon}(x_s)$ is a solution of Laguerre's equation

$$x_s\, y'' + (\epsilon + 1 - x_s)y' + \epsilon y = 0.$$

Consequently, we see that the right-hand side of (5.15) simplifies to

$$-x_s\, L_{\epsilon-2}^{\epsilon+2}(x_s) + (\epsilon + 1 - x_s) L_{\epsilon-1}^{\epsilon+1}(x_s) + (m+1) L_\epsilon^{\epsilon}(x_s)$$
$$= -x_s\left(L_\epsilon^{\epsilon}(x_s)\right)'' - (\epsilon + 1 - x_s)\left(L_\epsilon^{\epsilon}(x_s)\right)' + (m+1) L_\epsilon^{\epsilon}(x_s) = (m+1+\epsilon) L_\epsilon^{\epsilon}(x_s).$$

The result now follows from this identity and (5.15).

A degree count implies that all roots of $\left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)'$ must be simple.

We now show the following theorem which will establish (5.13) and (5.14).

Theorem 5.1 (see [31]). For $m+1+\epsilon = 0, m+1, m+2, m+3, \ldots$, the function $y = L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$ is a solution of (5.13), where $\lambda_{m+1+\epsilon}^{s}$ is given in (5.14).

Proof. The proof is straightforward when $m+1+\epsilon = 0$ so we assume $\epsilon \geq 0$. With $y = L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$, we see from Lemma 5.1(b) that

$$\ell_m^{III,\epsilon-1}[y](x_s) = -A_m^{III,\epsilon} B_m^{III,\epsilon}\left[L_{m,m+1+\epsilon}^{III,\epsilon-1}\right](x_s) + (-m + \epsilon - 1) L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$$
$$= -A_m^{III,\epsilon}\left[\frac{\left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)'}{L_m^{-\epsilon}(-x_s)}\right] + (-m + \epsilon - 1) L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$$

$$by\ definition\ of\ B_m^{III,\epsilon}$$

$$= -A_m^{III,\epsilon}\left[(m+1+\epsilon) L_\epsilon^{\epsilon}\right](x_s) + (-m + \epsilon - 1) L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) \text{ by Lemma 5.3}$$
$$= (m+1+\epsilon) L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) + (-m + \epsilon - 1) L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\ by\ (5.11) = (2\epsilon) y(x_s).$$

The next two results give new representations of the Type III exceptional Laguerre polynomials.

Theorem 5.2 (see [31]). For $(1+\epsilon) \in \mathbb{N}$

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) = (m+1+\epsilon)\int_0^{x_s} L_\epsilon^{\epsilon}(t) L_m^{-\epsilon}(-t)\,dt + (m+1)\binom{2+2\epsilon}{\epsilon}\binom{m-\epsilon-2}{m+1}. \tag{5.21}$$

Proof. This follows immediately from (5.20) and the normalization for the classical Laguerre polynomials

$$L_{1+\epsilon}^{\epsilon-1}(0) = \binom{2\epsilon}{1+\epsilon}, \tag{5.22}$$

and from (5.12), which gives

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(0) = (m+1)L_{\epsilon}^{\epsilon}(0)L_{m+1}^{-\epsilon-1}(0). \tag{5.23}$$

The following representation of the Type III exceptional Laguerre polynomials will be important for determining the location of their zeros.

Lemma 5.4 (see [31]). For $m, 1+\epsilon \in \mathbb{N}$ the Type III exceptional Laguerre polynomial $L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$ can be written as

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) = (2\epsilon)L_{\epsilon-1}^{\epsilon}(x_s)L_m^{-\epsilon}(-x_s) + (m+1)L_{\epsilon}^{\epsilon}(x_s)L_{m+1}^{-\epsilon}(-x_s) - (m+1+\epsilon)L_{\epsilon}^{\epsilon}(x_s)L_m^{-\epsilon}(-x_s). \quad (5.24)$$

Proof. Recall (5.12):

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) = x_s L_{\epsilon-1}^{\epsilon+1}(x_s)L_m^{-\epsilon}(-x_s) + (m+1)L_{\epsilon}^{\epsilon}(x_s)L_{m+1}^{-\epsilon-1}(-x_s).$$

From [28, p. 102, (5.1.14)],

$$x_s\left(L_{m+1+\epsilon}^{\epsilon-1}(x_s)\right)' = -(m+2\epsilon)L_{m+\epsilon}^{\epsilon-1}(x_s) + (m+1+\epsilon)L_{m+1+\epsilon}^{\epsilon-1}(x_s),$$

we see that

$$x_s L_{\epsilon-1}^{\epsilon+1}(x_s) = -x_s\left(L_{\epsilon}^{\epsilon}(x_s)\right)' = 2\epsilon L_{\epsilon-1}^{\epsilon}(x_s) - \epsilon L_{\epsilon}^{\epsilon}(x_s). \tag{5.25}$$

Likewise, from the identity (see [28, p. 102, (5.1.13)])

$$L_{m+1+\epsilon}^{\epsilon-1}(x_s) = L_{m+1+\epsilon}^{\epsilon}(x_s) - L_{m+\epsilon}^{\epsilon}(x_s),$$

we obtain

$$L_{m+1}^{-\epsilon-1}(-x_s) = L_{m+1}^{-\epsilon}(-x_s) - L_m^{-\epsilon}(-x_s). \tag{5.26}$$

Substituting (5.25) and (5.26) into (5.12) yields (5.24).

Remark 5.3 [31]. Our discussion to this point shows that if we take (5.24) (or (5.21)) as our definition of the Type III exceptional Laguerre polynomials, they are orthogonal polynomials for $\epsilon < 1$. Regardless of this parameter restriction, the polynomial defined in either (5.24) (or (5.21)) is of degree $m+1+\epsilon$.

As noted by [10], the type III weight expression is non-singular for a more general range of parameters. The necessary and sufficient condition for the denominator $L_m^{-\epsilon}(-x_s)$ t be non-zero for $x_s \geq 0$ is given in Szego [28, Section 6.73] as

$$(\epsilon+m-2)(\epsilon+m-3)\cdots\epsilon(\epsilon-1) > 0.$$

Thus, a natural question to ask is whether or not these polynomials are orthogonal, in some sense, for values of $\epsilon - 1 \notin (-1, 0)$. It would be interesting to look into this question even when some of the associated moments do not exist.

We note that the Type III exceptional Laguerre polynomials are negative at the origin; that is to say, for $\epsilon < 1$ and $(1+\epsilon) \in \mathbb{N}$, we have

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(0) < 0. \tag{5.27}$$

To see this, recall from Theorem 5.2 that

$$L_{m,m+1+\epsilon}^{III,\epsilon-1}(0) = (m+1)L_{\epsilon}^{\epsilon}(0)L_{m+1}^{-\epsilon-1}(0).$$

Now, in general,

$$L_{m+1+\epsilon}^{1+\epsilon}(0) = \binom{n+1+\epsilon}{n}\frac{\Gamma(n+\epsilon+2)}{\Gamma(\epsilon+2)n!} = \frac{(\epsilon+2)(3+\epsilon)\ldots(n+1+\epsilon)}{n!},$$

so $L_{\epsilon}^{\epsilon}(0) > 0$ because $\epsilon > 0$. Restating the assumption on the parameter range as $0 < \epsilon < 1$, we see that $-\epsilon+1+j > 0$ for $j = 0, 1, \ldots, m-1$; thus

$$L_{m+1}^{-\epsilon-1}(0) = \frac{-\epsilon(-\epsilon+1)(-\epsilon+2)\cdots(-\epsilon+m)}{(m+1)!} < 0. \tag{5.28}$$

From (5.28), the inequality in (5.27) now follows. The negativity of $L_{m,m+1+\epsilon}^{III,\epsilon-1}(0)$ turns out to be essential in our analysis (see Section 5.5) of determining the location of the roots of $L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)$.

5.3. The norms of the Type III exceptional Laguerre polynomials [31]

We now compute the norms of these Type III polynomials.

