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Research Paper

A Neumann problem For the Fracture Responses of An Isotropic Elastic Wedge with Free Radial Surfaces and Loaded Circular Surface Under Anti Plane Shear Deformation

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Abstract : *The problem involves analyzing the fracture responses of an isotropic elastic wedge with free radial surfaces and loaded circular surface. The loading gives rise to a two-dimensional Neumann boundary value problem for an equilibrium equation in terms of the displacement function. The problem is formulated using the complex potential theory and the solution of the problem accessed using the Mellin integral transform of the second kind through which a series form of the displacement and the stresses everywhere on the wedge material which defines the fracture fields are obtained .It is found that at the wedge apex,* $\theta = 0\,, r \to 0$ *, the only existing stress component is the tearing stress , which can induce material failure or fracture. The stress fields are used to articulate the order of singularity due to the configuration. Also, stresses are also obtained for various apex angles, which results to different configurations. It is also found that the wedge material remains bonded for a finite fracture fields, while fracture of the wedge material sets in when the fracture fields becomes infinite.*

Key Words: *Fracture Responses, Complex Potential Theory, Mellin Integral Transform, Fracture Fields, Free Loaded Radial Surface, Anti Plane Deformation.*

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I. INTRDUCTION

The deformation fields in elastic wedge materials have been studied in [1,2,3,4,7,8] under anti plane shear deformation with different loading conditions. Analytic solutions for problems for structures like wedges is of high interest in corresponding fracture analysis. Recently, the rate of failure of structural materials has induced the interest of researchers . This may be as a result of the applications of such material in engineering and structural designs. In view of this, wedges are of special interest. Anti plane deformation of a finite isotropic elastic wedge is of theoretical interest because it is a counterpart of in-plane deformation, therefore the interest in the investigation of the fields and the fracture responses for an isotropic elastic wedge material with radius a, apex angle ϕ and finite length in the direction perpendicular to the plane of the wedge, under the condition of anti plane shear deformation. The configuration is subjected to a traction-free radial edges and a circular surface traction loading, T. Fields in this work is a special case of other existing literatures.

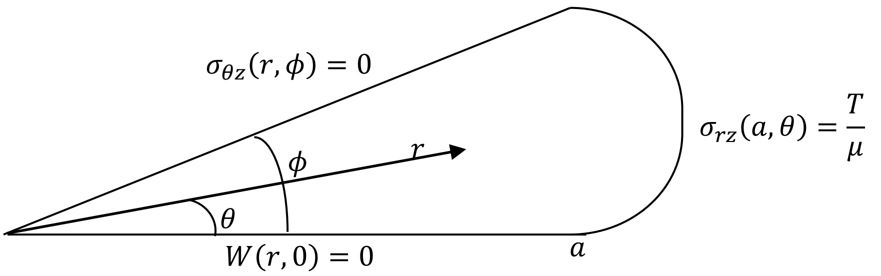

FIGURE 1 (GEOMETRY OF THE PROBLEM)

II. MATERIALS AND METHODS

In this paper, a parameterized elastic finite isotropic wedge structure was considered. The matrix was induced with an anti plane shear deformation, and subjected to free radial surfaces and a loaded circular surface, through which the fracture responses were analyzed. The schematic cross section of the wedge was shown in figure 1.The parameterized of the geometry was performed on its main parameters: stresses, displacements, apex angle, free radial surfaces and loaded circular surface.The methodology of this work is owing to the fact the functions in elastic solids satisfies the equilibrium equation, as well as boundary conditions. The task was implemented as a boundary value problem with the mathematical problem supplemented by boundary conditions. As a result of the nature of the boundary conditions, we adopted the complex potential theory in the formulation of the problem, and the finite Mellin transform of the second kind in conjunction with its inversion formula to obtain the fracture fields. Through these fields, the fracture responses and the condition for strength of stress singularities of the configuration was established.

III. PROBLEM FORMULATION

The governing boundary value problem is formulated in terms of the non vanishing components of the displacement function $W(r,\theta)$ which is a function of the in-plane coordinates , r, θ as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) W(r,\theta) = 0, 0 \le r \le a, 0 \le \theta \le \phi \tag{2.1}$$

The boundary conditions are

$$W(r,0) = 0 \tag{2.2}$$

$$\frac{\partial W(r,\phi)}{\partial \theta} = 0 \tag{2.3}$$

$$\frac{\partial W(a,\theta)}{\partial r} = \frac{T}{\mu} \tag{2.4}$$

where μ is the shear material property.

