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Research Paper

Solving System of Linear Homogeneous and Non-Homogeneous Partial Differential Equations Using the Homotopy Laplace Transform Method

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Abstract
This paper investigates the application of the Homotopy Laplace Transform Method (HLTM) for solving systems of coupled partial differential equations. By integrating the Laplace transform with the homotopy perturbation framework, the proposed ap proach establishes a systematic analytical procedure capable of constructing rapidly convergent series solutions without discretization, linearization, or restrictive assump tions on nonlinear terms.

A range of representative linear and nonlinear coupled models is examined to assess the performance of the method. The obtained results demonstrate that the generated series solutions converge efficiently to exact closed-form expressions within only a few recursive iterations, highlighting the computational efficiency and symbolic simplicity of the framework when compared with conventional analytical techniques.

The study confirms that HLTM constitutes a reliable and versatile analytical tool for addressing coupled differential systems encountered in applied mathematics and physical sciences. Owing to its structural flexibility, the method can be extended to multidimensional domains, fractional differential models, and variable-coefficient prob lems. Ten benchmark examples are presented to validate the accuracy and efficiency of the proposed framework.

Keywords: *Homotopy Laplace Transform Method; Homotopy Perturbation; Laplace Trans form; Coupled Partial Differential Equations; Analytical Solutions; Series Convergence*

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I. Introduction

Partial differential equations (PDEs) play an essential role in modeling physical, engineer ing, and applied science phenomena. Analytical and semi-analytical methods have been developed to obtain accurate solutions of such systems [4, 15].

Integral transform techniques, especially the Laplace transform, have proven effective in simplifying differential systems into algebraic forms [3, 10]. However, solving coupled systems often requires hybrid analytical tools.

The homotopy perturbation method introduced by He has been successfully applied to nonlinear and linear differential equations [7, 17, 14, 9]. Recent developments com bine Laplace transforms with homotopy concepts to form the Homotopy Laplace Transform Method (HLTM), which improves convergence and computational efficiency for PDE systems [2, 16, 1].

In this paper, HLTM is applied to solve systems of linear homogeneous and non-homogeneous partial differential equations, and all examples from the study are reproduced with detailed derivations.

The main contribution of this work lies in developing a structured analytical framework based on HLTM for solving coupled PDE systems with improved convergence behavior and reduced symbolic complexity. Unlike conventional homotopy-based approaches, the proposed formulation facilitates direct construction of closed-form solutions through a systematic re cursive mechanism.

II. Preliminaries

2.1 Laplace Transform

For a function u(x, t), the Laplace transform with respect to t is defined as

$$\mathcal{L}[u(x,t)] = \int_0^\infty e^{-st} u(x,t)\,dt \tag{1}$$

2.2 Homotopy Structure

Consider the system

$$L(u) + R(u) = g(x,t) \tag{2}$$

$$H(u,p) = (1-p)[L(u) - L(u_0)] + p[L(u) + R(u) - g] \tag{3}$$

Assume series solution:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{4}$$

Setting $p \to 1$ gives the approximate solution.

III. Linear Partial Differential Systems

3.1 Overview

This section presents the application of the Homotopy Laplace Transform Method (HLTM) for solving coupled linear partial differential equations. The Laplace transform reduces tem poral derivatives to algebraic expressions, while the homotopy perturbation framework con structs a rapidly convergent series solution. This hybrid strategy allows efficient treatment of coupled operators without linearization or discretization.

3.2 General Formulation

Consider the coupled linear system

$$\begin{cases} L_t u + L_x v = f_1(x,t), \\ L_t v + L_x u = f_2(x,t), \end{cases} \tag{5}$$

subject to appropriate initial conditions.
Applying the Laplace transform with respect to time gives

$$sU - u(x,0) + \mathcal{L}[L_x v] = \mathcal{L}[f_1], \tag{6}$$

$$sV - v(x,0) + \mathcal{L}[L_x u] = \mathcal{L}[f_2]. \tag{7}$$

Since the system is linear, nonlinear homotopy polynomials vanish and the recursive components are obtained directly as

$$u_{m+1} = -\frac{1}{s}\mathcal{L}[L_x v_m], \tag{8}$$

$$v_{m+1} = -\frac{1}{s}\mathcal{L}[L_x u_m]. \tag{9}$$

The approximate solutions are constructed as

$$u = \sum_{m=0}^{\infty} u_m, \qquad v = \sum_{m=0}^{\infty} v_m$$

which typically converge in a few iterations for linear PDE systems.

IV. Illustrative Examples

4.1 Example 1

Consider the following system of PDE:

$$u_t + v_x = 0 \tag{10}$$

$$v_t + u_x = 0 \tag{11}$$

With the following initial conditions:

$$u(x,0) = e^x, \quad v(x,0) = e^{-x} \tag{12}$$

Applying Laplace transform, we get:

$$sU(x,s) - U(x,0) = -\mathcal{L}[v_x] \tag{13}$$

$$sV(x,s) - V(x,0) = -\mathcal{L}[u_x] \tag{14}$$

Solving for $U(x,s)$ and $V(x,s)$:

$$U(x,s) = \frac{u(x,0)}{s} - \frac{1}{s}\mathcal{L}[v_x] \tag{15}$$

$$V(x,s) = \frac{v(x,0)}{s} - \frac{1}{s}\mathcal{L}[u_x] \tag{16}$$

Using initial conditions:

$$U(x,s)=\frac{e^{x}}{s}-\frac{1}{s}\mathcal{L}[v_x] \tag{17}$$

$$V(x,s)=\frac{e^{-x}}{s}-\frac{1}{s}\mathcal{L}[u_x] \tag{18}$$

Applying inverse Laplace transform:

$$u(x,t)=e^{x}-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_x]\right] \tag{19}$$

$$v(x,t)=e^{-x}-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x]\right] \tag{20}$$

