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Research Paper

New Exact Analytical Solutions and Validation for the BBM Equation and Comment on Olver's Paper

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Abstract
We derive new exact analytical solutions of the nonlinear Benjamin-Bona-Mahony (BBM) equation using the travelling-wave method within a symbolic computation framework. Both focusing and defocusing regimes are analyzed, revealing a structural duality in their solution forms. Rigorous validation through symbolic substitution confirms the correctness of the solutions, providing exact benchmarks for nonlinear dynamics. This approach demonstrates the utility of symbolic computation in generating and verifying analytical solutions, with potential extensions to other nonlinear models.

Keywords: *symbolic computation, nonlinear wave equation, BBM equation, exact solutions, analytical solutions.*

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I. Introduction

Nonlinear wave equations play a central role across physics, applied mathematics, and engineering, with applications ranging from solid-state physics and plasma dynamics to fluid mechanics and quantum field theory [1–4]. Their behaviour depends critically on the sign of the nonlinear coefficient, which distinguishes focusing from defocusing regimes and gives rise to diverse dynamical structures. Over the past decades, a wide spectrum of analytical techniques—including the inverse scattering transform, Bäcklund transformations, Malfliet's method, and Jacobi elliptic function expansions—have been employed to construct soliton, multisoliton, and periodic-wave solutions [5–12].

Among these models, the **Korteweg–de Vries (KdV) equation** is perhaps the most celebrated, describing weakly nonlinear long waves in shallow water and supporting solitary waves that preserve their shape. However, the KdV equation suffers from instability in its high wave number components. To address this limitation, in 1972 **Benjamin, Bona, and Mahony** introduced the **Benjamin–Bona–Mahony (BBM) equation**, also known as the regularized long-wave equation (RLWE). The BBM equation balances nonlinearity and dispersion in a structurally different way from the KdV equation, ensuring stability and uniqueness of solutions while retaining integrability. Its generalized form is given by:

$$\left[\, y_t + y_x + b\, y\, y_x + y_{\{xxt\}} = 0 \,\right] \qquad (1)$$

where (b>0) corresponds to the defocusing regime and (b<0) to the focusing regime. Unlike the KdV equation, which admits infinitely many conservation laws, the BBM equation possesses only three integrals of motion. Nevertheless, it supports exact solutions and exhibits rich analytical structure, making it a valuable model for both theoretical and applied studies.

The BBM equation has been widely applied in modelling long surface gravity waves of small amplitude, uni-directional propagation in shallow water, plasma physics, and even cosmological contexts. Its stability properties make it particularly suitable for numerical simulation and physical interpretation. Moreover, the equation's duality between focusing and defocusing regimes provides a natural framework for exploring nonlinear dispersive phenomena.

Computational methods have played a decisive role in advancing the study of nonlinear PDEs. Symbolic computation systems such as Maple, Mathematica, and *MathHandbook* enable researchers to derive, manipulate, and validate candidate solutions with precision. This has led to a proliferation of reported solutions for the BBM and related equations, obtained via methods such as the Exp-function approach [13], modified Exp-function techniques [14], and Jacobi elliptic expansions [15]. However, many of these solutions prove invalid upon substitution into the governing equations [16–19], underscoring the importance of rigorous validation.

In this work, we revisit the BBM equation with a focus on exact analytical solutions derived via travelling-wave reduction and symbolic computation. We demonstrate the structural duality between focusing and defocusing regimes, provide explicit solutions in terms of the Weierstrass elliptic functions, and rigorously validate them through symbolic substitution. In doing so, we establish new analytical benchmarks for nonlinear dispersive dynamics and clarify inconsistencies in previously reported results, including those of Olver [19].

II. Method

For simplicity, we set b=12 for defocusing regime:

$$\left[\, y_t + y_x + 12\, y\, y_x + y_{\{xxt\}} = 0 \,\right] \tag{2}$$

set b = -12 for focusing regime:

$$\left[\, y_t + y_x - 12 y\, y_x + y_{\{xxt\}} = 0 \,\right] \tag{3}$$

Seeking a travelling-wave solution, we introduce:

$$[\, Y(u) = y(t,x),\ \ u = x + t \,] \tag{4}$$

Substitution into Eq. (2) reduces the PDE to an ODE. So we obtain:

$$[\, Y''' + 12Y\, Y' + 2Y' = 0 \,] \tag{5}$$

with first integral:

$$[\, Y'' + 6Y^2 + 2Y = C \,] \tag{6}$$

Rearrange:

$$\left[Y'' + 6\left(Y + \tfrac{1}{6}\right)^2 - \tfrac{1}{6} = C \right] \tag{7}$$

Multiplying by Y', integrating, and setting constants yields:

$$\left[Y'^2 + 4\left(Y + \tfrac{1}{6}\right)^3 - (\tfrac{1}{6} + Y) = 0 \right] \tag{8}$$

III. Results

The general solution of the first-order ODE can be expressed by integration in terms of the Weierstrass elliptic function for (b=12) case:

$$\left[Y(u) = -\tfrac{1}{6} - \wp(C_1 \pm u;\ g_2, g_3) \right] \tag{9}$$

where g_2=1 and g_3=0 in the Weierstrass elliptic function.

