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Research Paper

Solving Systems of Nonlinear Homogeneous and Non-Homogeneous Partial Differential Equations Using the Homotopy Laplace Transform Method

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Abstract
This paper develops a rigorous semi-analytical framework for solving systems of nonlinear homogeneous and non-homogeneous partial differential equations via the Homotopy Laplace Transform Method (HLTM). The proposed approach combines the Laplace transform technique with the homotopy perturbation structure to generate rapidly convergent series solutions without linearization, discretization, or restrictive small-parameter assumptions.

The nonlinear operators are decomposed using He's polynomials, leading to a sys tematic and computationally efficient formulation that avoids the complexity associ ated with traditional decomposition techniques. A sufficient convergence criterion is established under a Lipschitz-type condition, ensuring the existence and uniqueness of the solution. Moreover, an explicit upper bound for the truncation error is derived, demonstrating geometric convergence of the solution series.

Several nonlinear coupled PDE systems are investigated to illustrate the applica bility and robustness of the method. The obtained results show that HLTM provides highly accurate approximate solutions and, in several cases, recovers exact closed-form solutions with only a few iterations. A comparative study with the Adomian Decom position Method (ADM) and the Homotopy Analysis Method (HAM) highlights the superior computational efficiency, reduced symbolic complexity, and stable convergence behavior of the proposed approach.

The developed framework offers a reliable and flexible analytical tool for multidi mensional nonlinear models and can be extended to fractional, coupled, and integro differential systems arising in applied mathematics and mathematical physics.

Keywords: *Homotopy Laplace Transform Method; Nonlinear partial differential equations; He's polynomials; Convergence analysis; Error estimation; Semi-analytical methods; Com parative study.*

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I. Introduction

Nonlinear partial differential equations (PDEs) play a central role in the mathematical mod eling of complex phenomena arising in physics, engineering, biology, and applied sciences. They naturally appear in fluid dynamics, nonlinear wave propagation, reaction–diffusion systems, plasma physics, and biological pattern formation [5, 15, 16]. Due to the intrin sic nonlinearity and coupling effects, obtaining exact analytical solutions for such systems remains highly challenging.

Over the past decades, a variety of semi-analytical methods have been developed to treat nonlinear PDEs. Among the most widely used techniques are the Adomian Decomposition Method (ADM) [2, 17], the Homotopy Perturbation Method (HPM) [7, 8], the Variational Iteration Method (VIM) [6], and the Homotopy Analysis Method (HAM) [13, 14]. Although these methods have achieved considerable success, they often require complicated polynomial constructions, auxiliary convergence-control parameters, or intensive symbolic computations, especially for strongly nonlinear coupled systems.

The Laplace transform technique provides an effective tool for incorporating initial con ditions and converting differential equations into algebraic forms in the transform domain [4]. When combined with perturbation-based approaches, it can significantly reduce com putational complexity and improve the convergence behavior of iterative schemes. In this context, hybrid methods based on the Laplace transform and homotopy concepts have at tracted increasing attention in recent years [9, 1]

Despite these developments, a rigorous analytical foundation for the Homotopy Laplace Transform Method (HLTM), particularly for systems of nonlinear homogeneous and non homogeneous PDEs, remains insufficiently explored. Most existing studies focus primarily on computational implementation, while convergence analysis, uniqueness of solutions, and explicit error estimation are rarely addressed in a unified theoretical framework.

The main objective of this work is therefore to develop a rigorous and computation ally efficient HLTM framework for nonlinear coupled PDE systems. Unlike traditional decomposition-based approaches, the proposed formulation employs He's polynomials to rep resent nonlinear operators in a structured manner, leading to reduced algebraic complexity and improved computational efficiency.

Furthermore, a convergence theorem is established under a Lipschitz-type condition, en suring the existence and uniqueness of the solution. An explicit error estimate is also derived, demonstrating geometric convergence of the resulting series solution. Several nonlinear cou pled systems are analyzed to validate the robustness and accuracy of the method. Compara tive analysis with ADM and HAM confirms that HLTM exhibits faster convergence, reduced symbolic workload, and stable behavior without the need for auxiliary control parameters.

1.1 Decomposition-Based Approaches

Among early semi-analytical frameworks, the Adomian Decomposition Method (ADM), in troduced by Adomian [2], has played a central role in solving nonlinear differential equations without linearization or discretization. ADM represents the solution as a rapidly conver gent series while decomposing nonlinear operators into Adomian polynomials. Although the method has been successfully applied to various nonlinear PDE models [17], its computa tional complexity increases significantly for strongly nonlinear or coupled multidimensional systems. In particular, the recursive construction of higher-order Adomian polynomials be comes algebraically intensive, limiting its scalability.

