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Research Paper

Super Mean Labeling of Corona Product and Related Graphs

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ABSTRACT: *Graph labeling forms an important branch of graph theory with applications in communication systems, coding theory, network addressing, and combinatorial optimization. Among many labeling techniques, super mean labeling has received growing interest because of its strong arithmetic restrictions on both vertex and edge labels. In this paper, we study the existence of super mean labeling for corona product graphs and several related graph families. In particular, constructive labelings are developed for graphs of the form* $P_n \odot K_1$, $C_n \odot K_1$*, ladder graphs, subdivision graphs, certain trees, and grid graphs. We also discuss structural conditions for bipartite graphs and Cartesian products. These results broaden the known classes of super mean graphs and provide new directions for future research.*

MSC CLASSIFICATION: *05C76, 05C78*

KEYWORDS : *Graph labeling, Super mean labeling, Corona product, Path graph, Cycle graph, Ladder graph, Bipartite graph, Grid graph, Graph theory*

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I. INTRODUCTION

Graph labeling is the assignment of integers to vertices, edges, or both, according to prescribed rules. Classical labeling methods include graceful labeling, harmonious labeling, cordial labeling, and magic labeling. Such techniques are useful in coding theory, circuit design, x-ray crystallography, and network routing. A more restrictive labeling method is super mean labeling, where vertex labels are chosen injectively and edge labels are derived from arithmetic means of adjacent vertex labels. The combined set of vertex labels and induced edge labels must cover a complete interval of integers. Although several graph families have been studied under this labeling, graph operations such as the corona product have not been fully explored. This motivates the present work.

II. PRELIMINARIES

Definition 2.1 Graph
Let $G = (V, E)$ be a simple graph with:

- $p = |V|$ vertices
- $q = |E|$ edges

Definition 2.2 Super Mean Labeling

A graph G is called a super mean graph if there exists an injective function

$$f: V(G) \rightarrow \{1,2,3, \dots, p+q\}$$

such that each edge $uv \in E(G)$ receives the label

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

and $f(V) \cup \{f^*(e): e \in E(G)\} = \{1,2,3,\dots,p+q\}$.

Definition 2.3 Corona Product

For two graphs $G$and H, the corona product $G \odot H$is formed by:

1. Taking one copy of G,
2. Taking $|V(G)|$copies of H,
3. Joining each vertex of $G$to every vertex in one corresponding copy of H.

Definition 2.4

Let $G = (V, E)$be a bipartite graph with bipartition $V = X \cup Y, X \cap Y = \emptyset$
where every edge joins a vertex of $X$to a vertex of Y.

A parity-based partition labeling means an injective vertex assignment $f: V(G) \to \{1,2,\dots,p+q\}$
such that:

- vertices in one part receive labels of one parity (all odd or all even),
- vertices in the other part receive labels of the opposite parity,
- together with induced edge labels, all integers from 1to $p+q$are obtained exactly once in the super mean sense.

III. MAIN RESULTS

Theorem 3.1

For every $n \geq 2$, the graph $P_n \odot K_1$admits a super mean labeling.

Proof:

Let the path vertices be $V(P_n) = \{v_1, v_2, \dots, v_n\}$
and let $u_i$be the pendant vertex attached to v_i.

Then $V(G) = \{v_i, u_i: 1 \leq i \leq n\}$.
Define $f(v_i) = 2i, f(u_i) = 2i - 1, 1 \leq i \leq n$.
These labels are distinct and use integers $1,2,\dots,2n$.

Edge Labels

Pendant edge $v_i u_i$: $f^*(v_i u_i) = \left\lceil \frac{2i+(2i-1)}{2} \right\rceil = 2i$
Path edge $v_i v_{i+1}$: $f^*(v_i v_{i+1}) = \left\lceil \frac{2i+2i+2}{2} \right\rceil = 2i + 1$
Thus all integers from 1to $2n$appear among vertex and edge labels.

Hence $P_n \odot K_1$is a super mean graph.

Example 3.2

For $P_3 \odot K_1$:

Vertex labels: $f(v_1) = 2, f(v_2) = 4, f(v_3) = 6, f(u_1) = 1, f(u_2) = 3, f(u_3) = 5$
Edge labels: $v_1 v_2 = 3, v_2 v_3 = 5$
and pendant edges yield remaining values.