Theorem 5.3 (see [31]). Suppose $\epsilon < 1$. The Type III exceptional Laguerre polynomials

$$\{L_{m,m+1+\epsilon}^{III,\epsilon-1} \mid m+1+\epsilon = 0, m+1, m+2, m+3, \ldots\}$$

are orthogonal in the Hilbert space $L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$ and the norms are explicitly given by

$$\left(\left\|L_{m,m+1+\epsilon}^{III,\epsilon-1}\right\|_m^{III,\epsilon-1}\right)^2 = \int_0^\infty \left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)^2 W_m^{III,\epsilon-1}(x_s)dx_s$$

$$= \begin{cases} \dfrac{(m+1+\epsilon)\Gamma(1+2\epsilon)}{\epsilon!} & if\ \epsilon \geq 0 \\ \dfrac{\Gamma(\epsilon)\Gamma(-\epsilon+1)m!}{\Gamma(m-\epsilon+1)} & if\ m+1+\epsilon = 0 \end{cases}. \tag{5.29}$$

Proof. We compute the norms in this proof; the orthogonality will follow directly from the selfadjointness of the operator $T_m^{III,\epsilon-1}$ in Theorem 5.8 in Section 5.6. The proof, when $\epsilon \geq 0$, rests on the following adjoint relationship for the $A_m^{III,\epsilon-1}$ and $B_m^{III,\epsilon-1}$ operators

$$B_m^{III,\epsilon-1}[f](x_s)g(x_s)W^{\epsilon-1}(x_s) + A_m^{III,\epsilon-1}[g](x_s)f(x_s)W_m^{III,\epsilon-2}(x_s) = \frac{d}{dx_s}\left(\frac{W^{\epsilon-1}(x_s)}{L_m^{-\epsilon+1}(-x_s)}f(x_s)g(x_s)\right), \quad (5.30)$$

Where we take $f = f(x_s), g = g(x_s)$ to be polynomials and where $W^{\epsilon-1}(x_s) = x_s^{\epsilon-1}e^{-x_s}$ is the classical Laguerre weight. To prove this, divide the left-hand side of (5.30) by

$$\frac{W^{\epsilon-1}(x_s)}{L_m^{-\epsilon+1}(-x_s)}f(x_s)g(x_s)$$

to obtain

$$L_m^{-\epsilon+1}(-x_s)B_m^{III,\epsilon-1}\frac{[f](x_s)}{f(x_s)} + \frac{A_m^{III,\epsilon-1}[g](x_s)}{x_s L_m^{-\epsilon+1}(-x_s)g(x_s)};$$

on the other hand, a tedious calculation shows that

$$\frac{d}{dx_s}\log\left(\frac{W^{\epsilon-1}(x_s)}{L_m^{-\epsilon+1}(-x_s)}f(x_s)g(x_s)\right)$$

$$= \frac{f'(x_s)}{f(x_s)} + \frac{x_s L_m^{-\epsilon+1}(-x_s)g'(x_s) + \left((\epsilon-1)L_m^{-\epsilon+1}(-x_s) - x_s L_m^{-\epsilon+1}(-x_s) - x_s\left(L_m^{-\epsilon+1}(-x_s)\right)'\right)g(x_s)}{x_s L_m^{-\epsilon+1}(-x_s)g(x_s)}$$

$$= \frac{f'(x_s)}{f(x_s)} + \frac{x_s L_m^{-\epsilon+1}(-x_s)g'(x_s) + \left(-x_s\left(L_{m+1}^{-\epsilon}(-x_s)\right)'' + (\epsilon-1-x_s)\left(L_{m+1}^{-\epsilon}(-x_s)\right)'\right)g(x_s)}{x_s L_m^{-\epsilon+1}(-x_s)g(x_s)}$$

$$= \frac{f'(x_s)}{f(x_s)} + \frac{x_s L_m^{-\epsilon+1}(-x_s)g'(x_s) - (m+1)L_{m+1}^{-\epsilon}(-x_s)g(x_s)}{x_s L_m^{-\epsilon+1}(-x_s)g(x_s)} \text{ from (5.8) with } \epsilon \to \epsilon-1$$

$$= L_m^{-\epsilon+1}(-x_s)B_m^{III,\epsilon-1}\frac{[f](x_s)}{f(x_s)} + \frac{A_m^{III,\epsilon-1}[g](x_s)}{x_s L_m^{-\epsilon+1}(-x_s)g(x_s)} \text{ by definition of } A_m^{III,\epsilon-1} \text{ and } B_m^{III,\epsilon-1}.$$

This establishes (5.30). Setting $f = A_m^{III,\epsilon}[L_{m+1+\epsilon}^{\epsilon}]$ and $g = L_{m+1+\epsilon}^{\epsilon}$ in (5.30), integrating and using Lemma 5.1, Part (a), and (5.11) gives

$$-(2m+\epsilon+2)\int_0^\infty \left(L_{m+1+\epsilon}^{\epsilon}(x_s)\right)^2 W^{\epsilon}(x_s)dx_s + \int_0^\infty \left(L_{m,2m+\epsilon+2}^{III,\epsilon-1}(x_s)\right)^2 W_m^{III,\epsilon-1}(x_s)dx_s$$

$$= -\left(\frac{L_{m,2m+\epsilon+2}^{III,\epsilon-1}(x_s)L_{m+1+\epsilon}^{\epsilon}(x_s)}{L_m^{-\epsilon}(-x_s)}W^{\epsilon}(x_s)\right)\Bigg|_{x_s=0}^{x_s=\infty}$$

$$= -\left(\frac{L_{m2,m+\epsilon+2}^{III,\epsilon-1}(x_s)L_{m+1+\epsilon}^{\epsilon-1}(x_s)}{L_m^{-\epsilon}(-x_s)}x_s^{\epsilon}e^{-x_s}\right)\Bigg|_{x_s=0}^{x_s=\infty} = 0 \quad (5.31)$$

Hence,

$$\int_0^\infty \left(L_{m,2m+\epsilon+2}^{III,\epsilon-1}(x_s)\right)^2 W_m^{III,1+\epsilon}(x_s)dx_s = (2m+\epsilon+2)\int_0^\infty \left(L_{m+1+\epsilon}^{\epsilon}(x_s)\right)^2 W^{\epsilon}(x_s)dx_s$$

Since (see [28, Chapter V, (5.1.1)])

$$\int_0^\infty \left(L_{m+1+\epsilon}^{\epsilon-1}(x_s)\right)^2 W^{\epsilon-1}(x_s)dx_s = \frac{\Gamma(m+1+2\epsilon)}{(m+1+\epsilon)!} \qquad (j \in N0),$$

we see from (5.31) that

$$\int_0^\infty \left(L_{m,2m+\epsilon+2}^{III,\epsilon-1}(x_s)\right)^2 W_m^{III,\epsilon-1}(x_s)dx_s = \frac{(2m+\epsilon+2)\Gamma(m+2\epsilon+2)}{(m+1+\epsilon)!}.$$

Replacing $2m+\epsilon+2$ by n yields

$$\int_0^\infty \left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)^2 W_m^{III,1+\epsilon}(x_s)dx_s = \frac{(m+1+\epsilon)\Gamma(1+2\epsilon)}{\epsilon!}$$

for $\epsilon \geq 0$, as required.

To prove the norm formula in (5.29) for $n = 0$, we first establish the following identity:

$$\int_0^\infty W_m^{III,\epsilon-1}(x_s)dx_s + m\int W_{m-1}^{III,\epsilon-2}(x_s)dx_s = -\frac{x_s^{\epsilon-1}e^{-x_s}}{L_m^{-\epsilon}(-x_s)L_{m-1}^{-\epsilon+1}(-x_s)}. \quad (5.32)$$

Let $\psi(x_s) = L_m^{-\epsilon}(-x_s)$ so that $\psi'(x_s) = L_{m-1}^{-\epsilon+1}(-x_s)$ and $\psi''(x_s) = L_{m-2}^{-\epsilon+1}(-x_s)$. Now $\psi(x_s)$ is a solution of the Laguerre differential equation

$$x_s\psi''(x_s) + (x_s - \epsilon + 1)\psi'(x_s) - m\psi(x_s) = 0.$$

Divide this differential equation by $\psi(x_s)\left(\psi'(x_s)\right)^2$ and rearrange to obtain

$$\frac{m}{\left(\psi'(x_s)\right)^2}+\frac{x_s}{\psi(x_s)}\left(\frac{1}{\psi'(x_s)}\right)'+\frac{\epsilon-1\,-\,x_s}{\psi(x_s)\psi'(x_s)}=\,0.$$

Multiplying by $x_s^{\epsilon-2}e^{-x_s}$ yieldsm

$$\frac{mx_s^{\epsilon}e^{-x_s}}{\left(\psi'(x_s)\right)^2}+\frac{x_s^{\epsilon-1}e^{-x_s}}{\psi(x_s)}\left(\frac{1}{\psi'(x_s)}\right)'+\frac{(x_s^{\epsilon-1}e^{-x_s})'}{\psi(x_s)\psi'(x_s)}=\,0. \tag{5.33}$$

