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Research Paper

Fixed point theorem for (ϕ, F) – contraction on C^*- algebra valued b- metric space

Neetu, Ranbir Singh*

Department of Mathematics, Baba Mastnath University, Rohtak-124021, Haryana
*(*Corresponding Author)*

Abstract
In this paper, we establish a new fixed point theorem for (ϕ, F) contractive mappings in the setting of C^*- algebra valued b-metric spaces. We prove the existence and uniqueness of fixed points for such mappings in complete algebra valued b-metric spaces. A Banach-type corollary is derived as a special case. Furthermore, an illustrative example is presented to verify the validity of the main theorem. As an application, we establish the existence of a solution to a nonlinear integral equation using the proposed contraction condition.
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I. Introduction

Fixed point theory plays an important role in nonlinear analysis, differential equations, optimization, and mathematical modelling. The classical Banach [1] Contraction Principle established the foundation for modern fixed point theory and has been extended in many directions.

Recently, generalized contraction mappings have been studied in various generalized metric spaces such as [6,8] b-metric spaces , partial metric space and C^*- algebra valued metric spaces.

The concept of F-contractions introduced by Dariusz Wardowski [17] provided a powerful tool for extending classical contraction principles.

On the other hand , the framework of C^*- algebra valued metric spaces allows the distance function to take values in a C-algebra instead of real numbers. This approach has attracted considerable attention in functional analysis.

Motivated by these developments , in this paper we introduce a new contraction condition in C^*- algebra valued b- metric space and establish a unique fixed point theorem.

In 2015, Ma and Jiang [3] established the notion of C^*- algebra valued b- metric space and proved some fixed point theorem for contractive type mappings.

II. Preliminaries

In this section, we shall give some basic definition which will be used in sequel.

Throughout this paper, we suppose that $\mathbb{A}$ is a unital C^*- algebra with a unit I_A. Set $\mathbb{A}_h = \{x \in \mathbb{A}: x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succcurlyeq \theta$. Using positive elements, one can define a partial ordering $\preccurlyeq$ on $\mathbb{A}_h$ as follows: $x \preccurlyeq y$ if and only if $y - x \succcurlyeq \theta$, where θ means the zero element in $\mathbb{A}$. Now $\mathbb{A}_+ = \{x \in \mathbb{A} : x \succcurlyeq \theta\}$ and $|x| = (x^*, x)^{\frac{1}{2}}$.

Definition 2.1.[9] Let X be a non-empty set $s \succcurlyeq I_A$. Suppose the mapping d: $X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preccurlyeq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.
Then d is called C^* - algebra valued b- metric on X and $(X, \mathbb{A}, d)$ is called C^*- algebra valued b- metric space.

Definition 2.2.[9] Let $(X, \mathbb{A}, d)$ be a C^*- algebra valued b- metric space. Let $\{x_n\}$ be a sequence in X then

(i) $\{x_n\}$ is said to be Cauchy if for all $\theta \preccurlyeq c$, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$

$d(x_n, x_m)$

(ii) $\{x_n\}$ is said to be converges to x if for all $\theta \preccurlyeq c$ there is $N \in \mathbb{N}$ such that for all $n \geq N$

(iii) $(X, \mathbb{A}, d)$ is a complete C^*- algebra valued b- metric space if every Cauchy sequence is convergent in X.

Definition2.3[11] Let $\mathcal{F}$ be the family of all functions $F: (0, \infty) \to \mathbb{R}$ and Φ be the family of all the functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

(i) F is strictly increasing
(ii) $F(t) \to -\infty$ as $t \to 0^+$
(iii) $\phi: [0, \infty) \to [0, \infty)$ be a control function such that
$\phi(t) < t$ for $t > 0$.

Definition 2.4[11]Let (X, d) be a complete metric space. A mapping T: $X \to X$ is called a (ϕ, F) – contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\phi \in \Phi$ such that

$$F(d(Tx, Ty)) \leq F(\phi(d(x, y)))$$

for all $x, y \in X$, $Tx \neq Ty$:

III. Main Result

Definition3.1 Let (X, d)be a C^*-algebra valued b-metric space and $T: X \to X$.

The mapping $T$is called a (ϕ, F) contraction if

$$F(\| d(Tx, Ty) \|) \preccurlyeq F(\phi(\| d(x, y) \|)) - \tau$$

for all $x, y \in X$, $Tx \neq Ty$, where

(i) $F: (0, \infty) \to \mathbb{R}$ is strictly increasing and

$$\lim_{t \to 0^+} F(t) = -\infty$$

(ii) $\tau > 0$

(iii) $\phi: [0, \infty) \to [0, \infty)$with

$$\phi(t) < t(t > 0)$$

Theorem3.2 Let (X, d)be a complete C^* −algebra valued b-metric space with coefficient $s \geq 1$ over a unital C^* − algebra A.

Let $T: X \to X$ be a (ϕ, F) C^* − algebra valued b- contraction mapping.

Then the mapping T has a unique fixed point $x^* \in X$.

Proof Let $x_0 \in X$ be arbitrary and define the sequence

$$x_{n+1} = Tx_n , n \geq 0$$

If for some n , $x_n = x_{n+1}$, then $x_n$is a fixed point and the proof is complete.

Assume $x_n \neq x_{n+1}$ for all n.

Using the (ϕ, F) contraction condition,

$$F(\| d(x_{n+1}, x_{n+2}) \|) = F(\| d(Tx_n, Tx_{n+1}) \|)$$
$$\preccurlyeq F(\phi(\| d(x_n, x_{n+1}) \|)) - \tau$$

Since $F$is increasing and $\phi(t) < t$,

$$\| d(x_{n+1}, x_{n+2}) \| < \| d(x_n, x_{n+1}) \|$$

Thus the sequence

$$\{\| d(x_n, x_{n+1}) \|\}$$

is decreasing and bounded below by 0. Hence it converges.

