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Research Paper

On the multiple completeness of eigen and associated vectors of the polynomial operator bundle in Hilbert space.

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The consideration of Caushy's problem with several initial conditions of the type

$$(E - A_0 - A_1 \frac{d}{dt} - \ldots - A_n \frac{d^n}{dt^n})x = 0 \qquad \textbf{(1)}$$

led to the research of the general spectral theory of operator bundle

$$A(\lambda) = A_0 + \lambda A_1 B + \lambda^2 A_2 B^2 + \ldots + \lambda^{n-1} A_{n-1} B^{n-1} + \lambda^n B^n, \qquad \textbf{(2)}$$

In turn, the study of the bundle (2) led to the research of the questions of the multiple completeness and multiple decompositions on eigen and associated vectors of this polynomial bundle [1],[11].

The bundle $L(\lambda)$ is known as the Keldysh bundle. It is known that n multiple completeness of eigen and associated vectors of bundle $L(\lambda)$ is true when the operators $A_i\,(i = 0,1,\ldots,n)$ are completely continuous, B is a self-adjoint completely continuous operator of finite order. The fundamental result of Keldysh [1] has been generalized by many mathematicians in different directions. Here we should note the works of J.E.Allakhverdiev[2], M.Q.Qasymov[3], A.Q.Kostyuchenko and Q.V.Radzievsky [4], Q.Radzievsky[5] and many others. Theorems about multiple decompositions on eigen and associated vectors with brackets of Keldysh bundle $L(\lambda)$ are proven in works of R.M.Dzhabarzadeh[6], V.N.Vizitey and A.S.Markus[7] .

But in all these works the operator at λ^n in (2), at best, is completely continuous normal of finite order, with the restrictions on the location of its spectrum, besides $KerB = \{0\}$.

In the case when linear operator A is not self-adjoint some results were given in [8],[9],[10].The summation by Abel method on eigen and associated vectors was studied by V.B.Lidsky [8]. In [9] for the polynomial bundle when the main operator is the sum of completely continuous self-adjoint of finite order p operator and the completely continuous operator whose order s less than p $(s < p)$.

In [10]it is proved the completeness of eigen and associated vectors of completely continuous of finite order operator if the norm of the imaginary part of this is sufficient small. In this article similar result will be proven for polynomial Keldysh bundle. We introduce some definitions [11],[12].

1.If for some nonzero vector y_0 we have $A(c)y_{0=} = y_0$,then y_0 is called an eigenvector of operator $A(\lambda)$,corresponding to eigenvalue c .

2. Vector y_k is called an $k-th$ -associated vector to the eigenvector y_0 if the following equalities

$$\begin{aligned} & y = A(c)y \\ & y_1 = A(c)y_1 + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y \\ & \ldots\ldots\ldots\ldots \\ & y_k = A(c)y_k + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y_{k-1} + \ldots + \frac{1}{k!}\frac{\partial^k A(c)}{\partial c^k}y \end{aligned} \qquad (3)$$

are fulfilled.

The system of linear independent eigen and associated vectors is called a chain of eigen and associated vectors of operator $A(\lambda)$,corresponding to eigenvalue c . The number of eigen and associated vectors in the chain is called a length of eigenvector y_0 . The set of all independent eigen and associated vectors, corresponding to all eigenvectors with the eigenvalue c ,is called the multiplicity of eigenvalue c .

3.M.V.Keldysh built the derivative systems with the help of the formulas

$$\left[\frac{d^k}{dt^k}e^{\lambda t}\left(x_k + \frac{1}{1!}x_{k-1} + \ldots + \frac{1}{k!}x_0\right)\right](t=0) \quad k = 1,2,\ldots,s \qquad (4)$$

4. System of eigen and associated vectors of operator bundle $A(\lambda)$ in space H

forms the $n-$ multiple complete system if any n elements $f_0, f_1, \ldots, f_{n-1}$ of the space H can be approximated with the help of linear combinations of elements $\{x_i^{(j)}\}_{i=1}^{\infty}, j = 0,1,2,\ldots,n-1$ in accordance with predetermined accuracy and the same coefficients, not depending on indices of elements $f_0, f_1, \ldots, f_{n-1}$.

5.The system of subspaces $\{M_k\}_{k=1}$ forms n -multiple basis if any element x may be presented in the form $x=\sum_{j=1}^{\infty}{}_1 x_j$ where x_j from M_j .