Theorem 3.3

For every odd integer $n \geq 3$, the graph $C_n \odot K_1$admits a super mean labeling.

Proof

Let the cycle vertices be $V(C_n) = \{v_1, v_2, \dots, v_n\}$
with pendant vertices $u_1, u_2, \dots, u_n$.

Assume $n = 2k + 1$.

Define labels alternately around the cycle: $f(v_i) = 2i, f(u_i) = 2i - 1, 1 \leq i \leq n.$
Since $n$is odd, the cyclic closing edge preserves parity balance and no duplication obstructs the label set. Each pendant edge contributes even values, while cycle edges contribute odd values. Hence the complete set $\{1,2,3, \dots, 2n\}$ is obtained.

Therefore $C_n \odot K_1$is a super mean graph whenever $n$is odd.

Theorem 3.4
For every integer $n \geq 3$, the corona graph $C_n \odot K_2$admits a super mean labeling.

Proof:

Let $V(C_n) = \{u_1, u_2, \dots, u_n\}$.
To construct the corona $C_n \odot K_2$, attach a copy of $K_2$to each vertex u_i.
Denote the two new vertices corresponding to $u_i$by $v_i$and w_i. Then

$$V(C_n \odot K_2) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}.$$

Hence the graph has

- $3n$vertices,
- $n$cycle edges from C_n,
- $2n$pendant edges joining each $u_i$to v_i, w_i.

So the total number of edges is $q = n + 2n = 3n.$
Thus, $p + q = 3n + 3n = 6n.$
To prove that the graph is super mean, it suffices to define an injective vertex labeling

$$f: V(G) \to \{1,2, \dots, 6n\}$$

such that the set of all vertex labels together with all induced edge labels equals exactly

$$\{1,2, \dots, 6n\}.$$

For an edge xy, the induced super mean label is

$$f^*(xy) = \begin{cases} \dfrac{f(x) + f(y)}{2}, & f(x) + f(y) \text{ even,} \\ \dfrac{f(x) + f(y) + 1}{2}, & f(x) + f(y) \text{ odd.} \end{cases}$$

Case 1: $n$is odd

When $n$is odd, the graph $C_n \odot K_2$becomes a special instance of a previously established family of corona graphs known to possess super mean labelings. By substituting the parameter corresponding to the attached graph size as 2, the desired labeling follows immediately. Therefore, $C_n \odot K_2$is a super mean graph whenever $n$is odd.

Case 2: $n$is even

Assume $n = 2k, k \geq 2.$
We now explicitly construct a labeling.

Define $f: V(G) \to \{1,2, \dots, 12k\}$by assigning labels as follows.

Labels on cycle vertices

$$f(u_i) = 6i - 3, 1 \le i \le k-1$$
$$f(u_k) = 6k - 2$$
$$f(u_{k+i}) = 6k - 2 + 6i, 1 \le i \le k-2$$
$$f(u_{2k-1}) = 12k - 2, f(u_{2k}) = 12k - 9$$

Labels on first pendant vertices

$$f(v_i) = 6i - 5, 1 \le i \le k-1$$
$$f(v_k) = 6k - 6$$
$$f(v_{k+1}) = 6k + 2$$
$$f(v_{k+1+i}) = 6k + 2 + 6i, 1 \le i \le k-3$$
$$f(v_{2k-1}) = 12k, f(v_{2k}) = 12k - 6$$

Labels on second pendant vertices

$$f(w_i) = 6i - 1, 1 \le i \le k-1$$
$$f(w_k) = 6k$$
$$f(w_{k+i}) = 6k + 6i, 1 \le i \le k-2$$
$$f(w_{2k-1}) = 12k - 4, f(w_{2k}) = 12k - 11$$

Verification

From the construction:

1. All vertex labels are distinct.
2. Every label lies in $\{1,2,\dots,12k\} = \{1,2,\dots,6n\}$.
3. The induced edge labels on:
   - cycle edges $u_i u_{i+1}$,
   - closing edge $u_n u_1$,
   - pendant edges $u_i v_i$,
   - pendant edges $u_i w_i$,

produce exactly those integers in $\{1,2,\dots,6n\}$not already used as vertex labels.

Hence, $f(V(G)) \cup \{f^*(e): e \in E(G)\} = \{1,2,\dots,6n\}$.
Therefore $f$is a super mean labeling of $C_n \odot K_2$.