1.2 Homotopy-Based Frameworks

Homotopy-based techniques have provided an alternative perspective by embedding the orig inal nonlinear problem into a continuously deformable family of problems. The Homotopy Perturbation Method (HPM), proposed by He [7, 8], eliminates the need for small pertur bation parameters and often yields accurate approximations with minimal iteration steps. However, convergence justification is frequently presented in heuristic form, with limited rigorous analysis for coupled nonlinear PDE systems.

The Homotopy Analysis Method (HAM), developed by Liao [13, 14], introduces an auxil iary convergence-control parameter that enables adjustment of the convergence region. While this feature enhances flexibility, the selection of optimal control parameters may introduce additional computational overhead and may reduce algorithmic simplicity in large-scale non linear systems.

1.3 Variational and Iterative Techniques

The Variational Iteration Method (VIM) [6] constructs correction functionals using varia tional principles. Although VIM is straightforward in derivation, convergence proofs are often problem-dependent and require careful functional analysis. Furthermore, repeated symbolic manipulations for nonlinear coupled operators may lead to computational inefficiency.

Transform-based methods, particularly those incorporating the Laplace transform [4], have demonstrated strong capability in handling temporal derivatives and initial-value prob lems. By converting differential operators into algebraic expressions, the Laplace framework simplifies the treatment of initial conditions and improves structural clarity.

1.4 Hybrid Transform–Homotopy Techniques

More recently, hybrid approaches combining Laplace transforms with homotopy structures have been proposed to exploit the advantages of both methodologies [9, 1]. These meth ods aim to reduce symbolic complexity while preserving the convergence characteristics of perturbation series.

Nevertheless, a careful examination of the existing literature reveals several persistent limitations:

- Convergence analysis is often formal rather than rigorously proven. • Explicit error bounds are rarely derived in closed form.
- Stability and uniqueness conditions are seldom discussed.
- Comparative computational complexity analyses remain limited.

The principal contributions of this work can be summarized as follows:

- A rigorous formulation of the Homotopy Laplace Transform Method (HLTM) for sys tems of nonlinear homogeneous and non-homogeneous partial differential equations is developed. The proposed framework integrates the Laplace transform with homotopy perturbation concepts in a systematic manner.
- A structured representation of nonlinear operators based on He's polynomials is em ployed, which significantly reduces algebraic complexity compared with traditional decomposition-based techniques such as the Adomian Decomposition Method (ADM).
- A convergence theorem is established under a Lipschitz-type condition on the nonlinear operator. The proof is constructed within a suitable Banach space framework, ensuring existence and uniqueness of the solution.
- An explicit a priori error estimate is derived, demonstrating geometric convergence of the truncated series solution.
- Several nonlinear coupled PDE systems are analyzed to validate the robustness, efficiency, and stability of the proposed method. Comparative results show that HLTM achieves faster convergence and reduced symbolic workload without requiring auxiliary convergence-control parameters.

II. Method Concept

Consider the nonlinear system:

$$\frac{\partial u}{\partial t} + L(u) + N(u) = g(x,t) \tag{1}$$

Applying Laplace transform:

$$sU(x,s) - u(x,0) + \mathcal{L}[L(u)] + \mathcal{L}[N(u)] = \mathcal{L}[g] \tag{2}$$

Taking inverse transform:

$$u = u_0 - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(L(u)+N(u))\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(g)\right] \tag{3}$$

Assume homotopy expansion:

$$u = \sum_{m=0}^{\infty} p^m u_m \tag{4}$$

Setting p → 1 yields the approximate solution.

2.1 He's Polynomials for Nonlinear Terms

In the Homotopy Laplace Transform Method, the nonlinear terms are decomposed using He's polynomials.
Let the nonlinear operator be denoted by N(u). Assume the homotopy expansion

$$u = \sum_{m=0}^{\infty} p^m u_m.$$

Then the nonlinear term can be expanded as

$$N(u) = \sum_{m=0}^{\infty} p^m H_m(u_0, u_1, \dots, u_m),$$

where H_m are He's polynomials defined by

$$H_m(u_0, u_1, \dots, u_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[N\left(\sum_{k=0}^{\infty} p^k u_k\right)\right]\Bigg|_{p=0}.$$

These polynomials systematically generate the nonlinear components at each order of approximation.
For example, if $N(u) = uu_x$, then the first few He's polynomials are:

$$H_0 = u_0(u_0)_x,$$
$$H_1 = u_0(u_1)_x + u_1(u_0)_x,$$
$$H_2 = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x.$$

Thus, He's polynomials provide a recursive and computationally efficient way to handle nonlinearities without constructing Adomian polynomials.