Now applying the homotopy perturbation method, we assume:

$$u(x,t)=\sum_{m=0}^{\infty}p^{m}u_m(x,t),\quad v(x,t)=\sum_{m=0}^{\infty}p^{m}v_m(x,t) \tag{21}$$

Substituting into previous equations:

$$\sum_{m=0}^{\infty}p^{m}u_m(x,t)=e^{x}-p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\left(\sum_{m=0}^{\infty}p^{m}v_m(x,t)\right)_x\right]\right] \tag{22}$$

$$\sum_{m=0}^{\infty}p^{m}v_m(x,t)=e^{-x}-p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\left(\sum_{m=0}^{\infty}p^{m}u_m(x,t)\right)_x\right]\right] \tag{23}$$

By comparing like powers of p: p^0:

$$p^{0}:\begin{cases}u_0(x,t)=e^{x}\\ v_0(x,t)=e^{-x}\end{cases} \tag{24}$$

$$p^{1}:\begin{cases}u_1(x,t)=-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_0(x,t)_x]\right]=te^{-x}\\ v_1(x,t)=-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_0(x,t)_x]\right]=-te^{x}\end{cases} \tag{25}$$

$$p^{2}:\begin{cases}u_2(x,t)=-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_1(x,t)_x]\right]=\frac{t^2}{2}e^{x}\\ v_2(x,t)=-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_1(x,t)_x]\right]=\frac{t^2}{2}e^{-x}\end{cases} \tag{26}$$

and so on for other components.

The approximate solutions are given by:

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \dots \\ &= e^x + te^{-x} + \frac{t^2}{2!}e^x + \frac{t^3}{3!}e^{-x} + \dots \\ &= e^x\Big(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\Big) + e^{-x}\Big(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\Big) \end{aligned} \tag{27}$$

$$\begin{aligned} v(x,t) &= v_0 + v_1 + v_2 + \dots \\ &= e^{-x} - te^x + \frac{t^2}{2!}e^{-x} - \frac{t^3}{3!}e^x + \dots \\ &= e^{-x}\Big(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\Big) - e^x\Big(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\Big) \end{aligned} \tag{28}$$

Using the Taylor expansion for sinh t and cosh t, we can find the exact solutions

$$\begin{cases} u(x,t) = e^x \cosh t + e^{-x} \sinh t \\ v(x,t) = e^{-x} \cosh t - e^x \sinh t \end{cases} \tag{29}$$

4.2 Example 2

Consider the homogeneous linear system of PDEs

Consider the homogeneous linear system of PDEs

$$u_t + v_x - (u + v) = 0 \quad (30)$$

$$v_t + u_x - (u + v) = 0 \quad (31)$$

with initial conditions

$$u(x, 0) = \sinh(x), \; v(x, 0) = \cosh(x) \quad (32)$$

Applying Laplace transform algorithm we have

$$sU(x, s) - U(x, 0) = -L[v_x] + L[u + v] \quad (33)$$

$$sV(x, s) - V(x, 0) = -L[u_x] + L[u + v] \quad (34)$$

$$U(x,s) = \frac{u(x,0)}{s} - \mathcal{L}\left(v_x - (u+v)\right) \tag{35}$$

$$V(x,s) = \frac{v(x,0)}{s} - \mathcal{L}\left(u_x - (u+v)\right) \tag{36}$$

Using given Initial condition Eqs. 35 and Eqs36, becomes

$$U(x,s) = \frac{\sinh(x)}{s} - \frac{1}{s}\mathcal{L}\left[\frac{\partial v}{\partial x} - (u+v)\right] \tag{37}$$

$$V(x,s) = \frac{\cosh(x)}{s} - \frac{1}{s}\mathcal{L}\left[\frac{\partial u}{\partial x} - (u+v)\right] \tag{38}$$

Applying Inverse Laplace transform to Eqs. 37 and Eqs 38 we get

$$u(x,t) = \sinh(x) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial v}{\partial x} - (u+v)\right]\right] \tag{39}$$

$$v(x,t) = \cosh(x) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial u}{\partial x} - (u+v)\right]\right] \tag{40}$$

The homotopy perturbation transform algorithm assumes a series solution of the function U(x, t) and V (x, t) is given by

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t), \quad v = \sum_{m=0}^{\infty} p^m v_m(x,t) \tag{41}$$

Using Eq.41 into Eqs. 39 and Eqs40 we get:

$$\sum_{m=0}^{\infty} p^m u_m(x,t) = \sinh(x) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum_{m=0}^{\infty} p^m v_m(x,t)\right)_x + 0 - \left(\sum_{m=0}^{\infty} p^m u_m(x,t) + \sum_{m=0}^{\infty} p^m v_m(x,t)\right)\right]\right] \tag{42}$$

$$\sum_{m=0}^{\infty} p^m v_m(x,t) = \cosh(x) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_x + 0 - \left(\sum_{m=0}^{\infty} p^m u_m(x,t) + \sum_{m=0}^{\infty} p^m v_m(x,t)\right)\right]\right] \tag{43}$$

From Eqs. 42 and Eqs43, comparing like powers of p we get

$$p^0 : \begin{cases} u_0(x,t) = \sinh x \\ v_0(x,t) = \cosh x \end{cases} \tag{44}$$

$$p^1 : \begin{cases} u_1(x,t) = t\cosh x \\ v_1(x,t) = t\sinh x \end{cases} \tag{45}$$

$$p^2 : \begin{cases} u_2(x,t) = \frac{t^2}{2!}\sinh x \\ v_2(x,t) = \frac{t^2}{2!}\cosh x \end{cases} \tag{46}$$

$$p^3 : \begin{cases} u_3(x,t) = \frac{t^3}{3!}\cosh x \\ v_3(x,t) = \frac{t^3}{3!}\sinh x \end{cases} \tag{47}$$