6.The completely continuous operator A has a finite order if the series of eigenvalues of operator $(A^*A)^{\frac{1}{2}}$ in some positive degrees converge. Lower boundary of such degrees is called an order of operator $=A$. Let B be a completely continuous operator, and

$$T=\frac{B+B^*}{2} \text{ and } S=\frac{B^*-B}{2} \text{ ,then } B=T-S .$$

We designate $r=\max\left\|(B+B^*\ )^{-k-1}(B^*-B)(B+B^*{}_0)^{--k}\right\|<1$

Theorem.

Let (2) be polynomial bundle and the following conditions:

a) $=A\ A_i\,(i=0,1,...,n)$ are completely continuous operators
b) B is completely continuous operator of finite order p
$KerB=0\}, \quad Ker(B+B^*)=\{0\}$

c) $\max\limits_k \left\|T^{k-1}ST^{-k}\right\| \ <1$

$$\left\|E-\ \bigcap_{k=n}^{1}(E-T^{k-1}ST^{-k})^{-1}\right\|<\mathrm{Sin}\frac{\pi}{np} \quad , \quad p>1 \qquad \textbf{(5)}$$

Then the multiple completeness of eigen and assiciated vectors of bundle (2) is true.

Proof.

After simple transformations operator ,standing at the parameter λ^i for each i in the bundle (2) takes the form

$$A_iB^i = A_i(T-S)^i = A\ (T-S)^{i-1}(T-S)T^{-i}T^{ii}$$
$$= A_1(T-S)^{i-1}T^{-i+1}\ (E-T^{i-1}ST^{-i})T^i \ =$$
$$= A_i\ (T-S)^{i-2}T^{-i+2}(E-T^{i-2}ST^{-i+1)})^{-i}(E-T^{i-1}SiT^{-i})T^{ii}.$$

Continuatuing these transformations, we get

$$A(_i(E-ST^{-1})(E-TST^{-2})...(E-T^iE-T^{i-1}ST^{-i1})T^i$$

We introduce the designations

$$M_i = (E - ST^{-1})(E - TST^{-2})...(E - T^i E - T^{i-1} ST^{-i1}$$

$$i = 1, 2, ..., n$$

$$M_n^{-1} = (E - T^{i-1} ST^{-n})^{--1} ...(E - TST^{-2}).^{-1}\ (E - ST^{-1})^{-1}$$

$$(E - ST^{-1})^{-1}$$

As the result of these transformations in the end we transform the bundle (2) in Hilbert space to the equation

$$(M_n^{-1} - M_n^{-1} A_0 - \lambda\ M_n^{-1} A_1 M_1 T - ... -$$
$$-\lambda^{n-1} M_n^{-1} A_{n-1} M_{n-1} T^{n-1} - \lambda^n T^{nn})x = 0$$

Due to the conditions of the Theorem, operators $M_i\ i(i = 1, 2, ..., n)$ and M_n^{-1} are bounded.

We are designating

$$C_i = M_n^{-1} A_i M_i\ (i = 1, 2, ..., n),\quad M_0 = 1$$

$$C_0 = M_n^{-i} A_0 + E - M_n^{-1}$$

All operators $C_i\ \ (i = 1, 2, ..., n-1)$ are completely continuous ,but operator $C_0 =\ M_n^{-i} A_0 + E - M_n^{-1}$ is the sum of completely continuous

operator $M_n^{-i} A_0$ and bounded operator $E - M_n^{-1}$ whose the norm is less than sin $\sin\frac{\pi}{np}$.

Finally we consider the polynomial bundle

$$C(\lambda) = +C_0 + \lambda C_1 T + \lambda^2 C_2 T^2 + ... + \lambda^{n-1} C_{n-1} T^{n-1} + \lambda T^{nn}, \qquad \textbf{(6)}$$

Polynomial bundle (6) is the famous Keldysh's bundle ,and it was investigated in [1],[2].The generalized of Keldysh's theorem allows to have a boundary part of operator ,standing at the parameter λ^0 So the conditions of generalized theorem [2] are fulfilled , and multiple completeness of eigen and associated vectors of (6)takes place.Further, it is not difficult to see that eigen and associated vectors of (2) and (6) coincide. Theorem is proven.

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