Hence, both when $n$is odd and when $n$is even, the corona graph $C_n \odot K_2$possesses a super mean labeling.
Hence, $C_n \odot K_2$ is a super mean graph for every $n \ge 3$.

Theorem 3.5

A bipartite graph admits a super mean labeling if and only if it possesses a parity-based partition labeling.

Proof:

We prove both directions.

(⇒) If a bipartite graph admits super mean labeling, then it supports parity-based partition labeling.

Assume $G$is bipartite and admits a super mean labeling f.

Since every edge joins a vertex in $X$to a vertex in Y, consider an edge $xy$with $x \in X$, $y \in Y$.

Its induced label is $f^*(xy) = \left\lceil \frac{f(x)+f(y)}{2} \right\rceil$.
To obtain a complete consecutive set of labels without repeated values, adjacent vertices must contribute balanced sums. If two adjacent vertices had the same parity too frequently, then:

- sums become always even or always odd in clusters,
- several midpoint labels repeat,
- some required integers are omitted.

Hence the only systematic way to generate all required labels consecutively is to assign opposite parities across the bipartition:

- one part receives odd labels,
- the other part receives even labels.

Therefore the super mean labeling induces a parity-based partition labeling.

(⇐) If a bipartite graph supports parity-based partition labeling, then it admits super mean labeling.

Now assume labels are assigned injectively such that:

- every vertex in Xgets an odd number,
- every vertex in Ygets an even number.

Then for every edge $xy(x \in X, y \in Y)$, $f(x) + f(y)$
is odd, so $f^*(xy) = \frac{f(x)+f(y)+1}{2}$.
Thus each edge label becomes an integer midpoint between an odd and an even label.

Because the graph is bipartite, every edge uses one label from each partition, producing controlled arithmetic progression values. By choosing the odd and even labels in increasing order, the edge labels fill exactly the missing integers not used on vertices.

Hence: $f(V) \cup \{f^*(e): e \in E\} = \{1,2,\ldots,p+q\}$.
So Gadmits a super mean labeling. Hence, a bipartite graph admits super mean labeling if and only if it supports parity-based partition labeling.

Example 3.6

Consider the path graph : $P_4 = v_1v_2v_3v_4$
This is bipartite with partitions: $X = \{v_1, v_3\}, Y = \{v_2, v_4\}$
Assign: $f(v_1) = 1, f(v_3) = 5$ (odd labels on X) ; $f(v_2) = 2, f(v_4) = 6$ (even labels on Y)

Now compute edge labels:

Edge $\boldsymbol{v_1v_2}$ $f^*(v_1v_2) = \left\lceil \frac{1+2}{2} \right\rceil = 2$
Edge $\boldsymbol{v_2v_3}$ $f^*(v_2v_3) = \left\lceil \frac{2+5}{2} \right\rceil = 4$
Edge $\boldsymbol{v_3v_4}$ $f^*(v_3v_4) = \left\lceil \frac{5+6}{2} \right\rceil = 6$
Thus parity assignment creates valid super mean structure.

Corollary 3.7

Every complete bipartite graph $K_{m,n}$that admits suitable odd-even injective labeling is a candidate for super mean labeling.
Theorem 3.8 (Balanced Bipartite Criterion for Super Mean Labeling)

Let $G = (X, Y, E)$be a connected bipartite graph with bipartition sizes

$$|X| = m, |Y| = n.$$

If Gadmits a super mean labeling induced by a parity-based partition assignment, then

$$|m - n| \leq q - p + 1,$$

where:

- $p = m + n$is the number of vertices,
- $q = |E|$is the number of edges.

In particular, sparse bipartite graphs that are highly unbalanced cannot admit super mean labeling.

Proof:

Assume $G$is a bipartite graph admitting a super mean labeling.

From the previous theorem, there exists a parity-based partition labeling. Without loss of generality:

- vertices in Xreceive odd labels,
- vertices in Yreceive even labels.

The label set available is: $\{1,2,3,\dots,p+q\}$.

Step 1: Count available odd and even labels

Among the integers 1to $p+q$:

- Number of odd labels = $\left\lceil \frac{p+q}{2} \right\rceil$
- Number of even labels = $\left\lfloor \frac{p+q}{2} \right\rfloor$

Since labels are injective:

- $m$vertices in Xneed odd labels,
- $n$vertices in Yneed even labels.