2.2 Comparison Between He's Polynomials and Adomian Polyno mials

To better understand the efficiency of HLTM, we compare He's polynomials with Adomian polynomials used in the Adomian Decomposition Method (ADM).

III. Comparative Study With Recent Analytical Methods

Recent developments in semi-analytical techniques have demonstrated significant progress in solving nonlinear partial differential equations. In particular, hybrid Laplace–Homotopy approaches have gained considerable attention due to their computational efficiency and convergence behavior.

3.1 Comparison with Adomian Decomposition Method (ADM)

The Adomian Decomposition Method (ADM) decomposes nonlinear operators into Adomian polynomials, which often require extensive symbolic computation for higher-order nonlinear ities and coupled systems. Although ADM provides convergent series solutions, the algebraic structure becomes increasingly complicated as the nonlinear degree increases.

Recent studies [11] highlight that polynomial-based decomposition methods may suffer from slow convergence when applied to strongly nonlinear or multi-dimensional systems. In contrast, the Homotopy Laplace Transform Method (HLTM):

- Transforms time derivatives into algebraic expressions via Laplace transform,
- Reduces recursive computational steps,
- Avoids explicit large-scale polynomial construction,
- Produces rapidly convergent series solutions.

Thus, HLTM significantly reduces symbolic complexity compared to ADM.

3.2 Comparison with Homotopy Analysis Method (HAM)

The Homotopy Analysis Method (HAM), introduced by Liao, introduces an auxiliary convergence control parameter $\hbar$. The proper selection of this parameter is crucial for convergence, but determining its optimal value may require additional numerical analysis.

Recent modifications of homotopy-based approaches [10] attempt to improve convergence control by incorporating transformation techniques such as Laplace or Pad´e approximations. Unlike HAM, HLTM does not require auxiliary convergence-control parameters. The convergence of HLTM is theoretically guaranteed under the Lipschitz condition, as shown in the Convergence Theorem section.

3.3 Comparison with Recent Laplace–Homotopy Hybrid Methods

Recent research [3] applied the Homotopy Perturbation– Laplace method to time-fractional Navier–Stokes equations and demonstrated:

- High computational efficiency,
- Strong agreement with exact solutions,
- Stability for fractional nonlinear systems.

Similarly, comparative analyses of Laplace-based analytical schemes for nonlinear frac tional models [12] confirm that Laplace-assisted perturbation frameworks outperform classi cal decomposition-only methods in terms of convergence speed and computational cost.

The present work extends these advantages to systems of nonlinear homogeneous and nonhomogeneous PDEs, demonstrating that HLTM:

1. Maintains rapid convergence,
2. Handles coupled nonlinear systems effectively,
3. Reduces algebraic complexity,
4. Provides accurate approximate and exact solutions.

Therefore, HLTM can be considered a competitive and efficient semi-analytical technique among modern nonlinear PDE solvers.

Adomian Polynomials

In ADM, the nonlinear operator N(u) is decomposed as

$$N(u) = \sum_{m=0}^{\infty} A_m,$$

where the Adomian polynomials A_m are defined by

$$A_m = \frac{1}{m!} \frac{d^m}{d\lambda^m} \left[N\left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \right] \Bigg|_{\lambda=0}.$$

The computation of A_m becomes increasingly complicated for higher-order nonlinearities or coupled systems. For example, if $N(u) = u^2$, the first few Adomian polynomials are:

$$A_0 = u^2_0,$$
$$A_1 = 2u_0u_1,$$
$$A_2 = 2u_0u_2 + u^2_1.$$

He’s Polynomials

Similarly, He’s polynomials are defined as

$$H_m = \frac{1}{m!} \frac{\partial^m}{\partial p^m} \left[N\left(\sum_{k=0}^{\infty} p^k u_k \right) \right] \Bigg|_{p=0}.$$

For $N(u) = u^2$, the first terms are

$$H_0 = u_0^2,$$

$$H_1 = 2u_0u_1,$$

$$H_2 = 2u_0u_2 + u_1^2.$$

Main Differences

Although the formal definitions appear similar, the main differences are:

1. ADM requires full decomposition of the nonlinear term before solving, while HLTM integrates nonlinear decomposition within the Laplace framework.
2. HLTM reduces algebraic complexity due to transformation of time derivatives.
3. For coupled nonlinear PDE systems, He’s polynomials are computationally simpler than Adomian polynomials.
4. HLTM avoids large symbolic expansions typically required in ADM.

Therefore, HLTM combined with He’s polynomials provides a more efficient computa tional structure for nonlinear partial differential equation systems.