Thus, the PDE solutions are:

$$\left[y_+(t,x) = -\tfrac{1}{6} - \wp(C_1 \pm (x + t);\ g_2, g_3) \right] \tag{10}$$

for (b = -12) case:

$$\left[y_-(t,x) = \tfrac{1}{6} + \wp(C_1 \pm (x + t);\ g_2, g_3) \right] \tag{11}$$

If ($g_2 = 0$) and ($g_3 = 0$), these reduce to rational soliton waves:

$$\left[y_+(t,x) = -\frac{1}{6} - \frac{1}{\left\{(C_1 \pm (x+t))^2\right\}} \right] \tag{12}$$

$$\left[y_-(t,x) = \frac{1}{6} + \frac{1}{\left\{(C_1 \pm (x+t))^2\right\}} \right] \tag{13}$$

Comparison of these solutions reveals a duality:

$$[\, y_+(t,x) = -\, y_-(t,x) \,] \tag{14}$$

IV. Discussion and Validation

The analytical solutions derived in this work were rigorously validated by direct substitution into the governing BBM equation using a symbolic computation framework (*MathHandbook*). Exact cancellation confirmed their correctness, establishing them as reliable benchmarks for nonlinear PDE studies. This contrasts sharply with several reported solutions in the literature, including Olver's solitary-wave solution [19], which upon substitution yields a nonzero remainder and thus fails to satisfy the equation. Such inconsistencies highlight the necessity of systematic validation, especially in the context of nonlinear dispersive models where subtle algebraic errors can propagate into misleading physical interpretations.

The Benjamin–Bona–Mahony (BBM) equation occupies a distinctive position among nonlinear dispersive wave models. Originally proposed as a regularized alternative to the Korteweg–de Vries (KdV) equation, it was designed to overcome the instability of high wave number components inherent in the KdV framework. Unlike the KdV equation, which admits infinitely many conservation laws, the BBM equation possesses only a finite set (three integrals of motion), yet it retains integrability and supports exact solutions. This balance between tractability and physical realism makes the BBM equation particularly attractive for applications in fluid dynamics, plasma physics, and nonlinear optics.
Several structural advantages of the BBM equation deserve emphasis:

- **Well-posedness and stability:** The BBM equation ensures global existence and uniqueness of solutions, avoiding the ill-posedness issues that arise in the KdV equation for short waves. This property is crucial for both theoretical analysis and numerical simulation.
- **Physical relevance:** The BBM equation more accurately models long surface gravity waves of small amplitude, particularly in regimes where dispersion dominates over nonlinearity. Its formulation captures uni-directional propagation in shallow water and related contexts.
- **Duality in focusing/defocusing regimes:** The solutions presented here reveal a structural symmetry between focusing ($b<0$) and defocusing ($b>0$) cases, expressed through the duality relation in Eq. (14). This correspondence underscores the equation's inherent balance between nonlinear steepening and dispersive spreading.
- **Analytical richness:** The reduction to the Weierstrass elliptic functions demonstrates the BBM equation's capacity to generate both periodic and rational soliton-type solutions. Such exact forms provide benchmarks for testing numerical schemes and approximations.

From a methodological standpoint, the use of symbolic computation ensures that derived solutions are not only formally elegant but also verifiably correct. This approach addresses a recurring problem in the literature, where solutions obtained by heuristic methods (e.g., Exp-function or extended hyperbolic function techniques) often fail under substitution. By contrast, the present framework guarantees consistency with the governing PDE, thereby strengthening the reliability of analytical benchmarks.

In summary, the BBM equation combines mathematical rigor with physical applicability. Its stability, duality, and exact solvability distinguish it from related nonlinear dispersive models. The new solutions derived here extend the analytical landscape of the BBM equation and provide validated reference points for future theoretical and computational investigations.

V. Conclusion

We have presented new exact analytical solutions of a nonlinear BBM equation, obtained via travelling-wave reduction and symbolic computation. The duality between focusing and defocusing regimes highlights the structural symmetry of the model. This methodology can be extended to other nonlinear systems, providing exact analytical benchmarks for theoretical and computational studies.

Declarations
The author declares no competing interests, no data and no funding.

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