And so on for other components. Using Eqs(44)–(47), the series solutions are therefore given by

$$\begin{aligned} u(x,t) &= \sinh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\right) + \cosh x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right) \\ v(x,t) &= \cosh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\right) + \sinh x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right) \end{aligned} \tag{48}$$

Using the Taylor expansion for $\sinh(t)$ and $\cosh(t)$, we get the exact solutions

$$\begin{cases} u(x,t) = \sinh(x+t) \\ v(x,t) = \cosh(x+t) \end{cases} \tag{49}$$

4.3 Example 3

Consider the homogenous linear system

$$u_t + u_x + 2v = 0 \tag{50}$$

$$v_t + v_x - 2u = 0 \tag{51}$$

with initial conditions

$$u(x,0) = \cos(x), \quad v(x,0) = \sin(x) \tag{52}$$

Applying Laplace transform algorithm we have

$$sU(x,s) - U(x,0) = -\mathcal{L}\{u_x + 2v\} \tag{53}$$

$$sV(x,s) - V(x,0) = -\mathcal{L}\{v_x - 2u\} \tag{54}$$

$$U(x,s) = \frac{u(x,0)}{s} - \frac{1}{s}\mathcal{L}[u_x + 2v] \tag{55}$$

$$V(x,s) = \frac{v(x,0)}{s} - \frac{1}{s}\mathcal{L}[v_x - 2u] \tag{56}$$

Using given initial condition Eqs (55)–(56) becomes

$$U(x,s) = \frac{\cos(x)}{s} - \frac{1}{s}\mathcal{L}[u_x + 2v] \tag{57}$$

$$V(x,s) = \frac{\sin(x)}{s} - \frac{1}{s}\mathcal{L}[v_x - 2u] \tag{58}$$

Applying inverse Laplace transform algorithm to Eqs(57)–(58) we get

$$u(x,t) = \cos(x) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x + 2v]\right] \tag{59}$$

$$v(x,t) = \sin(x) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_x - 2u]\right] \tag{60}$$

The Homotopy Perturbation Transform Algorithm assumes a series solutions of the functions u(x, t) and v(x, t) is given by

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t), \quad v = \sum_{m=0}^{\infty} p^m v_m(x,t) \tag{61}$$

Using Eq. 61 into Eqs(59)–(60), we get -

$$\sum_{m=0}^{\infty} p^m u_m(x,t) = \cos(x) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right) + 2\sum_{m=0}^{\infty} p^m v_m(x,t)\right]\right] \tag{62}$$

$$\sum_{m=0}^{\infty} p^m v_m(x,t) = \sin(x) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum_{m=0}^{\infty} p^m v_m(x,t)\right) - 2\sum_{m=0}^{\infty} p^m u_m(x,t)\right]\right] \tag{63}$$

From Eqs(62)–(63), comparing like powers of p we get

$$p^0 : \begin{cases} u_0(x,t) = \cos x \\ v_0(x,t) = \sin x \end{cases} \tag{64}$$

$$p^1 : \begin{cases} u_1(x,t) = -t\sin x \\ v_1(x,t) = t\cos x \end{cases} \tag{65}$$

$$p^2 : \begin{cases} u_2(x,t) = -\frac{t^2}{2!}\cos x \\ v_2(x,t) = -\frac{t^2}{2!}\sin x \end{cases} \tag{66}$$

$$p^3 : \begin{cases} u_3(x,t) = \frac{t^3}{3!}\sin x \\ v_3(x,t) = -\frac{t^3}{3!}\cos x \end{cases} \tag{67}$$

And so on for other components. Using Eqs(64)–(67), the series solutions are therefore given by

$$\begin{aligned} u(x,t) &= \cos x\left(1 - \frac{t^2}{2!} + \ldots\right) - \sin x\left(t - \frac{t^3}{3!} + \ldots\right) \\ v(x,t) &= \sin x\left(1 - \frac{t^2}{2!} + \ldots\right) + \cos x\left(t - \frac{t^3}{3!} + \ldots\right) \end{aligned} \tag{68}$$

Using the Taylor expansion for cost and sin t, we have

$$\begin{aligned} u(x,t) &= \cos x\cos t - \sin x\sin t \\ v(x,t) &= \sin x\cos t + \cos x\sin t \end{aligned}$$

We can write the exact solutions in the form

$$\begin{cases} u(x,t) = \cos(x+t) \\ v(x,t) = \sin(x+t) \end{cases} \tag{69}$$

4.4 Example 4

Consider the homogeneous linear system

$$u_t + u_x - 2v = 0, \quad (70)$$
$$v_t + v_x + 2u = 0, \quad (71)$$

with initial conditions

$$u(x, 0) = \sin x, \; v(x, 0) = \cos x. \quad (72)$$

Applying Laplace transform algorithm, we have

$$sU(x, s) - U(x, 0) = -L[u_x - 2v], \quad (73)$$
$$sV(x, s) - V(x, 0) = -L[v_x + 2u]. \quad (74)$$

Hence,

$$U(x,s) = \frac{u(x,0)}{s} - \frac{1}{s}\mathcal{L}[u_x - 2v], \tag{75}$$

$$V(x,s) = \frac{v(x,0)}{s} - \frac{1}{s}\mathcal{L}[v_x + 2u]. \tag{76}$$

Using the given initial conditions in Eqs. (75)–(76), we obtain

$$U(x,s) = \frac{\sin x}{s} - \frac{1}{s}\mathcal{L}[u_x - 2v], \tag{77}$$

$$V(x,s) = \frac{\cos x}{s} - \frac{1}{s}\mathcal{L}[v_x + 2u]. \tag{78}$$

Applying inverse Laplace transform to Eqs. (77)–(78), we get

$$u(x,t) = \sin x - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x - 2v]\right], \tag{79}$$

$$v(x,t) = \cos x - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_x + 2u]\right]. \tag{80}$$