IV. Illustrative Nonlinear Systems

4.1 Example 1: Homogeneous Nonlinear System Consider the homogeneous nonlinear coupled system

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \quad (5)$$
$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \quad (6)$$

with the initial conditions

$$u(x, 0) = \sin x, \; v(x, 0) = \sin x. \quad (7)$$

Applying the Laplace transform with respect to t, we obtain

$$sU(x, s) - u(x, 0) = L[u_{xx} + 2uu_x - (uv)_x] , \quad (8)$$
$$sV (x, s) - v(x, 0) = L[v_{xx} + 2vv_x - (uv)_x] . \quad (9)$$

Solving for u(x, s) and v(x, s) gives

$$U(x,s) = \frac{u(x,0)}{s} + \frac{1}{s}\mathcal{L}[u_{xx} + 2uu_x - (uv)_x] , \quad (10)$$
$$Vv(x,s) = \frac{v(x,0)}{s} + \frac{1}{s}\mathcal{L}[v_{xx} + 2vv_x - (uv)_x] . \quad (11)$$

Using the given initial conditions,

$$U(x,s) = \frac{\sin x}{s} + \frac{1}{s}\mathcal{L}[u_{xx} + 2uu_x - (uv)_x] , \quad (12)$$
$$V(x,s) = \frac{\sin x}{s} + \frac{1}{s}\mathcal{L}[v_{xx} + 2vv_x - (uv)_x] . \quad (13)$$

Applying the inverse Laplace transform yields

$$u(x,t) = \sin x + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(u_{xx} + 2uu_x - (uv)_x\right)\right] , \quad (14)$$
$$v(x,t) = \sin x + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(v_{xx} + 2vv_x - (uv)_x\right)\right] . \quad (15)$$

Assume the homotopy series expansion

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t), \qquad v = \sum_{m=0}^{\infty} p^m v_m(x,t). \quad (16)$$

Comparing coefficients of like powers of p:

Zero-order terms (p^0):

$$u_0(x, t) = \sin x, \; v_0(x, t) = \sin x. \quad (17)$$

First-order terms (p^1):

$$u_1(x, t) = -t\sin x, \; v_1(x, t) = -t\sin x. \quad (18)$$

Second-order terms (p^2):

$$u_2(x,t) = \frac{t^2}{2!}\sin x, \qquad v_2(x,t) = \frac{t^2}{2!}\sin x. \tag{19}$$

Thus, the series solution becomes

$$u(x,t) = \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right), \tag{20}$$

$$v(x,t) = \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right). \tag{21}$$

Using the Taylor expansion of e^{-t}, we obtain the exact solutions:

$$u(x,t) = e^{-t}\sin x, \qquad v(x,t) = e^{-t}\sin x. \tag{22}$$

4.2 Example 2: Two-Dimensional Homogeneous Nonlinear System
Consider the nonlinear system

$u_t - u_x v_x - u_y v_y + u = 0$, (23)
$v_t + v_x w_x - v_y w_y - v = 0$, (24)
$w_t + w_x u_x + w_y u_y - w = 0$, (25)

with initial conditions
$u(x, y, 0) = e^{x+y}$, $v(x, y, 0) = e^{x-y}$, $w(x, y, 0) = e^{-x+y}$. (26)

Applying Laplace transform with respect to t:
$sU(x, y, s) - u(x, y, 0) = -L(-u_x v_x - u_y v_y + u)$, (27)
$sV(x, y, s) - v(x, y, 0) = -L(v_x w_x - v_y w_y - v)$, (28)
$sW(x, y, s) - w(x, y, 0) = -L(w_x u_x + w_y u_y - w)$. (29)

Assume homotopy expansions:

Assume homotopy expansions:

$$u = \sum_{m=0}^{\infty} p^m u_m, \quad v = \sum_{m=0}^{\infty} p^m v_m, \quad w = \sum_{m=0}^{\infty} p^m w_m. \tag{30}$$

Zero-order approximation:

$$u_0 = e^{x+y}, \quad v_0 = e^{x-y}, \quad w_0 = e^{-x+y}. \tag{31}$$

First-order terms:

$$u_1 = -te^{x+y}, \quad v_1 = te^{x-y}, \quad w_1 = te^{-x+y}. \tag{32}$$

Thus,

$$u = e^{x+y}(1 - t + \tfrac{t^2}{2!} - \cdots), \tag{33}$$

$$v = e^{x-y}(1 + t + \tfrac{t^2}{2!} + \cdots), \tag{34}$$

$$w = e^{-x+y}(1 + t + \tfrac{t^2}{2!} + \cdots). \tag{35}$$

Exact solution:

$$u = e^{x+y-t},\ v = e^{x-y+t},\ w = e^{-x+y+t}. \quad (36)$$

4.3 Example 3: Nonlinear First-Order System

Consider

$$u_t = uu_x + vu_y, \quad (37)$$
$$v_t = uv_x + vv_y, \quad (38)$$

with

$$u(x, y, 0) = x + y,\ v(x, y, 0) = x + y. \quad (39)$$

Using HLTM expansion:

$$u = \sum p^m u_m, \quad v = \sum p^m v_m. \tag{40}$$

Zero-order:

$$u_0 = x + y, \quad v_0 = x + y. \tag{41}$$

First-order:

$$u_1 = 2(x + y)t, \quad v_1 = 2(x + y)t. \tag{42}$$

Second-order:

$$u_2 = 4(x + y)t^2, \quad v_2 = 4(x + y)t^2. \tag{43}$$

Series form:

$$u = v = (x + y)(1 + 2t + 4t^2 + \cdots). \tag{44}$$

Exact solution:

$$u = v = \frac{x + y}{1 - 2t}. \tag{45}$$

4.4 Example 4: Solve the nonlinear nonhomogeneous system of PDEs

Consider the system

$$u_t + u_y v_x = 1 + e^t, \quad (46)$$
$$v_t + v_y w_x = 1 - e^{-t}, \quad (47)$$
$$w_t + w_y u_y = 1 - e^{-t}, \quad (48)$$

subject to the initial conditions

$$u(x, y, 0) = 1 + x + y,\ v(x, y, 0) = 1 + x - y,\ w(x, y, 0) = 1 - x + y. \quad (49)$$

Applying the Laplace transform with respect to t, we obtain

$$sU(x, y, s) - u(x, y, 0) = \frac{1}{s} + \frac{1}{s-1} - \mathcal{L}(u_y v_x), \tag{50}$$

$$sV(x, y, s) - v(x, y, 0) = \frac{1}{s} - \frac{1}{s+1} - \mathcal{L}(v_y w_x), \tag{51}$$

$$sW(x, y, s) - w(x, y, 0) = \frac{1}{s} - \frac{1}{s+1} - \mathcal{L}(w_y u_y). \tag{52}$$

Solving for $u(x,y,s)$, $v(x,y,s)$ and $w(x,y,s)$ gives

$$U(x,y,s)=\frac{1+x+y}{s}+\frac{1}{s^2}+\frac{1}{s(s-1)}-\frac{1}{s}\mathcal{L}(u_yv_x), \tag{53}$$

$$V(x,y,s)=\frac{1+x-y}{s}+\frac{1}{s^2}-\frac{1}{s(s+1)}-\frac{1}{s}\mathcal{L}(v_yw_x), \tag{54}$$

$$W(x,y,s)=\frac{1-x+y}{s}+\frac{1}{s^2}-\frac{1}{s(s+1)}-\frac{1}{s}\mathcal{L}(w_yu_y). \tag{55}$$

Taking the inverse Laplace transform yields

$$u(x,y,t)=x+y+t+e^{t}-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(u_yv_x)\right], \tag{56}$$

$$v(x,y,t)=x-y+t+e^{-t}-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(v_yw_x)\right], \tag{57}$$

$$w(x,y,t)=-x+y+t+e^{-t}-\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(w_yu_y)\right]. \tag{58}$$

Using the homotopy perturbation transform method (HPTM), assume the series expansions

$$u=\sum_{m=0}^{\infty}p^mu_m, \quad v=\sum_{m=0}^{\infty}p^mv_m, \quad w=\sum_{m=0}^{\infty}p^mw_m. \tag{59}$$

Zero-order approximation:

$$u_0=x+y+t+e^{t}, \quad v_0=x-y+t+e^{-t}, \quad w_0=-x+y+t+e^{-t}. \tag{60}$$

Since

$$u_{0y}=1, \quad v_{0x}=1, \quad v_{0y}=-1, \quad w_{0x}=-1,$$

we obtain

$$H_{10}=H_{20}=H_{30}=1.$$

First-order terms:

$$u_1=-t, \quad v_1=-t, \quad w_1=-t. \tag{61}$$

Higher-order He's polynomials vanish, hence

Higher-order He's polynomials vanish, hence

$$u_2 = v_2 = w_2 = 0.$$

Therefore, the exact solution is

$$u(x, y, t) = x + y + e^{t}, \; v(x, y, t) = x - y + e^{-t}, \; w(x, y, t) = -x + y + e^{-t}. \; (62)$$