Since

$$\frac{x_s^{\epsilon-1}e^{-x_s}}{\left(\psi(x_s)\right)^2}+\frac{x_s^{\epsilon-1}e^{-x_s}}{\psi'(x_s)}\left(\frac{1}{\psi(x_s)}\right)'\,\equiv\,0,$$

we see that (5.33) can be rewritten as

$$\frac{x_s^{\epsilon-1}e^{-x_s}}{\left(\psi(x_s)\right)^2}+\frac{mx_s^{\epsilon-2}e^{-x_s}}{\left(\psi'(x_s)\right)^2}=-\left(\frac{x_s^{\epsilon-1}e^{-x_s}}{\psi'(x_s)}\left(\frac{1}{\psi(x_s)}\right)'+\frac{x_s^{\epsilon-1}e^{-x_s}}{\psi(x_s)}\left(\frac{1}{\psi'(x_s)}\right)'+\frac{(x_s^{\epsilon-1}e^{-x_s})'}{\psi(x_s)\psi'(x_s)}\right). \tag{5.34}$$

From the product rule for derivatives, notice that

$$\frac{x_s^{\epsilon-1}e^{-x_s}}{\psi'(x_s)}\left(\frac{1}{\psi(x_s)}\right)'+\frac{x_s^{\epsilon-1}e^{-x_s}}{\psi(x_s)}\left(\frac{1}{\psi'(x_s)}\right)'+\frac{(x_s^{\epsilon-1}e^{-x_s})'}{\psi(x_s)\psi'(x_s)}$$

$$=\left(\left(\frac{1}{\psi(x_s)}\right)\cdot\left(\frac{1}{\psi'(x_s)}\right)\cdot x_s^{1+\epsilon}e^{-x_s}\right)'=\left(\frac{x_s^{1+\epsilon}e^{-x_s}}{L_m^{-\epsilon}(-x_s)L_{m-1}^{-\epsilon+1}(-x_s)}\right)'. \tag{5.35}$$

Substituting (5.35) into (5.34) and using the definition of $W_m^{III,\epsilon-1}(x_s)$, we obtain

$$W_m^{III,\epsilon-1}(x_s)+mW_{m-1}^{III,\epsilon}(x_s)\;=-\left(x_s^{\epsilon-1}e^{-x_s}L_m^{-\epsilon}(-x_s)L_{m-1}^{-\epsilon+1}(-x_s)\right)';$$

integrating this expression now yields (5.32). Applying this relation inductively yields

$$\int W_m^{III,\epsilon-1}(x_s)dx_s\;=\;m!\,(-1)^m\int e^{-x_s}\,x_s^{\epsilon-1-m}dx_s-\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}j!\frac{x_s^{\epsilon-1-j}\,e^{-x_s}}{L_{m-j-1}^{-\epsilon+1+j}(-x_s)L_{m-j}^{-\epsilon+j}(-x_s)}. \tag{5.36}$$

For $r\;>\;0, let\;C_r\;=\;C_1+C_2+C_3$ denote the contour given by the ray

$$C_1=\{x_s\;-ir:0\le x_s<\infty\}$$

oriented from right to left, by the left-side semi-circle

$$C_2=\left\{re^{it}:\frac{\pi}{2}\le t\le\frac{3\pi}{2}\right\}$$

oriented clockwise, and by the ray

$$C_3\;=\{x_s\;+ir:0\le x_s<\infty\}$$

oriented from left to right. The zeros of $L_m^{-\epsilon}(-x_s)$ are all negative, and so by taking r sufficiently small, the contour C_r can be made to not include these negative roots. Observe that the integrand denominator is the square of a polynomial with simple roots. Hence the residuesof the integrand at the roots of $L_m^{-\epsilon}(-x_s)$ vanish, which means that it suffices to impose the condition that $r\;>\;0$ and that $L_m^{-\epsilon}(r\,)\ne 0$. With this assumption,

$$\int_0^\infty W_m^{III,\epsilon-1}(x_s)dx_s\;=\frac{1}{1\;-\;e^{2\pi i(\epsilon-1)}}\int_{C_r}\frac{(-z)^{\epsilon-1}e^{\pi i(\epsilon-1)}e^{-z}}{\left(L_m^{-\epsilon}(-z)\right)^2}\,dz$$

where $(-z)^{\epsilon-1}$ denotes the principal branch of the power function. By deforming C_r we can rewrite the latter as a Mellin–Barnes integral, namely

$$\int_0^\infty W_m^{III,\epsilon-1}(x_s)dx_s\;=\frac{i}{2\sin(\pi(1+\epsilon))}\int_{-r-i\infty}^{-r+i\infty}\frac{(-z)^{\epsilon-1}e^{-z}}{\left(L_m^{-\epsilon}(-z)\right)^2}\,dz. \tag{5.37}$$

Applying the same procedure to the usual integral representation of the Γ-function gives

$$\frac{i}{2\sin(\pi a)}\int_{-r-i\infty}^{-r+i\infty}(-z)^a e^{-z}\,dz\;=\;\Gamma(1+a) \tag{5.38}$$

valid for all non-integral values of a and all $r>0$. Applying (5.37) and (5.38) with $a\;=\;\epsilon-1-\,m$ to (5.36) gives

$$\int_0^\infty W_m^{III,\epsilon-1}(x_s)dx_s\;=\;m!\,(-1)^m\Gamma(1+\epsilon-1-m)\;=\frac{\Gamma(\epsilon)\Gamma(-\epsilon+1)m!}{\Gamma(m-\epsilon+1)}$$

This completes the proof of the theorem.

5.4. The completeness of the Type III exceptional Laguerre polynomials [31]

In preparation for the proof of completeness of $\{L_{m,m+1+\epsilon}^{III,\epsilon-1}\mid m+1+\epsilon\;=0,m\;+\;1,m\;+2,m\;+\;3,\dots\}$, we remind the reader that the set $\mathcal{P}$ denotes the vector space of all polynomials with complex coefficients in the real variable x_s and, for $m+1+\epsilon\in\mathbb{N}_0$, let $\mathcal{P}_{m+1+\epsilon}$ denote the vector space of all $p\in\mathcal{P}$ with degree $\le\;m+1+\epsilon$. The following lemma is critical for our argument; a proof can be found in [15, Lemma 3, p. 416].

Lemma 5.5. Suppose $\eta(x_s)$ is a polynomial such that $\eta(x_s)\ne 0$ for all $x_s\;\ge\;0$. Then, for $\epsilon\;>\;0$, the subspace

$$\eta\,\mathcal{P}:=\{\eta(x_s)p(x_s)\mid p\in\mathcal{P}\}$$

is dense in $L^2\left((0,\infty);x_s^{\epsilon-1}e^{-x_s}\right)$.

We now show the following completeness result.

Theorem 5.4 (see [31]). The set of Type III exceptional Laguerre polynomials

$$\{L_{m,m+1+\epsilon}^{III,\epsilon-1} | m+1+\epsilon = 0, m+1, m+2, m+3, \dots\}$$

forms a complete orthogonal set of polynomials in the Hilbert space $L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$.

Proof. The proof that we give of completeness is similar to the proof that we give of completeness in Theorem 5.9; we give the full proof since some essential ingredients of the proof below are different.