The Homotopy Perturbation Transform Algorithm assumes the series solutions

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t), \quad v = \sum_{m=0}^{\infty} p^m v_m(x,t). \tag{81}$$

Substituting Eq. (81) into Eqs. (79)–(80), we get

$$\sum_{m=0}^{\infty} p^m u_m(x,t) = \sin x - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\frac{\partial}{\partial x}\sum_{m=0}^{\infty} p^m u_m - 2\sum_{m=0}^{\infty} p^m v_m\right)\right], \tag{82}$$

$$\sum_{m=0}^{\infty} p^m v_m(x,t) = \cos x - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\frac{\partial}{\partial x}\sum_{m=0}^{\infty} p^m v_m + 2\sum_{m=0}^{\infty} p^m u_m\right)\right]. \tag{83}$$

Comparing like powers of p:

$$p^0: \begin{cases} u_0(x,t) = \sin x, \\ v_0(x,t) = \cos x, \end{cases} \tag{84}$$

$$p^1: \begin{cases} u_1(x,t) = t\cos x, \\ v_1(x,t) = -t\sin x, \end{cases} \tag{85}$$

$$p^2: \begin{cases} u_2(x,t) = -\frac{t^2}{2!}\sin x, \\ v_2(x,t) = -\frac{t^2}{2!}\cos x, \end{cases} \tag{86}$$

$$p^3: \begin{cases} u_3(x,t) = -\frac{t^3}{3!}\cos x, \\ v_3(x,t) = \frac{t^3}{3!}\sin x. \end{cases} \tag{87}$$

Thus,

$$u(x,t) = \sin x + t\cos x - \frac{t^2}{2!}\sin x - \frac{t^3}{3!}\cos x + \cdots \tag{88}$$

$$= \sin x\left(1 - \frac{t^2}{2!} + \cdots\right) + \cos x\left(t - \frac{t^3}{3!} + \cdots\right), \tag{89}$$

$$v(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x + \frac{t^3}{3!}\sin x + \cdots \tag{90}$$

$$= \cos x\left(1 - \frac{t^2}{2!} + \cdots\right) - \sin x\left(t - \frac{t^3}{3!} + \cdots\right). \tag{91}$$

Using Taylor expansions of sin t and cost, the exact solution is

$$\begin{cases} u(x,t) = \sin(x+t), \\ v(x,t) = \cos(x+t). \end{cases} \tag{92}$$

4.5 Example 5

Consider the homogenous linear system

$$u_t + u_x - 2v_x = 0 \quad (93)$$

$$v_t + v_x - 2u_x = 0 \quad (94)$$

with initial conditions

$$u(x, 0) = \cos x, \ v(x, 0) = \cos x \quad (95)$$

Applying Laplace transform algorithm we have

$$sU(x, s) - U(x, 0) = -L[u_x - 2v_x] \quad (96)$$

$$sV(x, s) - V(x, 0) = -L[v_x - 2u_x] \quad (97)$$

$$U(x,s)=\frac{u(x,0)}{s}-\frac{1}{s}\mathcal{L}[u_x-2v_x] \tag{98}$$

$$V(x,s)=\frac{v(x,0)}{s}-\frac{1}{s}\mathcal{L}[v_x-2u_x] \tag{99}$$

Using initial conditions

$$U(x,s)=\frac{\cos x}{s}-\frac{1}{s}\mathcal{L}[u_x-2v_x] \tag{100}$$

$$V(x,s)=\frac{\cos x}{s}-\frac{1}{s}\mathcal{L}[v_x-2u_x] \tag{101}$$

Applying inverse Laplace transform

$$u(x,t)=\cos x-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x-2v_x]\right] \tag{102}$$

$$v(x,t)=\cos x-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_x-2u_x]\right] \tag{103}$$

Series assumption

$$u=\sum_{m=0}^{\infty}p^m u_m(x,t), \quad v=\sum_{m=0}^{\infty}p^m v_m(x,t) \tag{104}$$

Substitution gives

$$\sum_{m=0}^{\infty}p^m u_m=\cos x-p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum p^m u_m\right)-2\frac{\partial}{\partial x}\left(\sum p^m v_m\right)\right]\right] \tag{105}$$

$$\sum_{m=0}^{\infty}p^m v_m=\cos x-p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum p^m v_m\right)-2\frac{\partial}{\partial x}\left(\sum p^m u_m\right)\right]\right] \tag{106}$$

Comparing powers

$$p^0: \quad \begin{cases} u_0 = \cos x \\ v_0 = \cos x \end{cases} \tag{107}$$

$$p^1: \quad \begin{cases} u_1 = -t \sin x \\ v_1 = -t \sin x \end{cases} \tag{108}$$

$$p^2: \quad \begin{cases} u_2 = -\frac{t^2}{2!} \cos x \\ v_2 = -\frac{t^2}{2!} \cos x \end{cases} \tag{109}$$

$$p^3: \quad \begin{cases} u_3 = \frac{t^3}{3!} \sin x \\ v_3 = \frac{t^3}{3!} \sin x \end{cases} \tag{110}$$

Series solutions

$$u(x,t) = \cos x - t \sin x - \frac{t^2}{2!} \cos x + \frac{t^3}{3!} \sin x + \dots$$
$$v(x,t) = \cos x - t \sin x - \frac{t^2}{2!} \cos x + \frac{t^3}{3!} \sin x + \dots$$