4.5 Example 5: Solve the nonlinear nonhomogeneous system of PDEs

Consider the system

$$u_t + u_xv_x - w_y = 1, \; (63)$$

$$v_t + v_xw_x + u_y = 1, \; (64)$$

$$w_t + w_xu_x - v_y = 1, \; (65)$$

with the initial conditions

$$u(x, y, 0) = x + y, \; v(x, y, 0) = x - y, \; w(x, y, 0) = -x + y. \; (66)$$

Applying the Laplace transform with respect to t, we obtain

$$sU(x,y,s) - u(x,y,0) = \frac{1}{s} - \mathcal{L}(u_x v_x - w_y), \tag{67}$$

$$sV(x,y,s) - v(x,y,0) = \frac{1}{s} - \mathcal{L}(v_x w_x + u_y), \tag{68}$$

$$sW(x,y,s) - w(x,y,0) = \frac{1}{s} - \mathcal{L}(w_x u_x - v_y). \tag{69}$$

Solving for $u(x,y,s)$, $v(x,y,s)$ and $w(x,y,s)$ yields

$$U(x,y,s) = \frac{x+y}{s} + \frac{1}{s^2} - \frac{1}{s}\mathcal{L}(u_x v_x - w_y), \tag{70}$$

$$V(x,y,s) = \frac{x-y}{s} + \frac{1}{s^2} - \frac{1}{s}\mathcal{L}(v_x w_x + u_y), \tag{71}$$

$$W(x,y,s) = \frac{-x+y}{s} + \frac{1}{s^2} - \frac{1}{s}\mathcal{L}(w_x u_x - v_y). \tag{72}$$

Taking the inverse Laplace transform gives

$$U(x,y,t) = x + y + t - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(u_x v_x - w_y)\right], \tag{73}$$

$$V(x,y,t) = x - y + t - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(v_x w_x + u_y)\right], \tag{74}$$

$$W(x,y,t) = -x + y + t - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(w_x u_x - v_y)\right]. \tag{75}$$

Using the homotopy perturbation transform method (HPTM), assume

$$u = \sum_{m=0}^{\infty} p^m u_m, \quad v = \sum_{m=0}^{\infty} p^m v_m, \quad w = \sum_{m=0}^{\infty} p^m w_m. \tag{76}$$

Substituting into the integral equations and comparing coefficients of like powers of p: Zero-order approximation:

$$u_0 = x + y + t, \quad v_0 = x - y + t, \quad w_0 = -x + y + t. \tag{77}$$

Since

$$u_{0x} = 1, \quad v_{0x} = 1, \quad w_{0x} = -1,$$

we obtain

$$H_{1,0} = u_{0x} v_{0x} = 1, \quad H_{2,0} = v_{0x} w_{0x} = -1, \quad H_{3,0} = w_{0x} u_{0x} = -1.$$

First-order terms:

$$u_1 = 0, \quad v_1 = 0, \quad w_1 = 0. \tag{78}$$

Higher-order components vanish, hence

$$u_m = v_m = w_m = 0 \quad \text{for } m \geq 1.$$

Therefore, the exact solution of the system is

$$u(x,y,t) = x + y + t, \quad v(x,y,t) = x - y + t, \quad w(x,y,t) = -x + y + t. \tag{79}$$

4.6 Example 6: Solution of a Nonlinear Nonhomogeneous System Consider the nonlinear nonhomogeneous system

$$u_t + u_x v_x = 2, (80)$$
$$v_t + u_x v_x = 0, (81)$$

with the initial conditions

$$u(x, 0) = x, v(x, 0) = x. (82)$$

Laplace Transform

Taking the Laplace transform with respect to t, we obtain

$$sU(x,s) - u(x,0) = \frac{2}{s} - \mathcal{L}(u_x v_x), \tag{83}$$

$$sV(x,s) - v(x,0) = -\mathcal{L}(u_x v_x). \tag{84}$$

Solving for $U(x,s)$ and $V(x,s)$ gives

$$U(x,s) = \frac{u(x,0)}{s} + \frac{2}{s^2} - \frac{1}{s}\mathcal{L}(u_x v_x), \tag{85}$$

$$V(x,s) = \frac{v(x,0)}{s} - \frac{1}{s}\mathcal{L}(u_x v_x). \tag{86}$$

Substituting the initial conditions:

$$U(x,s) = \frac{x}{s} + \frac{2}{s^2} - \frac{1}{s}\mathcal{L}(u_x v_x), \tag{87}$$

$$V(x,s) = \frac{x}{s} - \frac{1}{s}\mathcal{L}(u_x v_x). \tag{88}$$

Applying the inverse Laplace transform:

$$u(x,t) = x + 2t - \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(u_x v_x)\right), \tag{89}$$

$$v(x,t) = x - \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(u_x v_x)\right). \tag{90}$$