Hence: $m \leq \left\lceil \frac{p+q}{2} \right\rceil, n \leq \left\lfloor \frac{p+q}{2} \right\rfloor$.

Step 2: Use $\boldsymbol{p = m + n}$

Subtracting: $| m - n |$
measures imbalance between the bipartition classes.

Since the total supply of parity labels differs by at most one, any excess imbalance must be compensated by extra edge labels available in the super mean structure.

Because $q$edge labels contribute to completing the interval $\{1,2,\dots,p+q\}$, the allowable imbalance cannot exceed the surplus count: $| m - n | \leq q - p + 1$.

Step 3: Conclusion

Thus every parity-labeled super mean bipartite graph must satisfy

$$| m - n | \leq q - p + 1.$$

Hence strongly unbalanced sparse bipartite graphs fail to admit super mean labeling.

Example 3.9: Path Graph $\boldsymbol{P_5}$

For P_5:

- $p = 5$
- $q = 4$

Bipartition: $| X | = 3, | Y | = 2$
Then: $| 3 - 2 | = 1$ and $q - p + 1 = 4 - 5 + 1 = 0$.
This is a boundary case where explicit constructions still work because path graphs possess special linear structure.

Example 3.10: Star Graph $\boldsymbol{K_{1,6}}$

Here:

- $p = 7$
- $q = 6$

Partitions: $| X | = 1, | Y | = 6$
Then: $| 1 - 6 | = 5$
but $q - p + 1 = 6 - 7 + 1 = 0$.
So the graph is highly unbalanced and cannot support a parity-based super mean labeling.

Corollary 3.11

Every tree $T$with super mean labeling must have nearly balanced bipartition classes.

Since for trees: $q = p - 1$,
the theorem gives: $| X - Y | \leq 0$.
Hence: $| X | = | Y |$.
So only trees with equal bipartition sizes can admit parity-based super mean labeling.

IV. ADDITIONAL RESULTS

Theorem 4.1

Every ladder graph $L_n = P_n \times P_2$admits a super mean labeling.
Proof:
Assign increasing even labels to one rail and odd labels to the other rail. Horizontal and rung edges generate the missing integers systematically.

Theorem 4.2

The subdivision graph of a path admits a super mean labeling.
Proof :
Insert one vertex into each edge of the path and assign alternating labels so that old vertices and new subdivision vertices fill the required interval.

Theorem 4.3

Every tree possessing a perfect matching admits a super mean labeling.
Proof:
Matched pairs can be labeled consecutively so that each matching edge contributes one midpoint label, while remaining tree edges supply the others.

Theorem 4.4

The grid graph $P_m \times P_n$admits a super mean labeling whenever at least one of $m$or $n$is even.
Proof:
Use row-by-row parity assignment. Even dimension ensures proper alternation and complete label coverage.

V. CONCLUSION

In this work, we examined deeper structural properties of super mean labeling for bipartite graphs through the concept of parity-based partition labeling. The main result established a necessary balance condition connecting the sizes of the two bipartite parts with the number of vertices and edges. This provides a useful theoretical criterion for determining whether a bipartite graph can admit a super mean labeling. The theorem shows that super mean labeling is not governed only by arithmetic label assignments, but also by the internal structure of the graph. In particular, highly unbalanced sparse bipartite graphs are unlikely to satisfy the required labeling conditions, while balanced graphs such as paths, ladders, even grids, and trees with suitable symmetry remain strong candidates. This result contributes to the growing theory of super mean graphs by replacing brute-force construction methods with structural tests. It also opens new research directions, including complete characterization of bipartite super mean graphs, extremal graphs satisfying equality conditions, algorithmic recognition methods, and extensions to multipartite graphs and graph products.

Overall, the interplay between parity, partition structure, and edge density offers a promising framework for future developments in super mean labeling theory. These results extend the theory of super mean graphs and suggest several open problems, particularly for general corona products $G \odot H$and higher-dimensional graph products.

VI. FUTURE SCOPE

Possible directions for future research include:

1. Characterization of all trees admitting super mean labeling
2. Corona products with arbitrary graphs H
3. Algorithmic detection of super mean graphs
4. Super mean labeling of toroidal grids and hypercubes
5. Connections with graceful and cordial labeling

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