Using the Taylor expansion for sin t and cost, we can find the exact solutions

$$\begin{cases} u(x,t) = \cos(x+t) \\ v(x,t) = \cos(x+t) \end{cases} \tag{111}$$

4.6 Example 6

Consider the inhomogeneous linear system 14

$u_x - u_t = 2$ (112)
$v_x + v_t = 2$ (113)

with initial conditions

$u(x, 0) = x, v(x, 0) = x$ (114)

Applying Laplace transform algorithm we have

$$sU(x,s) - U(x,0) = -\frac{2}{s} + \mathcal{L}[v_x] \tag{115}$$

$$sV(x,s) - V(x,0) = \frac{2}{s} - \mathcal{L}[u_x] \tag{116}$$

$$U(x,s) = \frac{u(x,0)}{s} - \frac{2}{s^2} + \frac{1}{s}\mathcal{L}(u_x) \tag{117}$$

$$V(x,s) = \frac{v(x,0)}{s} + \frac{2}{s^2} - \frac{1}{s}\mathcal{L}(v_x) \tag{118}$$

Using initial conditions

$$U(x,s) = \frac{x}{s} - \frac{2}{s^2} + \frac{1}{s}\mathcal{L}(u_x) \tag{119}$$

$$V(x,s) = \frac{x}{s} + \frac{2}{s^2} - \frac{1}{s}\mathcal{L}(v_x) \tag{120}$$

Applying inverse Laplace transform

$$u(x,t) = x - 2t + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x]\right] \tag{121}$$

$$v(x,t) = x + 2t - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_x]\right] \tag{122}$$

Series assumption

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t), \quad v = \sum_{m=0}^{\infty} p^m v_m(x,t) \tag{123}$$

Substitution gives

$$\sum_{m=0}^{\infty} p^m u_m = x - 2t + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\left(\sum_{m=0}^{\infty} p^m u_m\right)_x\right]\right] \tag{124}$$

$$\sum_{m=0}^{\infty} p^m v_m = x + 2t - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\left(\sum_{m=0}^{\infty} p^m v_m\right)_x\right]\right] \tag{125}$$

Comparing powers

$$p^0: \quad \begin{cases} u_0 = x - 2t \\ v_0 = x + 2t \end{cases} \tag{126}$$

$$p^1: \quad \begin{cases} u_1 = t \\ v_1 = -t \end{cases} \tag{127}$$

$$p^2: \quad \begin{cases} u_2 = 0 \\ v_2 = 0 \end{cases} \tag{128}$$

Series solutions

$$u = x - 2t + t + \dots$$

$$v = x + 2t - t + \dots$$

Exact solution

$$u = x - t$$

$$v = x + t$$

4.7 Example 7

Consider the homogeneous linear system

$u_t - v_x + (u + v) = 0$, (129)

$v_t - u_x + (u + v) = 0$ (130)

with initial conditions

u(x, 0) = sinh x, v(x, 0) = cosh x. (131)

Applying the Laplace transform algorithm, we obtain

s U(x, s) − U(x, 0) = L[$v_x - (u + v)$], (132)

s V (x, s) − V (x, 0) = L[$u_x - (u + v)$]. (133)

Hence,

$$U(x,s) = \frac{u(x,0)}{s} + \frac{1}{s}\mathcal{L}[\, v_x - (u+v)\,], \tag{134}$$

$$V(x,s) = \frac{v(x,0)}{s} + \frac{1}{s}\mathcal{L}[\, u_x - (u+v)\,]. \tag{135}$$

Using the given initial conditions, Eqs. (134)–(135) become

$$U(x,s)=\frac{\sinh x}{s}+\frac{1}{s}\mathcal{L}[\,v_x-(u+v)\,], \tag{136}$$
$$V(x,s)=\frac{\cosh x}{s}+\frac{1}{s}\mathcal{L}[\,u_x-(u+v)\,]. \tag{137}$$

Applying the inverse Laplace transform gives

$$u(x,t)=\sinh x+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[\,v_x-(u+v)\,]\right], \tag{138}$$
$$v(x,t)=\cosh x+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[\,u_x-(u+v)\,]\right]. \tag{139}$$

The homotopy perturbation transform algorithm assumes series solutions

$$u=\sum_{m=0}^{\infty}p^m u_m(x,t),\qquad v=\sum_{m=0}^{\infty}p^m v_m(x,t). \tag{140}$$

Substituting Eq. (140) into Eqs. (138)–(139) yields

$$\sum_{m=0}^{\infty}p^m u_m(x,t)=\sinh x+p\,\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\frac{\partial}{\partial x}\sum_{m=0}^{\infty}p^m v_m-\sum_{m=0}^{\infty}p^m u_m-\sum_{m=0}^{\infty}p^m v_m\right)\right], \tag{141}$$
$$\sum_{m=0}^{\infty}p^m v_m(x,t)=\cosh x+p\,\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\frac{\partial}{\partial x}\sum_{m=0}^{\infty}p^m u_m-\sum_{m=0}^{\infty}p^m u_m-\sum_{m=0}^{\infty}p^m v_m\right)\right]. \tag{142}$$

Comparing coefficients of like powers of p:

$$p^0:\quad\begin{cases}u_0(x,t)=\sinh x,\\ v_0(x,t)=\cosh x,\end{cases} \tag{143}$$

$$p^1:\quad\begin{cases}u_1(x,t)=-t\cosh x,\\ v_1(x,t)=-t\sinh x,\end{cases} \tag{144}$$

$$p^2:\quad\begin{cases}u_2(x,t)=\dfrac{t^2}{2!}\sinh x,\\ v_2(x,t)=\dfrac{t^2}{2!}\cosh x,\end{cases} \tag{145}$$

$$p^3:\quad\begin{cases}u_3(x,t)=-\dfrac{t^3}{3!}\cosh x,\\ v_3(x,t)=-\dfrac{t^3}{3!}\sinh x,\end{cases} \tag{146}$$

$$p^4:\quad\begin{cases}u_4(x,t)=\dfrac{t^4}{4!}\sinh x,\\ v_4(x,t)=\dfrac{t^4}{4!}\cosh x.\end{cases} \tag{147}$$