$Let\ \epsilon > 0$ and let $f \in L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$. Define

$$\tilde{f}(x_s) := \frac{f(x_s)}{L_m^{-\epsilon}(-x_s)};$$

clearly

$$\|f\|_m^{III,\epsilon-1} = \|\tilde{f}\|^{\epsilon-1},$$

where $\|.\|^{\epsilon-1}$ denotes the norm in $L^2\big((0,\infty);\ x_s^{\epsilon-1}e^{-x_s}\big)$. Hence $\tilde{f} \in L^2\big((0,\infty);\ x_s^{\epsilon-1}e^{-x_s}\big)$. From Lemma 5.5, with

$$\eta(x_s) = L_m^{-\epsilon+1}(-x_s),$$

there exists $p \in \mathcal{P}$, say with $deg(p) = m+1+\epsilon$, such that

$$\left\|\tilde{f}(x_s) - L_m^{-\epsilon}(-x_s)p(x_s)\right\|^{\epsilon-1} < \epsilon.$$

Hence it follows that

$$\begin{aligned}
\epsilon^2 &> \int_0^\infty \left|\frac{f(x_s)}{L_m^{-\epsilon}(-x_s)} - L_m^{-\epsilon}(-x_s)p(x_s)\right|^2 x_s^{\epsilon-1}e^{-x_s}dx_s \\
&= \int_0^\infty \left|\frac{f(x_s) - \big(L_m^{-\epsilon}(-x_s)\big)^2 p(x_s)}{L_m^{-\epsilon}(-x_s)}\right|^2 x_s^{\epsilon-1}e^{-x_s}dx_s \\
&= \int_0^\infty \left|f(x_s) - \big(L_m^{-\epsilon}(-x_s)\big)^2 p(x_s)\right|^2 W_m^{III,\epsilon-1}(x_s)dx_s \\
&= \left(\left\|f(x_s) - \big(L_m^{-\epsilon}(-x_s)\big)^2 p(x_s)\right\|_m^{III,\epsilon}\right)^2.
\end{aligned}$$

Notice that

$$\big(L_m^{-\epsilon}(-x_s)\big)^2 p \in \mathcal{F}_{1+\epsilon+3m}, \tag{5.39}$$

where

$$\mathcal{F}_{1+\epsilon+3m} := \{P \in \mathcal{P}_{1+\epsilon+3m} \mid P'\big(-(x_s)_j\big) = 0\ (j = 1,2,\dots,m)\},$$

and where $\{(x_s)_j\}_{j=1}^m \subset (0,\infty)$ are the simple roots of the Laguerre polynomial $L_m^{-\epsilon}(x_s)$. We now show that $\big(L_m^{-\epsilon}(-x_s)\big)^2 p(x_s) \in \mathcal{E}_{1+\epsilon+3m}$, where

$$\mathcal{E}_{1+\epsilon+3m} := \text{span}\{L_{m,j}^{III,\epsilon-1} \mid j = 0, m+1, m+2, \dots, 1+\epsilon+3m\};$$

this will complete the proof of the theorem. Note that

$$dim\,(\mathcal{E}_{1+\epsilon+3m}) = dim\,(\mathcal{F}_{1+\epsilon+3m}) = 2m+2+\epsilon. \tag{5.40}$$

Since the span of eigenfunctions of an operator is an invariant subspace of that operator, we see that

$$\ell_m^{III,\epsilon-1}[\mathcal{E}_{1+\epsilon+3m}] \subset \mathcal{E}_{1+\epsilon+3m}.$$

In particular, if $P \in \mathcal{E}_{1+\epsilon+3m}$, then

$$\ell_m^{III,\epsilon-1}[P](x_s) = -x_sP''(x_s) + \left(-\epsilon + x_s + 2x_s\frac{\big(L_m^{-\epsilon}(-x_s)\big)'}{L_m^{-\epsilon}(-x_s)}\right)P'(x_s) + (-m+\epsilon-1)P(x_s) \in \mathcal{P}.$$

Consequently, the term

$$\frac{2x_s\big(L_m^{-\epsilon}(-x_s)\big)'}{L_m^{-\epsilon}(-x_s)}P'(x_s)$$

must be a polynomial. Since $L_m^{-\epsilon}(x_s)$ is a classical Laguerre polynomial, its roots $\{(x_s)_j\}_{j=1}^m \subset (0,\infty)$ are simple so it follows that $P'\big(-(x_s)_j\big) = 0$ for $j = 1,2,\dots,m$. Thus,

$$\mathcal{E}_{1+\epsilon+3m} \subset \mathcal{F}_{1+\epsilon+3m}.$$

From (5.40) we see, in fact, that

$$\mathcal{E}_{1+\epsilon+3m} = \mathcal{F}_{1+\epsilon+3m}$$

From (5.39), it follows that

$$\big(L_m^{-\epsilon}(-x_s)\big)^2 p(x_s) \in \mathcal{E}_{1+\epsilon+3m},$$

and this completes the proof.

5.5. Location of roots and root asymptotics of the Type III exceptional Laguerre polynomials

The following theorem gives us exact location of the m + k (real) roots of the Type III exceptional Laguerre polynomial $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$.

Theorem 5.5 (see [31]). Suppose $m, 1+\epsilon \in \mathbb{N}$ and $\epsilon < 1$. The Type III exceptional Laguerre polynomial $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ has $1+\epsilon$ positive roots which interlace the roots of $L^{\epsilon}_{\epsilon}(x_s)$ and m negative roots which interlace the roots of $L^{-\epsilon}_{m}(-x_s)$. More precisely, let $\{(x_s)^{\epsilon}_{\epsilon,i}\}^{\epsilon}_{i=1} \subset (0,\infty)$ denote the roots of the Laguerre polynomial $L^{\epsilon}_{\epsilon}(x_s)$ (known to be simple), and let $\{z^{-\epsilon}_{m,i}\}^{m}_{i=1} \subset (-\infty, 0)$ denote the (simple) roots of the Laguerre polynomial $L^{-\epsilon}_{m}(-x_s)$, with both sets ordered as follows:

$$z^{-\epsilon}_{m,m} < z^{-\epsilon}_{m,m-1} < \cdots < z^{-\epsilon}_{m,1} < 0 < (x_s)^{\epsilon}_{\epsilon,1} < (x_s)^{\epsilon}_{\epsilon,2} < \cdots < (x_s)^{\epsilon}_{\epsilon,\epsilon}.$$

Then

(a) each of the k intervals

$$\left(0,(x_s)^{\epsilon}_{\epsilon,1}\right), \left((x_s)^{\epsilon}_{\epsilon,1},(x_s)^{(\epsilon)}_{(\epsilon,2)}\right), \dots, \left((x_s)^{\epsilon}_{\epsilon,\epsilon-1},(x_s)^{\epsilon}_{\epsilon,\epsilon}\right), \left((x_s)^{\epsilon}_{\epsilon,\epsilon},\infty\right)$$

contains exactly one root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}$;

(b) each of the m intervals

$$\left(-\infty, z^{-\epsilon}_{m,m}\right), \left(z^{-\epsilon}_{m,m}, z^{-\epsilon}_{m,m-1}\right), \dots, \left(z^{-\epsilon}_{m,2}, z^{-\epsilon}_{m,1}\right)$$

contains exactly one root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}$.

Proof. The key identity in establishing both (a) and (b) is the identity given in (5.24), namely

$$L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s) = 2\epsilon L^{\epsilon}_{\epsilon-1}(x_s)L^{-\epsilon}_{m}(-x_s) + (m+1)L^{\epsilon}_{\epsilon}(x_s)L^{-\epsilon}_{m+1}(-x_s) - (m+1+\epsilon)L^{\epsilon}_{\epsilon}(x_s)L^{-\epsilon}_{m}(-x_s).$$

We first prove part (a). Letting $x_s = (x_s)^{\epsilon}_{\epsilon,i} (i = 1,2,\dots,\epsilon)$ yields

$$L^{III,\epsilon-1}_{m,m+1+\epsilon}\left((x_s)^{\epsilon}_{\epsilon,i}\right) = 2(1+\epsilon)L^{\epsilon}_{\epsilon-1}\left((x_s)^{\epsilon}_{\epsilon,i}\right)L^{-\epsilon}_{m}\left(-(x_s)^{\epsilon}_{\epsilon,i}\right). \tag{5.41}$$

Since $L^{-\epsilon}_{m}(-x_s)$ has no roots in $(0,\infty)$ and $L^{-\epsilon}_{m}(0) > 0$, we see that

$$L^{-\epsilon}_{m}\left(-(x_s)^{\epsilon}_{\epsilon,i}\right) > 0 \qquad (i = 1,2,\dots,\epsilon).$$

Furthermore, from the classical theory, the roots of $L^{\epsilon}_{\epsilon}(x_s)$ and $L^{\epsilon}_{\epsilon-1}(x_s)$ interlace and since $L^{\epsilon}_{\epsilon-1}(0) > 0$, we see that

$$L^{\epsilon}_{\epsilon-1}\left((x_s)^{\epsilon}_{\epsilon,1}\right) > 0.$$

Hence, from (5.41), we deduce that

$$\operatorname{sgn}\left(L^{III,\epsilon-1}_{m,m+1+\epsilon}\left((x_s)^{\epsilon}_{\epsilon,i}\right)\right) = sgn\left(L^{\epsilon}_{\epsilon-1}\left((x_s)^{\epsilon}_{\epsilon,i}\right)\right) = (-1)^{i+1}(i = 1,\dots,\epsilon). \tag{5.42}$$