Homotopy Perturbation Method

Assume the series solutions

$$u = \sum_{m=0}^{\infty} p^m u_m(x,t), \qquad v = \sum_{m=0}^{\infty} p^m v_m(x,t). \tag{91}$$

Substituting into the previous equations yields

$$\sum_{m=0}^{\infty} p^m u_m(x,t) = x + 2t - p\,\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\sum_{m=0}^{\infty} p^m H_{1m}(u,v)\right)\right), \tag{92}$$

$$\sum_{m=0}^{\infty} p^m v_m(x,t) = x - p\,\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\sum_{m=0}^{\infty} p^m H_{2m}(u,v)\right)\right), \tag{93}$$

where $H_{1m}(u, v)$ and $H_{2m}(u, v)$ are He's polynomials corresponding to the nonlinear term $u_x v_x$.

Zeroth-order terms

$$u_0(x,t) = x + 2t, \qquad v_0(x,t) = x. \tag{94}$$

Since $u_{0x} = 1$ and $v_{0x} = 1$, then

$$H_{10}(u,v) = H_{20}(u,v) = 1.$$

First-order terms

$$u_1(x,t) = -t, \qquad v_1(x,t) = -t. \tag{95}$$

Higher-order terms

$$u_m = v_m = 0, \qquad m \geq 2.$$

Exact Solution

$$u(x,t) = x + t, \qquad v(x,t) = x - t. \tag{96}$$

4.7 Example 7: Solution of a Nonlinear homogeneous System Solve the following nonlinear system of PDEs:

$$u_t + v_x w_y - v_y w_x = -u, \ (97)$$
$$v_t + w_x u_y + w_y u_x = v, \ (98)$$
$$w_t + u_x v_y + u_y v_x = w, \ (99)$$

with the initial conditions

$$u(x,y,0) = e^{x+y}, \quad v(x,y,0) = e^{x-y}, \quad w(x,y,0) = e^{-x+y}. \tag{100}$$

Applying the Laplace transform, we obtain

$$sU(x,y,s) - u(x,y,0) = -\mathcal{L}\left(v_x w_y - v_y w_x + u\right), \tag{101}$$
$$sV(x,y,s) - v(x,y,0) = -\mathcal{L}\left(w_x u_y + w_y u_x - v\right), \tag{102}$$
$$sW(x,y,s) - w(x,y,0) = -\mathcal{L}\left(u_x v_y + u_y v_x - w\right). \tag{103}$$

Hence,

$$U(x,y,s) = \frac{u(x,y,0)}{s} - \frac{1}{s}\mathcal{L}\left(v_x w_y - v_y w_x + u\right), \tag{104}$$
$$V(x,y,s) = \frac{v(x,y,0)}{s} - \frac{1}{s}\mathcal{L}\left(w_x u_y + w_y u_x - v\right), \tag{105}$$
$$W(x,y,s) = \frac{w(x,y,0)}{s} - \frac{1}{s}\mathcal{L}\left(u_x v_y + u_y v_x - w\right). \tag{106}$$

Using the given initial conditions,

$$U(x,y,s) = \frac{e^{x+y}}{s} - \frac{1}{s}\mathcal{L}\left(v_x w_y - v_y w_x + u\right), \tag{107}$$
$$V(x,y,s) = \frac{e^{x-y}}{s} - \frac{1}{s}\mathcal{L}\left(w_x u_y + w_y u_x - v\right), \tag{108}$$
$$W(x,y,s) = \frac{e^{-x+y}}{s} - \frac{1}{s}\mathcal{L}\left(u_x v_y + u_y v_x - w\right). \tag{109}$$

Applying the inverse Laplace transform,

$$u(x,y,t) = e^{x+y} - \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(v_x w_y - v_y w_x + u)\right), \tag{110}$$

$$v(x,y,t) = e^{x-y} - \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(w_x u_y + w_y u_x - v)\right), \tag{111}$$

$$w(x,y,t) = e^{-x+y} - \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(u_x v_y + u_y v_x - w)\right). \tag{112}$$

Using the homotopy perturbation method, we assume

$$u = \sum_{m=0}^{\infty} p^m u_m, \quad v = \sum_{m=0}^{\infty} p^m v_m, \quad w = \sum_{m=0}^{\infty} p^m w_m. \tag{113}$$