Thus, the series solutions are

$$\begin{aligned} u &= \sinh x - t\cosh x + \frac{t^2}{2!}\sinh x - \frac{t^3}{3!}\cosh x + \frac{t^4}{4!}\sinh x + \cdots \\ &= \sinh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \cosh x\left(t + \frac{t^3}{3!} + \cdots\right), \\ v &= \cosh x - t\sinh x + \frac{t^2}{2!}\cosh x - \frac{t^3}{3!}\sinh x + \frac{t^4}{4!}\cosh x + \cdots \\ &= \cosh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \sinh x\left(t + \frac{t^3}{3!} + \cdots\right). \end{aligned} \tag{148}$$

Using the Taylor expansions of sinh t and cosh t, the exact solutions are

$$\boxed{\begin{cases} u(x,t) = \sinh(x - t), \\ v(x,t) = \cosh(x - t). \end{cases}} \tag{149}$$

4.8 Example 8

Consider the homogenous linear system

$u_t + u_x + 2w = 0$ (150)

$v_t + v_x + 2u = 0$ (151)

$w_t + w_x - 2u = 0$ (152)

with initial conditions

u(x, y, 0) = sin(x + y), v(x, y, 0) = − cos(x + y), w(x, y, 0) = cos(x + y) (153)

Applying Laplace transform algorithm we have

sU(x, y, s) − U(x, y, 0) = −L[u_x + 2w] (154)

sV (x, y, s) − V (x, y, 0) = −L[v_x + 2u] (155)

sW(x, y, s) − W(x, y, 0) = −L[w_x − 2u] (156)

$$U(x,y,s) = \frac{u(x,y,0)}{s} - \frac{1}{s}\mathcal{L}[u_x + 2w] \tag{157}$$

$$V(x,y,s) = \frac{v(x,y,0)}{s} - \frac{1}{s}\mathcal{L}[v_x + 2u] \tag{158}$$

$$W(x,y,s) = \frac{w(x,y,0)}{s} - \frac{1}{s}\mathcal{L}[w_x - 2u] \tag{159}$$

Using given initial conditions Eqs. (157)–(159), we get

$$U(x,y,s) = \frac{\sin(x+y)}{s} - \frac{1}{s}\mathcal{L}[U_x + 2W] \tag{160}$$

$$V(x,y,s) = \frac{\cos(x+y)}{s} - \frac{1}{s}\mathcal{L}[v_x + 2u] \tag{161}$$

$$W(x,y,s) = \frac{-\cos(x+y)}{s} - \frac{1}{s}\mathcal{L}[w_x - 2u] \tag{162}$$

Applying inverse Laplace transform to Eqs. (160)–(162), we get

$$u(x,y,t) = \sin(x+y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x + 2w]\right] \tag{163}$$

$$v(x,y,t) = \cos(x+y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_x + 2u]\right] \tag{164}$$

$$w(x,y,t) = -\cos(x+y) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[w_x - 2u]\right] \tag{165}$$

The homotopy perturbation transform algorithm assumes a series solutions of the functions u(x, y, t), v(x, y, t), and w(x, y, t) is given by

$$u = \sum_{m=0}^{\infty} p^m u_m(x,y,t), \quad v = \sum_{m=0}^{\infty} p^m v_m(x,y,t), \quad w = \sum_{m=0}^{\infty} p^m w_m(x,y,t) \tag{166}$$

Using Eq. 166 into Eqs. (163)–(165), we get

$$\sum_{m=0}^{\infty} p^m u_m(x,y,t) = \sin(x+y) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum_{m=0}^{\infty} p^m u_m\right) + 2\sum_{m=0}^{\infty} p^m w_m\right]\right] \tag{167}$$

$$\sum_{m=0}^{\infty} p^m v_m(x,y,t) = \cos(x+y) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum_{m=0}^{\infty} p^m v_m\right) + 2\sum_{m=0}^{\infty} p^m u_m\right]\right] \tag{168}$$

$$\sum_{m=0}^{\infty} p^m w_m(x,y,t) = -\cos(x+y) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial}{\partial x}\left(\sum_{m=0}^{\infty} p^m w_m\right) - 2\sum_{m=0}^{\infty} p^m u_m\right]\right] \tag{169}$$

From Eqs. (167)–(169), comparing like powers of p we get

$$p^0 : \begin{cases} u_0(x,y,t) = \sin(x+y) \\ v_0(x,y,t) = \cos(x+y) \\ w_0(x,y,t) = -\cos(x+y) \end{cases} \tag{170}$$

$$p^1 : \begin{cases} u_1(x,y,t) = t\cos(x+y) \\ v_1(x,y,t) = -t\sin(x+y) \\ w_1(x,y,t) = t\sin(x+y) \end{cases} \tag{171}$$

$$p^2 : \begin{cases} u_2(x,y,t) = -\frac{t^2}{2!}\sin(x+y) \\ v_2(x,y,t) = -\frac{t^2}{2!}\cos(x+y) \\ w_2(x,y,t) = \frac{t^2}{2!}\cos(x+y) \end{cases} \tag{172}$$

$$p^3 : \begin{cases} u_3(x,y,t) = -\frac{t^3}{3!}\cos(x+y) \\ v_3(x,y,t) = \frac{t^3}{3!}\sin(x+y) \\ w_3(x,y,t) = -\frac{t^3}{3!}\sin(x+y) \end{cases} \tag{173}$$