It follows that $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ has a root in each of the $\epsilon - 1$ intervals

$$\left((x_s)^{\epsilon}_{\epsilon,1},(x_s)^{\epsilon}_{\epsilon,2}\right), \left((x_s)^{\epsilon}_{\epsilon,2},(x_s)^{\epsilon}_{\epsilon,3}\right), \dots, \left((x_s)^{\epsilon}_{\epsilon,\epsilon-1},(x_s)^{\epsilon}_{\epsilon,\epsilon}\right).$$

From (5.27), $L^{III,\epsilon-1}_{m,m+1+\epsilon}(0) < 0$; hence, from (5.42) with $i = 1$, we see that there is another root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ in the interval $(0,(x_s)^{\epsilon}_{\epsilon,1})$. Lastly, from (5.20), we see that $x_s = (x_s)^{\epsilon}_{\epsilon,\epsilon-1}$ and $x_s = (x_s)^{\epsilon}_{\epsilon,\epsilon}$ are the two right-most extreme point of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$. We already showed that there is a zero of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ between these two extreme points. Therefore, regardless of whether $x_s = (x_s)^{\epsilon}_{\epsilon,\epsilon}$ is a relative maximum or relative minimum point, the graph of $y = L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ necessarily must cross the x-axis once more at a point $x^*_s > (x_s)^{\epsilon}_{\epsilon,\epsilon}$. Therefore, $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ has an additional zero in the interval $\left((x_s)^{\epsilon}_{\epsilon,\epsilon},\infty\right)$. Summarizing, we have shown that $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ has $(1+\epsilon)$ distinct, positive roots.

The proof of (b) is similar. In this case, from (5.24), we see that

$$L^{III,\epsilon-1}_{m,m+1+\epsilon}\left(z^{-\epsilon}_{m,i}\right) = (m+1)L^{\epsilon}_{\epsilon}\left(z^{-\epsilon}_{m,i}\right)L^{-\epsilon}_{m+1}\left(-z^{-\epsilon}_{m,i}\right). \tag{5.43}$$

Since $L^{\epsilon}_{\epsilon}(x_s)$ has ϵ positive roots and $L^{\epsilon}_{\epsilon}(0) > 0$, we see that

$$L^{\epsilon}_{\epsilon}\left(z^{-\epsilon}_{m,i}\right) > 0 \qquad (i = 1,2,\dots,m). \tag{5.44}$$

Moreover, since $L^{-\epsilon}_{m+1}(0) > 0$, it follows from the interlacing property of the roots of $L^{-\epsilon}_{m}(x_s)$and $L^{-\epsilon}_{m+1}(x_s)$ that

$$L^{-\epsilon}_{m+1}\left(-z^{-\epsilon}_{m,i}\right) = (-1)^{i} \qquad (i = 1,2,\dots,m).$$

Hence, from (5.43) and (5.44), we see that

$$\operatorname{sgn}\left(L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)\left(z^{-\epsilon}_{m,i}\right)\right) = \operatorname{sgn}\left(L^{-\epsilon}_{m+1}\left(-z^{-\epsilon}_{m,i}\right)\right) = (-1)^{i}(i = 1,\dots,m).$$

This implies that each of the $m - 1$ intervals

$$\left(z^{-\epsilon}_{m,m}, z^{-\epsilon}_{m,m-1}\right), \dots, \left(z^{-\epsilon}_{m,2}, z^{-\epsilon}_{m,1}\right)$$

contains a root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ .We claim that there is an additional root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ in the interval $\left(-\infty, z^{-\epsilon}_{m,m}\right)$. Indeed, from (5.20), $x_s = z^{-\epsilon}_{m,m}$ is the left-most extreme point of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ so, as in part (a), there must be another root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ at a point $z^* < z^{-\epsilon}_{m,m}$. This completes the proof that $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ has m roots in $(-\infty, 0)$. Combining this fact with part (a), we have found all $m + k$ roots of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ and this completes the proof of the theorem.

Remark 5.4 [31]. When $\epsilon = 0$, $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ has one positive root; the exact location of this root cannot be specifically identified. When $m = 1$, the Laguerre polynomial $L_1^{-\epsilon}(-x_s)$ has a unique root $z_{1,1}^{-\epsilon} < 0$. In this case, the above theorem indicates there is a unique root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ in the interval $(-\infty, z_{1,1}^{-\epsilon})$.
We call the m negative roots of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ above the 'exceptional' roots of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$. We now discuss the asymptotic behavior of the roots as $\epsilon \to \infty$.
Theorem 5.6 (see [31]). As $\epsilon \to \infty$:

(a) The exceptional roots of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ converge to the roots of $L_m^{-\epsilon}(-x_s)$.

(b) The first positive root of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ tends to zero.

Proof. Recall, from (5.12), that

$$L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s) = x_s L_{\epsilon-1}^{\epsilon+1}(x_s) L_m^{-\epsilon}(-x_s) + (m+1) L_\epsilon^{\epsilon}(x_s) L_{m+1}^{-\epsilon-1}(-x_s). \qquad (5.45)$$

Now, the outer ratio asymptotics for the classical Laguerre polynomials give

$$\frac{L_\epsilon^{\epsilon}(x_s)}{L_{\epsilon-1}^{\epsilon+1}(x_s)} \simeq \left(-\frac{x_s}{1+\epsilon}\right)^{\frac{1}{2}} + O\left(\frac{1}{1+\epsilon}\right), \qquad \text{as } \epsilon \to \infty$$

with convergence uniform on compact sets that avoid the positive real axis. Therefore, dividing the identity in (5.45) by $L_{\epsilon-1}^{\epsilon+1}(x_s)$ and taking the limit as $\epsilon \to \infty$, we obtain

$$\frac{L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)}{L_{\epsilon-1}^{\epsilon+1}(x_s)} \to x_s L_m^{-\epsilon}(-x_s), \qquad \text{as } \epsilon \to \infty.$$

Part (a) follows by Hurwitz's theorem [28, Theorem 1.91.3] and Theorem 5.5.
By [28, Theorem 6.31.3], the smallest zero of the classical Laguerre polynomial $L_\epsilon^{\epsilon}(x_s)$ goes to zero as $\epsilon \to \infty$. Therefore, by part (a) of Theorem 5.5, the smallest positive zero of $L^{III,\epsilon-1}_{m,m+1+\epsilon}(x_s)$ goes to zero, also.

5.6. Type III exceptional Laguerre spectral analysis [31]

The maximal domain associated with the differential expression $\ell_m^{III,\epsilon-1}[.]$, given in either (5.3) or (5.5), in the Hilbert space $L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$ is defined to be

$$\Delta_m^{III,\epsilon-1} := \left\{ f : (0,\infty) \to \mathbb{C} \;\middle|\; f, f' \in AC_{loc}(0,\infty);\; f, \ell_m^{III,\epsilon-1}[f] \in L^2\left((0,\infty); W_m^{III,\epsilon-1}\right) \right\}. \qquad (5.46)$$

The associated maximal operator

$$T_{1,m}^{III,\epsilon-1} : \mathcal{D}(T_{1,m}^{III,\epsilon-1}) \subset L^2\left((0,\infty); W_m^{III,\epsilon-1}\right) \to L^2\left((0,\infty); W_m^{III,\epsilon-1}\right),$$

is defined to be

$$T_{1,m}^{III,\epsilon-1} f = \ell_m^{III,\epsilon-1}[f]$$
$$f \in \mathcal{D}(T_{1,m}^{III,\epsilon-1}) := \Delta_m^{III,\epsilon-1}.$$

For $f, g \in \Delta_m^{III,\epsilon-1}$, Green's formula may be written as

$$\int_0^\infty \ell_m^{III,\epsilon-1}[f](x_s) g(x_s) W_m^{III,\epsilon-1}(x_s)dx_s = [f,g]_m^{III,\epsilon-1}(x_s)\Big|_{x_s=0}^{x_s=\infty} + \int_0^\infty f(x_s)\ell_m^{III,\epsilon-1}[g](x_s) W_m^{III,\epsilon-1}(x_s)dx_s,$$

where $[.,.]_m^{III,\epsilon-1}(.)$ is the sesquilinear form defined by

$$[f,g]_m^{III,\epsilon-1}(x_s) := x_s^{\epsilon} e^{-x_s} \left(L_m^{-\epsilon}(-x_s)\right)^2 \left(f(x_s)g'(x_s) - f'(x_s)g(x_s)\right) \quad (0 < x_s < \infty)$$

and where

$$[f,g]_m^{III,\epsilon-1}(x_s)\Big|_{x_s=0}^{x_s=\infty} := [f,g]_m^{III,\epsilon-1}(\infty) - [f,g]_m^{III,\epsilon-1}(0) := \lim_{x_s\to\infty}[f,g]_m^{III,\epsilon-1}(x_s) - \lim_{x_s\to 0^+}[f,g]_m^{III,\epsilon-1}(x_s).$$