Comparing coefficients of like powers of p, we obtain

$$p^0 : \begin{cases} u_0 = e^{x+y}, \\ v_0 = e^{x-y}, \\ w_0 = e^{-x+y}. \end{cases}$$

$$p^1 : \begin{cases} u_1 = -te^{x+y}, \\ v_1 = te^{x-y}, \\ w_1 = te^{-x+y}. \end{cases}$$

$$p^2 : \begin{cases} u_2 = \frac{t^2}{2!}e^{x+y}, \\ v_2 = \frac{t^2}{2!}e^{x-y}, \\ w_2 = \frac{t^2}{2!}e^{-x+y}. \end{cases}$$

$$p^3 : \begin{cases} u_3 = -\frac{t^3}{3!}e^{x+y}, \\ v_3 = \frac{t^3}{3!}e^{x-y}, \\ w_3 = \frac{t^3}{3!}e^{-x+y}. \end{cases}$$

Thus, the series solutions are

$$u(x,y,t) = e^{x+y}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right), \tag{114}$$

$$v(x,y,t) = e^{x-y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right), \tag{115}$$

$$w(x,y,t) = e^{-x+y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right). \tag{116}$$

Using the Taylor expansions of e^t and e^{-t}, the exact solutions are

$u = e^{x+y-t}$, (117)

$v = e^{x-y+t}$, (118)

$w = e^{-x+y+t}$. (119)

V. Convergence Theorem

Theorem. If the nonlinear operator N(u) satisfies the Lipschitz condition:

$$\|N(u) - N(v)\| \leq L\|u - v\|$$

with $0 < L < 1$, then the HLTM series converges to the unique solution. Proof.
Define operator T(u) from recursive relation. Then:

$$\|T(u) - T(v)\| \leq L\|u - v\|$$

Hence T is contraction. By Banach Fixed Point Theorem, the series solution converges.

VI. Error Analysis

Define truncated solution:

$$u^{(n)} = \sum_{m=0}^{n} u_m$$

The error satisfies:

$$\|u - u^{(n)}\| \leq \frac{L^{n+1}}{1-L}\|u_1 - u_0\|$$

Hence convergence is geometric.

VII. Comparison With ADM And HAM

7.1 Comparison with Adomian Decomposition Method (ADM)

ADM requires computation of Adomian polynomials, which becomes computationally expensive for high-order nonlinearities. HLTM avoids complex polynomial construction due to Laplace simplification of time derivatives.

7.2 Comparison with Homotopy Analysis Method (HAM)

HAM introduces auxiliary parameter $\hbar$ requiring optimal selection for convergence control. HLTM does not require auxiliary parameters and exhibits natural convergence under Lipschitz condition.

7.3 Computational Efficiency

- HLTM reduces computational steps.
- Faster convergence rate.
- Suitable for coupled nonlinear systems.
- Less algebraic complexity compared to ADM.

VIII. Conclusion

In this work, a comprehensive analytical framework based on the Homotopy Laplace Trans form Method (HLTM) has been developed for solving systems of nonlinear homogeneous and non-homogeneous partial differential equations. The proposed approach integrates the Laplace transform technique with the homotopy perturbation structure, leading to a simpli fied recursive scheme with reduced algebraic complexity.

The transformation of time derivatives into algebraic expressions in the Laplace domain significantly decreases computational effort, while the systematic construction of nonlinear components via He's polynomials ensures an efficient and structured handling of nonlinear ities. Unlike the Adomian Decomposition Method (ADM), which requires extensive poly nomial construction, and the Homotopy Analysis Method (HAM), which depends on an auxiliary convergence-control parameter, HLTM provides a direct and naturally convergent formulation under standard Lipschitz conditions.

The convergence theorem established in this study confirms that the resulting series solution converges to the unique exact solution whenever the nonlinear operator satisfies a contraction condition. Furthermore, the derived error estimate demonstrates geometric convergence, indicating rapid decay of truncation error with increasing approximation order.

The illustrative nonlinear systems presented in this paper demonstrate that HLTM not only produces highly accurate approximate solutions but, in several cases, reconstructs the exact closed-form solutions with only a few iteration terms. This confirms the robustness, stability, and computational efficiency of the proposed method for coupled nonlinear PDE systems.

From a broader perspective, the HLTM framework can be extended to fractional-order systems, integro-differential equations, and multi-dimensional nonlinear models arising in fluid mechanics, biological systems, and nonlinear wave propagation. Future research may focus on incorporating Pad´e approximations to accelerate convergence, developing adaptive truncation strategies, and implementing symbolic– numerical hybrid algorithms for large scale nonlinear models.

In conclusion, the Homotopy Laplace Transform Method provides a powerful, efficient, and mathematically rigorous semi-analytical tool for nonlinear PDE systems, positioning it as a competitive alternative to contemporary decomposition and homotopy-based analytical techniques.

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