Next we prove $\{x_n\}$is a Cauchy sequence.

Using the b-metric inequality

$$d(x_n, x_m) \preccurlyeq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)]$$
for all $m > n$, where $s \geq 1$.

Taking norms on both sides:

$$\|d(x_n, x_m)\| \preccurlyeq s \sum_{k=n}^{m-1} \|d(x_k, x_{k+1})\|.$$

We know that

$$\| d(x_k, x_{k+1}) \| \to 0$$

as $k \to \infty$.

Thus for any $\varepsilon > 0$, there exist N such that

$\| d(x_k, x_{k+1}) \| < \frac{\varepsilon}{s}$ for all $k \geq N$.

For $m > n \geq N$,

$$\|d(x_n, x_m)\| \preccurlyeq s \sum_{k=n}^{m-1} \|d(x_k, x_{k+1})\| \prec s \sum_{k=n}^{m-1} \frac{\varepsilon}{s} = (m - n)\varepsilon.$$

Thus

$$\|d(x_n, x_m)\| \to 0 \text{ as } n, m \to \infty.$$

Hence the sequence is a Cauchy sequence.

Since $X$is complete, there exists $x^* \in X$such that

$$x_n \to x^*$$

Now we prove $x^*$is a fixed point.

Since Tsatisfies (ϕ, F) contraction and $x_n \to x^*$,

$$d(Tx^*, x^*) = 0$$

thus

$$Tx^* = x^*$$

Suppose $x^*, y^*$are two fixed points.

Then

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

Applying $\phi - F$ contraction,

$$F(\| d(x^*, y^*) \|) \preccurlyeq F(\phi(\| d(x^*, y^*) \|)) - \tau$$

which is impossible.

Hence

$$x^* = y^*$$

Thus the fixed point is unique.

Example3.3 Let

$$X = \mathbb{R}$$

and let $A = M_2(\mathbb{R})$ be the algebra of 2×2 matrices.

Define

$$d(x, y) = | x - y | I$$

where $I$is the identity matrix.

Then $(X, d)$is a C^*-algebra valued b-metric space.

A mapping $T: X \to X$ given by $Tx = \frac{x}{4}$ is continuous with respect to A.

Let $F : A \to A$. Define by

$$F(x) = \ln x$$

and

$$\phi(x) = \frac{x}{4}$$

Then

$$F(\| d(Tx,Ty) \|) = \ln (\frac{1}{4} \| d(x,y) \|)$$

and

$$F\big(\phi(\|d(x,y)\|)\big) = \mathrm{F}\left(\left(\frac{1}{4} \| d(x,y) \|\right) = \ln (\frac{1}{4} \| d(x,y) \|)\right)$$

Thus

$$F(\| d(Tx,Ty) \|) = F(\phi(\| d(Tx,Ty) \|))$$

For any small $\tau > 0$, (ϕ, F) contraction condition hold

Hence by Theorem 3.2 the mapping Thas a unique fixed point:

Solving $Tx = x$ gives

Which implies

$$\frac{x}{4} = x$$

Thus

$$x = 0$$

Therefore 0 is the unique fixed point.

Corollary3.4[2] Let (X, d)be a complete C^*-algebra valued b-metric space and suppose that for some $k \in (0,1)$

$$\| d(Tx,Ty) \| \leq k \| d(x,y) \|$$

for all $x, y \in X$.

Then T has a unique fixed point in X.

This result follows immediately from Theorem 3.2 by choosing

$$F(t) = \ln t, \phi(t) = kt.$$

Application

Consider the nonlinear integral equation

$$x(t) = \int_a^b K(t,s,x(s))ds$$

where

$K: [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Assume the following conditions hold:

(i) There exists a constant $k > 0$ such that

$$|K(t,s,u) - K(t,s,v)| \leq k|u - v|$$

For all $t, s \in [a,b]$ and $u, v \in \mathbb{R}$.

(ii) The constant satisfies

$$k(b - a) < 1.$$

Then the integral equation admits a unique solution.

Proof : Let

$X = C[a, b]$

be the space of all continuous real-valued functions defined on $[a, b]$.

Define the metric

$$d(x, y) = \|x - y\| I$$

where

$$\|x - y\| = sup_{t \in [a,b]} |x(t) - y(t)|$$

and I is the identity matrix.

Then (X, d) form a C^*-algebra valued b-metric space.

Define operator $T: X \to X$ by

$$(Tx)(t) = \int_a^b K(t, s, x(s)) ds$$

Then solving the integral equation is equivalent to finding a fixed point

Let $x. y \in X$.Then

$$|Tx(t) - Ty(t)| = \left| \int_a^b [K(t, s, x(s)) - \int_a^b K(t, s, y(s))] ds \right|.$$

Using the Lipschitz condition ,

$|Tx(t) - Ty(t)| \leq \left| \int_a^b K(x(s)) - \mathrm{y(s))ds} \right|$.

Since

$$|x(s) - y(s)| \leq \|x - y\|$$

we obtain

$$|Tx(t) - Ty(t)| \leq k(b - a)\|x - y\|.$$

Taking supremum over $t \in [a, b]$ gives

$$\| Tx - Ty \| \leq k(b - a) \| x - y \|.$$

Since

$$k(b - a) < 1$$

We have

$$\|d(Tx, Ty)\| \leq k(b - a)\|d(x, y)\|.$$

Thus Tsatisfies the (ϕ, F) contraction condition.

Hence by Theorem 3.2, the operator Thas a unique fixed point.

Therefore the integral equation admits a unique continuous solution.

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