The adjoint of the maximal operator in $L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$ is the minimal operator $T_{0,m}^{III,\epsilon-1}$, defined by

$$T_{0,m}^{III,\epsilon-1} f = \ell_m^{III,\epsilon-1}[f]$$
$$f \in \mathcal{D}(T\, III,\epsilon-1)_{0,m}) := \{ f \in \Delta_m^{III,\epsilon-1} \mid [f,g]_m^{III,\epsilon-1}\Big|_{x_s=0}^{x_s=\infty} = 0 \text{ for all } g \in \Delta_m^{III,\epsilon-1} \}.$$

Both endpoints $x_s = 0$ and $x_s = \infty$ are singular points of $\ell_m^{III,\epsilon-1}[.]$. In fact, $x_s = 0$ is a regular singular endpoint in the sense of Frobenius and $x_s = \infty$ is an irregular singular endpoint. The associated Frobenius indicial equation at $x_s = 0$ is $r(r + \epsilon - 1) = 0$. Consequently, two linearly independent solutions of $\ell_m^{III,\epsilon-1}[y] = 0$ will behave asymptotically like

$$z_1(x_s) := 1 \qquad \text{and} \qquad z_2(x_s) := x_s^{\epsilon-1}$$

near $x_s = 0$. Since $\epsilon < 1$, it is clear that both solutions are in $L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$; in other words, $\ell_m^{III,\epsilon-1}[.]$ is in the limit-circle case at $x_s = 0$.
For the analysis at the irregular singular endpoint, $x_s = \infty$, we obtain two linearly independent solutions using the standard reduction of order method. Solving the differential equation $\ell_m^{III,\epsilon-1}[y](x_s) = 0$ we have a basis of solutions $\{y_1(x_s), y_2(x_s)\}$, where

$$y_1(x_s) = 1 \in L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$$

and

$$y_2(x_s) = \int_a^{x_s} \frac{e^t\left(L_m^{-\epsilon}(t)\right)^2}{t^\epsilon}\,dt \qquad (\epsilon \geq 0 \text{ is arbitrary}).$$

Mimicking the proof of the Type I case in Section 3.2, we find that $y_2 \notin L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$ .Consequently, we obtain the following result on the deficiency indices of$T_{0,m}^{III,\epsilon-1}$.

Theorem 5.7 [31]. Let $T_{0,m}^{III,\epsilon-1}$ be the minimal operator in $L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$ generated by the Type III exceptional Laguerre differential expression $\ell_m^{III,\epsilon-1}[.]$. For $\epsilon < 1$, the deficiency index of $T_{0,m}^{III,\epsilon-1}$ is $(1, 1)$.

For $\epsilon < 0$, we must impose one boundary condition at $x_s = 0$ in order to obtain a self-adjoint extension of the minimal operator $T_{0,m}^{III,\epsilon-1}$. We seek to find that self-adjoint operator $T_m^{III,\epsilon-1}$ which has the Type III polynomials $\{ L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s) | \, m+1+\epsilon = 0, m+1, m+2, m+3, \ldots\}$ as eigenfunctions.

Note that $x_s - \epsilon - 1 \in L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$ since $\epsilon < 0$. A calculation shows that

$$\ell_m^{III,\epsilon-1}[x_s^{-\epsilon+1}] = \frac{-2(\epsilon-1)x_s^{-\epsilon+1}L_{m-1}^{-\epsilon+1}(-x_s)}{L_m^{-\epsilon}(-x_s)} - m x_s^{-\epsilon+1}.$$

Since

$$\int_0^\infty \left|\frac{x_s^{-\epsilon+1}L_{m-1}^{-\epsilon+1}(-x_s)}{L_m^{-\epsilon}(-x_s)}\right|^2 \frac{x_s^{\epsilon-1}e^{-x_s}}{\left(L_m^{-\epsilon}(-x_s)\right)^2}dx_s \leq \frac{1}{\left(L_m^{-\epsilon}(0)\right)^4}\int_0^\infty \left(L_{m-1}^{-\epsilon+1}(-x_s)\right)^2 x_s^{-\epsilon+1}e^{-x_s}dx_s < \infty,$$

we see that $\ell_m^{III,\epsilon-1}[x_s - \epsilon + 1] \in L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$. Consequently, $x_s - \epsilon - 1 \in \Delta_m^{III,\epsilon-1}$ for $\epsilon < 0$. Moreover, the calculation

$$[x_s^{-\epsilon+1}, 1]_m^{III,\epsilon-1}(0) = (\epsilon-1)\lim_{x_s\to 0}\frac{e^{-x_s}}{\left(L_m^{-\epsilon}(-x_s)\right)^2} \neq 0,$$

proves that $1 \notin \mathcal{D}(T_{0,m}^{III,\epsilon-1})$, the minimal domain, and thus we can use the function 1 as an appropriate Glazman boundary function. For $f \in \Delta_m^{III,\epsilon-1}$, further calculations show that

$$0 = [f,1]_m^{III,\epsilon-1}(0) = \lim_{x_s\to 0^+} x_s^\epsilon f'(x_s)$$

and

$$\lim_{x_s\to 0^+} x_s^\epsilon \left(L_{m,m+1+\epsilon}^{III,\epsilon-1}(x_s)\right)' = 0.$$

Summarizing, and using Theorem 5.4, we obtain the following theorem.

Theorem 5.8 [31]. Suppose $\epsilon < 1$. The operator

$$T_m^{III,\epsilon-1} : \mathcal{D}(T_m^{III,\epsilon-1}) \subset L^2\left((0,\infty); W_m^{III,\epsilon-1}\right) \to L^2\left((0,\infty); W_m^{III,\epsilon-1}\right),$$

defined bv

$$T_m^{III,\epsilon-1}f = \ell_m^{III,\epsilon-1}[f]$$

$$f \in \mathcal{D}(T_m^{III,\epsilon-1}) := \left\{f \in \Delta_m^{III,\epsilon-1} \,\middle|\, \lim_{x_s\to 0^+} x_s^\epsilon f'(x_s) = 0\right\}, \tag{5.47}$$

is a self-adjoint extension of the minimal operator $T_{0,m}^{III,\epsilon-1}$ in $L^2\left((0,\infty); W_m^{III,\epsilon-1}\right)$ having the Type III exceptional Laguerre polynomials

$$\{L_{m,m+1+\epsilon}^{III,\epsilon-1} \,|\, m+1+\epsilon = 0, m+1, m+2, m+3, \ldots\}$$

as a complete set of eigenfunctions. Moreover, the spectrum of $T_m^{III,\epsilon-1}$ consists only of eigenvalues and is given explicitly by

$$\sigma(T_m^{III,\epsilon-1}) = \{2\epsilon \,|\, m+1+\epsilon = 0, m+1, m+2, m+3, \ldots\}.$$

5.7. Spectral equivalence of the type I and the type III operators [31]

Above we mentioned that, formally, the type I and type III differential expressions are related by the gauge transformation (5.4). From the point of view of spectral theory, the multiplication operator is an isometry between two weighted Hilbert spaces, and the gauge transformation is interpreted as intertwining relation between two unbounded, self-adjoint operators. Thus, we must first establish

Proposition 5.1 (see [31]). Suppose that $\epsilon < 1$, and set $\widehat{(1+\epsilon)} = \epsilon - 1$. Then the multiplication operator

$$M(x_s^{1-\epsilon}) : f(x_s) \mapsto x_s^{1-\epsilon}f(x_s), f \in L^2\left((0,\infty); W_m^{I,1-\epsilon}\right)$$

is an isometry $L^2\left((0,\infty); W_m^{I,1-\epsilon}\right) \to L^2\left((0,\infty); W_m^{III,\widehat{1-\epsilon}}\right)$.