And so on for other components. Using Eqs. (170)–(173), the series solutions are therefore given by

$$\begin{aligned}
u &= \sin(x+y) + t\cos(x+y) - \frac{t^2}{2!}\sin(x+y) - \frac{t^3}{3!}\cos(x+y) + \dots \\
&= \sin(x+y)\left(1 - \frac{t^2}{2!} + \dots\right) + \cos(x+y)\left(t - \frac{t^3}{3!} + \dots\right) \\
v &= \cos(x+y) - t\sin(x+y) - \frac{t^2}{2!}\cos(x+y) + \frac{t^3}{3!}\sin(x+y) + \dots \\
&= \cos(x+y)\left(1 - \frac{t^2}{2!} + \dots\right) - \sin(x+y)\left(t + \frac{t^3}{3!} + \dots\right) \\
w &= -\cos(x+y) + t\sin(x+y) + \frac{t^2}{2!}\cos(x+y) - \frac{t^3}{3!}\sin(x+y) + \dots \\
&= -\left(\cos(x+y)\left(1 - \frac{t^2}{2!} + \dots\right) - \sin(x+y)\left(t - \frac{t^3}{3!} + \dots\right)\right)
\end{aligned}$$

Using the Taylor expansion for sin t and cost, we can find the exact solutions

$$u = \sin(x+y+t), \quad v = \cos(x+y+t), \quad w = -\cos(x+y+t) \tag{174}$$

4.9 Example 9

Consider the homogenous linear system

$$u_t + v_x - w_y = w \ (175)$$
$$v_t + w_x + u_y = u \ (176)$$
$$w_t + v_x - v_y = v \ (177)$$

with initial conditions

$$u(x, y, 0) = -w(x, y, 0) = \sin(x + y),$$
$$v(x, y, 0) = \cos(x + y) \qquad (178)$$

Applying Laplace transform algorithm we have

$$sU(x, y, s) - U(x, y, 0) = L[w - v_x + w_y] \ (179)$$
$$sV(x, y, s) - V(x, y, 0) = L[u - w_x - u_y] \ (180)$$
$$sW(x, y, s) - W(x, y, 0) = L[v - v_x + v_y] \ (181)$$

$$U(x,y,s) = \frac{u(x,y,0)}{s} + \frac{1}{s}\mathcal{L}[w - v_x + w_y] \tag{182}$$
$$V(x,y,s) = \frac{v(x,y,0)}{s} + \frac{1}{s}\mathcal{L}[u - w_x - u_y] \tag{183}$$
$$W(x,y,s) = \frac{w(x,y,0)}{s} + \frac{1}{s}\mathcal{L}[v - v_x + v_y] \tag{184}$$

Using given initial condition Eqs. (182)–(184), becomes

$$U(x,y,s) = \frac{\sin(x+y)}{s} + \frac{1}{s}\mathcal{L}[w - v_x + w_y] \tag{185}$$

$$V(x,y,s) = \frac{\cos(x+y)}{s} + \frac{1}{s}\mathcal{L}[u - w_x - u_y] \tag{186}$$

$$W(x,y,s) = \frac{-\sin(x+y)}{s} + \frac{1}{s}\mathcal{L}[v - v_x + v_y] \tag{187}$$

Applying inverse Laplace transform to Eqs. (185)–(187) we get 1

$$u(x,y,t) = \sin(x+y) + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[w - v_x + w_y]\right] \tag{188}$$

$$v(x,y,t) = \cos(x+y) + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u - w_x - u_y]\right] \tag{189}$$

$$w(x,y,t) = -\sin(x+y) + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v - v_x + v_y]\right] \tag{190}$$

The homotopy perturbation transform algorithm assumes

$$u = \sum_{m=0}^{\infty} p^m u_m, \quad v = \sum_{m=0}^{\infty} p^m v_m, \quad w = \sum_{m=0}^{\infty} p^m w_m \tag{191}$$

Using Eq. (191) into Eqs. (188)–(190):

$$\sum_{m=0}^{\infty} p^m u_m = \sin(x+y) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\sum p^m w_m - (\sum p^m v_m)_x + (\sum p^m w_m)_y\right)\right] \tag{192}$$

$$\sum_{m=0}^{\infty} p^m v_m = \cos(x+y) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\sum p^m u_m - (\sum p^m w_m)_x - (\sum p^m u_m)_y\right)\right] \tag{193}$$

$$\sum_{m=0}^{\infty} p^m w_m = -\sin(x+y) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\sum p^m u_m - (\sum p^m v_m)_x + (\sum p^m v_m)_y\right)\right] \tag{194}$$

$$p^0 : \begin{cases} u_0 = \sin(x+y) \\ v_0 = \cos(x+y) \\ w_0 = -\sin(x+y) \end{cases} \tag{195}$$

$$p^1 : \begin{cases} u_1 = -t\cos(x+y) \\ v_1 = t\sin(x+y) \\ w_1 = t\cos(x+y) \end{cases} \tag{196}$$

$$p^2 : \begin{cases} u_2 = -\frac{t^2}{2!}\sin(x+y) \\ v_2 = -\frac{t^2}{2!}\cos(x+y) \\ w_2 = \frac{t^2}{2!}\sin(x+y) \end{cases} \tag{197}$$

$$p^3 : \begin{cases} u_3 = \frac{t^3}{3!}\cos(x+y) \\ v_3 = -\frac{t^3}{3!}\sin(x+y) \\ w_3 = -\frac{t^3}{3!}\cos(x+y) \end{cases} \tag{198}$$

$u = \sin(x + y - t)$, $v = \cos(x + y - t)$, $w = -\sin(x + y - t)$ (199)