Proof. This follows directly by an examination of the two weights:

$$W_m^{I,1-\epsilon}(x_s) = \frac{x_s^{1-\epsilon}e^{-x_s}}{\left(L_m^{-\epsilon}(-x_s)\right)^2}, \qquad W_m^{III,\widehat{1-\epsilon}}(x_s) = \frac{x_s^{\widehat{1-\epsilon}}e^{-x_s}}{\left(L_m^{\widehat{-1-\epsilon}-1}(-x_s)\right)^2} = \frac{x_s^{-1-\epsilon}e^{-x_s}}{\left(L_m^{-\epsilon}(-x_s)\right)^2}.$$

The next task is to define the appropriate self-adjoint extension of the type-I minimal operator. From (3.10) we can interchange the roles of 1 and $x_s^{-1+\epsilon}$ to obtain another interesting self-adjoint operator $S_m^{I,1-\epsilon}$, generated by $\ell_m^{I,1-\epsilon}[.]$, in $L^2\left((0,\infty); W_m^{I,1-\epsilon}\right)$ with different boundary conditions. Observing that

$$0=[f,x_s^{-1-\epsilon}]_m^{I,1-\epsilon}(0) \Leftrightarrow \lim_{x_s\to 0^+}\left(x_s f'(x_s)+(1-\epsilon)f(x_s)\right)=0, \tag{5.48}$$

we are now in position to prove the following theorem regarding the self-adjoint operatorSI,αm , defined below in (5.49), in the space $L^2\left((0,\infty); W_m^{I,1-\epsilon}\right)$. This operator $S_m^{I,1-\epsilon}$ is only quasiisospectral to the classical Laguerre operator in the sense that the commutation transformation that relates the classical Laguerre operator to its Type III counterpart represents a state-adding transformation in the sense of [4].

Theorem 5.9 (see [31]). Suppose $\epsilon<1$. Define

$$S_m^{I,1-\epsilon}:\mathcal{D}\left(S_m^{I,1-\epsilon}\right)\subset L^2\left((0,\infty); W_m^{I,1-\epsilon}\right)\to L^2\left((0,\infty); W_m^{I,1-\epsilon}\right)$$

By

$$\begin{cases} S_m^{I,1-\epsilon}f=\ell_m^{I,1-\epsilon}[f] \\ \mathcal{D}\left(S_m^{I,1-\epsilon}\right)=\left\{f\in\Delta_m^{I,1-\epsilon}\,\middle|\,\lim_{x_s\to 0^+}\left(x_s f'(x_s)+(1-\epsilon)f(x_s)\right)=0\right\}\end{cases}. \tag{5.49}$$

Then $S_m^{I,1-\epsilon}$ is self-adjoint in $L^2\left((0,\infty); W_m^{I,1-\epsilon}\right)$. Furthermore,

$$\{x_s^{-1+\epsilon}L_{m,m+1+\epsilon}^{III,-1+\epsilon}(x_s)\mid m+1+\epsilon=0,m+1,m+2,m+3,\ldots\}$$

forms a complete set of (orthogonal) eigen-functions of $S_m^{I,\epsilon-1}$ in $L^2\left((0,\infty); W_m^{I,\epsilon-1}\right)$, where

$$\{L_{m,m+1+\epsilon}^{III,\epsilon-1}\mid m+1+\epsilon=0,m+1,m+2,m+3,\ldots\}\quad(\epsilon<1)$$

is the sequence of Type III exceptional Laguerre polynomials which we introduce below in Section 5. Finally,

$$\sigma\left(S_m^{I,\epsilon-1}\right)=\sigma_p\left(S_m^{I,\epsilon-1}\right)=\{2\epsilon\mid m+1+\epsilon=0,m+1,m+2,m+3,\ldots\}.$$

Proof. The intertwining relation (5.4) implies that the multiplication operator $M(x_s^{\epsilon-1})$ is also an isometry between the maximal domains $\Delta_m^{I,\epsilon-1}\to\Delta_m^{III,-\epsilon+1}$; see (3.6) and (5.46) for the relevant definitions.

Our next claim is that $f\in\Delta_m^{I,\epsilon-1}$ satisfies the type I boundary condition shown in (5.49) if and only if $M(x_s^{\epsilon-1})(f)$ satisfies the type III boundary condition shown in (5.47). To prove this claim, let $f\in\Delta_m^{I,\epsilon-1}$, set $\hat f(x_s)=x_s^{\epsilon-1}f(x_s)$, and observe that

$$x_s^{\widehat{\epsilon-1+1}}\hat f'(x_s)=x_s^{-\epsilon+2}\left((\epsilon-1)x_s^{\epsilon-2}f(x_s)+x_s^{\epsilon-1}f'(x_s)\right)=(\epsilon-1)f(x_s)+x_sf'(x_s).$$

Therefore

$$S_m^{I,\epsilon-1}=M(x_s^{\epsilon-1})\circ S_m^{III,\epsilon-1}\circ M(x_s^{-\epsilon+1}),$$

where the multiplication operators refer to the isometries between $L^2\left((0,\infty); W_m^{I,\epsilon-1}\right)$ and $L^2\left((0,\infty); W_m^{I,\widehat{\epsilon-1}}\right)$ restricted to $\mathcal{D}\left(S_m^{I,\epsilon-1}\right)$ and $\mathcal{D}\left(S_m^{III,\epsilon-1}\right)$, respectively. The present claims nowfollow by Theorem 5.8.

Appendix. Examples of type III polynomials (see [31])

The following is a list of a few Type III exceptional Laguerre polynomials $\{L_{m,m+1+\epsilon}^{III,\epsilon-1}\}$for various values of m and n. A similar list of Type I and II exceptional Laguerre polynomials can be found in [21].

For $m=1$ we have:

$$L_{1,0}^{III,\epsilon-1}(x_s)=1$$

$$L_{1,2}^{III,\epsilon-1}(x_s)=x_s^2-2(\epsilon-1)x_s+(\epsilon-1)\epsilon$$

$$L_{1,3}^{III,\epsilon-1}(x_s)=-x_s^3+3\epsilon x_s^2-3(\epsilon-1)(\epsilon+1)x_s+\epsilon(\epsilon-1)(\epsilon+1)$$

$$L_{1,4}^{III,\epsilon-1}(x_s)=\frac{1}{2}x_s^4-2(\epsilon+1)x_s^3+(\epsilon+2)(3\epsilon-1)x_s^2-(2\epsilon-2)(\epsilon+1)(\epsilon+2)x_s+\frac{1}{2}\epsilon(\epsilon-1)(\epsilon+1)(\epsilon+2)$$

$$L_{1,5}^{III,\epsilon-1}(x_s)=-\frac{1}{6}x_s^5+\frac{5}{6}(\epsilon+2)x_s^4-\frac{5}{6}(\epsilon+3)(2\epsilon+1)x_s^3+\frac{5}{6}(\epsilon+2)(\epsilon+3)(2(\epsilon-1)+1)x_s^2-\frac{5}{6}(\epsilon-1)(\epsilon+1)(\epsilon+2)(\epsilon+3)x_s+\frac{1}{6}(\epsilon-1)\epsilon(\epsilon+1)(\epsilon+2)(\epsilon+3).$$

For $m=2$ we obtain:

$$L_{2,0}^{III,\epsilon-1}(x_s)=1$$

$$L_{2,3}^{III,\epsilon-1}(x_s)=\frac{1}{2}x_s^3-\frac{3(\epsilon-2)}{2}x_s^2+\frac{3(\epsilon-1)(\epsilon-2)}{2}x_s-\frac{(\epsilon-1)(\epsilon-2)\epsilon}{2}$$

$$L_{2,4}^{III,\epsilon-1}(x_s)=-\frac{1}{2}x_s^4+2(\epsilon-1)x_s^3-(\epsilon-2)(3\epsilon+1)x_s^2+2(\epsilon-1)(\epsilon-2)(\epsilon+1)x_s-\frac{(\epsilon-1)(\epsilon-2)\epsilon(\epsilon+1)}{2}$$

$$L_{2,5}^{III,\epsilon-1}(x_s) = \frac{1}{4}x_s^5 - \frac{5\epsilon}{4}x_s^4 + \frac{5}{2}((\epsilon-1)^2 + 2\epsilon - 3)x_s^3 - \frac{5(\epsilon-2)\epsilon(\epsilon+2)}{2}x_s^2 + \frac{5(\epsilon-1)(\epsilon-2)(\epsilon+1)(\epsilon+2)}{4}x_s - \frac{(\epsilon-1)(\epsilon-2)\epsilon(\epsilon+1)(\epsilon+2)}{4}$$

$$L_{2,6}^{III,\epsilon-1}(x_s) = -\frac{1}{12}x_s^6 + \frac{\epsilon+1}{2}x_s^5 - \frac{5(\epsilon-1)^2 + 19(\epsilon-1) + 6}{4}x_s^4 + \frac{(\epsilon+1)(\epsilon+3)(5(\epsilon-1)-3)}{3}x_s^3 - \frac{(\epsilon-2)(\epsilon+2)(\epsilon+3)(5\epsilon-1)}{4}x_s^2 + \frac{(\epsilon-1)(\epsilon-2)(\epsilon+1)(\epsilon+2)(\epsilon+3)}{2}x_s - \frac{(\epsilon-1)(\epsilon-2)\epsilon(\epsilon+1)(\epsilon+2)(\epsilon+3)}{12}.$$

For $m = 3$:

$$L_{3,0}^{III,\epsilon-1}(x_s) = 1$$

$$L_{3,4}^{III,\epsilon-1}(x_s) = \frac{1}{6}x_s^4 - \frac{2(\epsilon-3)}{3}x_s^3 + (\epsilon-2)(\epsilon-3)x_s^2 - \frac{2(\epsilon-1)(\epsilon-2)(\epsilon-3)}{3}x_s + \frac{(\epsilon-1)(\epsilon-3)(\epsilon-2)\epsilon}{6}$$

$$L_{3,5}^{III,\epsilon-1}(x_s) = -\frac{1}{6}x_s^5 + \frac{5(\epsilon-2)}{6}x_s^4 - \frac{5(\epsilon-3)(2\epsilon-1)}{6}x_s^3 + \frac{5(\epsilon-3)(\epsilon-2)(2\epsilon+1)}{6}x_s^2 - \frac{5(\epsilon-1)(\epsilon-3)(\epsilon-2)(\epsilon+1)}{6}x_s + \frac{(\epsilon-1)(\epsilon-3)(\epsilon-2)\epsilon(\epsilon+1)}{6}$$

$$L_{3,6}^{III,\epsilon-1}(x_s) = \frac{1}{12}x_s^6 - \frac{(\epsilon-1)}{2}x_s^5 + \frac{5(\epsilon-1)^2 + \epsilon - 13}{4}x_s^4 - \frac{(\epsilon-1)(\epsilon-3)(5\epsilon+8)}{3}x_s^3 + \frac{(\epsilon-3)(\epsilon-2)(\epsilon+2)(5\epsilon+1)}{4}x_s^2 - \frac{(\epsilon-1)(\epsilon-3)(\epsilon-2)(\epsilon+1)(\epsilon+2)}{2}x_s + \frac{(\epsilon-1)(\epsilon-3)(\epsilon-2)\epsilon(\epsilon+1)(\epsilon+2)}{12}$$

$$L_{3,7}^{III,\epsilon-1}(x_s) = -\frac{1}{36}x_s^7 + \frac{7\epsilon}{36}x_s^6 - \frac{7((\epsilon-1)^2 + 2\epsilon - 4)}{12}x_s^5 + \frac{35\epsilon((\epsilon-1)^2 + 2\epsilon - 8)}{36}x_s^4 - \frac{7(\epsilon-3)(\epsilon+3)(5(\epsilon-1)^2 + 10(\epsilon-1) - 3)}{36}x_s^3 + \frac{7(\epsilon-3)(\epsilon-2)\epsilon(\epsilon+2)(\epsilon+3)}{12}x_s^2 - \frac{7(\epsilon-1)(\epsilon-3)(\epsilon-2)(\epsilon+1)(\epsilon+2)(\epsilon+3)}{36}x_s + \frac{(\epsilon-1)(\epsilon-3)(\epsilon-2)\epsilon(\epsilon+1)(\epsilon+2)(\epsilon+3)}{36}.$$

References

[1] M. Atia, L.L. Littlejohn, J. Stewart, The spectral theory of the X1-Laguerre polynomials, Adv. Dyn. Syst. Appl. 8 (2) (2013) 81–92.
[2] S. Bochner, U¨ ber Sturm–Liouvillesche polynomsysteme, Math. Z. 29 (1929) 730–736.
[3] P. Borwein, T. Erd´elyi, The fullM¨untz theorem in C[0, 1] and L1[0, 1], J. Lond. Math. Soc. (2) 54 (1996) 102–110.
[4] P. Deift, Applications of a commutation formula, Duke Math. J (1978) 267–310.
[5] F. Deutsch, Best Approximation in Inner Product Spaces, in: CMS Books in Mathematics, Springer-Verlag, New York, 2001.
[6] A.J. Duran, Exceptional Meixner and Laguerre orthogonal polynomials, J. Approx. Theory 184 (2014) 176–208.
[7] D. Dutta, P. Roy, Conditionally exactly solvable potentials and exceptional orthogonal polynomials, J. Math. Phys. 51 (2010) 042101.
[8] A. Erd´elyi, et al., Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, 1953.
[9] Y. Grandati, Solvable rational extensions of the isotonic oscillator, Ann. Phys. 326 (2011) 2074–2090.
[10] Y. Grandati, Rational extensions of solvable potentials and exceptional orthogonal polynomials, J. Physics 343 (2012) 012041.
[11] D. G´omez-Ullate, N. Kamran, R. Milson, Structure theorems for linear and non-linear differential operators admitting invariant polynomial subspaces, Discrete Contin. Dyn. Syst. 18 (2007) 85–106.
[12] D. G´omez-Ullate, N. Kamran, R. Milson, An extended class of orthogonal polynomials defined by a Sturm–Liouville problem, J. Math. Anal. Appl. 359 (2009) 352–367.
[13] D. G´omez-Ullate, N. Kamran, R. Milson, An extension of Bochner's problem: exceptional invariant subspaces, J. Approx. Theory 162 (2010) 987–1006.
[14] D. G´omez-Ullate, N. Kamran, R. Milson, Exceptional orthogonal polynomials and the Darboux transformation, J. Phys. A 43 (2010) 434016.
[15] D. G´omez-Ullate, N. Kamran, R. Milson, Two-step Darboux transformations and exceptional Laguerre polynomials, J. Math. Anal. Appl. 387 (2012) 410–418.

[16] D. G´omez-Ullate, N. Kamran, R. Milson, On orthogonal polynomials spanning a non-standard flag, Contemp. Math. 563 (2012) 51–71.
[17] D. G´omez-Ullate, N. Kamran, R. Milson, A conjecture on exceptional orthogonal polynomials, Found. Comput. Math. 13 (2013) 615–666.
[18] D. G´omez-Ullate, F. Marcell´an, R. Milson, Asymptotic and interlacing properties of zeros of exceptional Jacobi and Laguerre polynomials, J. Math. Anal. Appl. 399 (2013) 480–495.
[19] C.L. Ho, Dirac(-Pauli), Fokker–Planck equations and exceptional Laguerre polynomials, Ann. Phys. 326 (2011) 2074–2090.
[20] C.L. Ho, S. Odake, R. Sasaki, Properties of the exceptional (Xℓ) Laguerre and Jacobi polynomials, SIGMA Symmetry Integrability Geom. Methods Appl. 7 (2011) 24. article 107.
[21] C.L. Ho, R. Sasaki, Zeros of the exceptional Laguerre and Jacobi polynomials, ISRN Math. Phys. (2012) 27. Article ID 920475.
[22] B. Midya, B. Roy, Exceptional orthogonal polynomials and exactly solvable potentials in position dependent mass Schr¨odinger Hamiltonians, Phys. Lett. A 373 (45) (2009) 4117–4122.
[23] M.A. Naimark, Linear Differential Operators II, Frederick Ungar Publishing Co., New York, 1968.
[24] S. Odake, R. Sasaki, Another set of infinitely many exceptional (Xℓ) Laguerre polynomials, Phys. Lett. B 684 (2009) 414–417.
[25] S. Odake, R. Sasaki, Infinitely many shape invariant potentials and new orthogonal polynomials, Phys. Lett. B 679 (2010) 173–176.
[26] C. Quesne, Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry, J. Phys. A 41 (2008) 392001.
[27] C. Quesne, Solvable rational potentials and exceptional orthogonal polynomials in supersymmetric quantum mechanics, SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009) 24. article 084.
[28] G. Szeg¨o, Orthogonal Polynomials, fourth ed., in: Colloquium Publications, vol. 23, American Mathematical Society, Providence, RI, 1974.
[29] T. Tanaka, N-fold supersymmetry and quasi-solvability associated with X2-Laguerre polynomials, J. Math. Phys. 51 (2010) 032101.
[30] A. Zettl, Sturm–Liouville Theory, in: Mathematical Surveys and Monographs, vol. 121, American Mathematical Society, Providence, RI, 2005.
[31] Constanze Liawa, Lance L. Littlejohna, Robert Milsonb,∗, Jessica Stewarta, The spectral analysis of three families of exceptional Laguerre polynomials, Journal of Approximation Theory 202 (2016) 5–41.