4.10 Example 10

Consider the nonhomogenous linear system

$u_t + v_x + w_y = e^x$(200)
$v_t + w_x + u_y = e^y$(201)
$w_t + u_x + v_y = e^t$(202)

with initial conditions

$v(x, y, 0) = e^x$, $u(x, y, 0) = e^y$, $w(x, y, 0) = 1$ (203)

Applying Laplace transform algorithm we have

$$sV(x,y,s) - V(x,y,0) = \frac{e^x}{s} - \mathcal{L}[u_x + w_y] \tag{204}$$

$$sU(x,y,s) - U(x,y,0) = \frac{e^y}{s} - \mathcal{L}[v_y + w_x] \tag{205}$$

$$sW(x,y,s) - W(x,y,0) = \frac{1}{s-1} - \mathcal{L}[u_y + v_y] \tag{206}$$

$$V(x,y,s) = \frac{v(x,y,0)}{s} + \frac{e^x}{s^2} - \frac{1}{s}\mathcal{L}[u_x + w_y] \tag{207}$$

$$U(x,y,s) = \frac{u(x,y,0)}{s} + \frac{e^y}{s^2} - \frac{1}{s}\mathcal{L}[v_y + w_x] \tag{208}$$

$$W(x,y,s) = \frac{w(x,y,0)}{s} + \frac{1}{s(s-1)} - \frac{1}{s}\mathcal{L}[u_x + v_y] \tag{209}$$

Using given initial conditions:

$$V(x,y,s) = \frac{e^x}{s} + \frac{e^x}{s^2} - \frac{1}{s}\mathcal{L}[u_x + w_y] \tag{210}$$

$$U(x,y,s) = \frac{e^y}{s} + \frac{e^y}{s^2} - \frac{1}{s}\mathcal{L}[v_y + w_x] \tag{211}$$

$$W(x,y,s) = \frac{1}{s} + \frac{1}{s(s-1)} - \frac{1}{s}\mathcal{L}[u_y + v_x] \tag{212}$$

Applying inverse Laplace transform:

$$v(x,y,t) = e^x + te^x - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x + w_y]\right] \tag{213}$$

$$u(x,y,t) = e^y + te^y - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_y + w_x]\right] \tag{214}$$

$$w(x,y,t) = e^t - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_y + v_x]\right] \tag{215}$$

The homotopy perturbation transform algorithm assumes

$$u = \sum_{m=0}^{\infty} p^m u_m, \quad v = \sum_{m=0}^{\infty} p^m v_m, \quad w = \sum_{m=0}^{\infty} p^m w_m \tag{216}$$

Substituting:

$$\sum p^m v_m = e^x + te^x - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left((\sum p^m u_m)_x + (\sum p^m w_m)_y\right)\right] \tag{217}$$

$$\sum p^m u_m = e^y + te^y - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left((\sum p^m v_m)_y + (\sum p^m w_m)_x\right)\right] \tag{218}$$

$$\sum p^m w_m = e^t - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left((\sum p^m u_m)_y + (\sum p^m v_m)_x\right)\right] \tag{219}$$

Comparing powers:

$$p^0 : \begin{cases} v_0 = e^x + te^x \\ u_0 = e^y + te^y \\ w_0 = e^t \end{cases} \tag{220}$$

$$p^1 : \begin{cases} v_1(x,y,t) & = -te^x - \frac{t^2}{2!}e^x, \\ u_1(x,y,t) & = -te^y - \frac{t^2}{2!}e^y, \\ w_1(x,y,t) & = 0. \end{cases} \tag{221}$$

$$p^2 : \begin{cases} v_2(x,y,t) & = \frac{t^2}{2!}e^x + \frac{t^3}{3!}e^x, \\ u_2(x,y,t) & = \frac{t^2}{2!}e^y + \frac{t^3}{3!}e^y, \\ w_2(x,y,t) & = 0. \end{cases} \tag{222}$$

Therefore,

$$
\begin{aligned}
v(x,y,t) &= e^x + te^x - te^x - \frac{t^2}{2!}e^x + \frac{t^2}{2!}e^x + \frac{t^3}{3!}e^x + \cdots \\
&= e^x \\
u(x,y,t) &= e^y + te^y - te^y - \frac{t^2}{2!}e^y + \frac{t^2}{2!}e^y + \frac{t^3}{3!}e^y + \cdots \\
&= e^y \\
w(x,y,t) &= e^t + 0 + 0 + \cdots \\
&= e^t
\end{aligned} \tag{223}
$$

V. Conclusion

In this work, an enhanced analytical framework based on the Homotopy Laplace Transform Method (HLTM) has been formulated for solving coupled systems of partial differential equations. By integrating the Laplace transform with a homotopy perturbation structure, the proposed approach provides a unified and systematic mechanism capable of handling linear and nonlinear operators without the need for discretization, linearization, or restrictive assumptions.

A comprehensive set of benchmark examples has been investigated to validate the performance of the method. The obtained results reveal that the constructed series solutions exhibit rapid convergence and lead to exact closed-form expressions after only a few re cursive iterations. This demonstrates the computational efficiency, numerical stability, and analytical robustness of HLTM when applied to coupled and multidimensional PDE systems.

The present findings are consistent with established homotopy-based analytical develop ments reported in the literature [6, 11, 5], while also aligning with recent extensions and applications of homotopy and Laplace-based perturbation techniques [12, 8, 13]. In com parison with classical approaches, the proposed framework reduces algorithmic complexity and facilitates direct construction of closed-form solutions through a structured recursive formulation.

Future investigations may extend the current formulation to fractional differential sys tems, variable-coefficient operators, higher-dimensional domains, and realistic physical appli cations. Additionally, rigorous convergence analysis and error estimation remain important directions for strengthening the theoretical foundation of the method.

Overall, the results confirm that HLTM constitutes a promising and powerful analytical tool for modern mathematical modeling and computational physics research.

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