The non vanishing stress components are $\sigma_{rz}(r,\theta) and\ \sigma_{\theta z}(r,\theta)$ which are related to the displacement gradients through the constitutive equation for an isotropic material undergoing anti plane deformation are

$$\sigma_{r,z}(r,\theta) = \mu \frac{\partial W(r,\theta)}{\partial r} \tag{2.5}$$

$$\sigma_{\theta z}(r,\theta) = \frac{\mu}{r}\frac{\partial W(r,\theta)}{\partial \theta} \tag{2.6}$$

IV. SOLUTION OF THE PROBLEM

It is known [3] that the solution to (2.1) for a finite wedge may be accomplish by means of the finite Mellin transform of the second kind for $W(r,\theta)$ defined by

$$\widetilde{W_1}(s,\theta) = \int_0^a \left(\frac{a^{2s}}{r^{s+1}} + r^{s-1}\right) W(r,\theta) dr \tag{3.1}$$

where s is a complex transform parameter and $\widetilde{W}_1(s,\theta)$ is the Mellin transform of the second kind of $W(r,\theta)$. If $\widetilde{W_1}(s,\theta)$ is found,then $W(r,\theta)$ is obtained by the use of the Mellin inversion formula defined by

$$W(r,\theta) = \frac{(-1)^j}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{W}_1 (s,\theta) r^{-s} ds \tag{3.2}$$

The application of (3.1) to (2.1) yields

$$\frac{d^2}{d\theta^2}\widetilde{W}_1(s,\theta) + s^2\widetilde{W}_1(s,\theta) = -2a^{s+1}\frac{\partial W(a,\theta)}{\partial r} \tag{3.3}$$

provided

$$\lim_{r\to 0}\left[(a^{2s}r^{1-s} + r^{s+1})\frac{\partial W(r,\theta)}{\partial r} + s(r^s - a^{2s}r^s)W(r,\theta)dr\right] = 0 \tag{3.4}$$

The asymptotic behaviors of the fracture fields

$$W(r,\theta) = O\left(r^{1-\lambda_s}\right), 0 < \lambda_s < 1, as\ r \to 0 \tag{3.5}$$

and

$\sigma_{rz}(r,\theta) = \sigma_{\theta z}(r,\theta) = O\left(r^{\lambda_s}\right), 0 < \lambda_s < 1, as\, r \to 0$ (3.6)
are derived utilizing the range of values of c in (3.2) which is the strip of regularity of $W(r,\theta)$ which are otained from (3.4).
Use made of (3.1) and (3.3) yields

$$\left(\frac{d^2}{d\theta^2} + s^2\right)\widetilde{W}_1(s,\theta) = -2a^{s+1}\frac{T}{\mu} \quad (3.7)$$

$$\widetilde{W}_1(s,0) = 0 \quad (3.8)$$

$$\frac{d\theta(s,\phi)}{d\theta} = 0 \quad (3.9)$$

$$\frac{d\widetilde{W}_1}{dr}(a,\theta) = \frac{T}{\mu} \quad (3.10)$$

To solve (3.7), we assume the complementary solution as

$$\widetilde{W}_{1c}(s,\theta) = A(s)sin\theta s + B(s)cos\theta s \quad (3.11)$$

and its particular solution as

$$\widetilde{W}_{1p} = constant, k \quad (3.12)$$

Utilizing (3.7) and (3.12) we get

$$k = -\frac{2a^{s+1}}{s^2}\frac{T}{\mu} \quad (3.13)$$

Then , the general solution of (3.7) is of the form

$$\widetilde{W}_1(s,\theta) = \widetilde{W}_{1c} + \widetilde{W}_{1p} \quad (3.14)$$

The task now is to find the arbitrary constants $A(s)\; and\; B(s)$ in (3.11), which is now in the form

$$\widetilde{W}_1(s,\theta) = A(S)sin\theta s + B(s)cos\theta s - \frac{2a^{s+1}}{s^2}\frac{T}{\mu} \quad (3.15)$$

Employing the boundary conditions on the radial edges of the wedge material, the coefficients in (3.15) are articulated in terms of the other coefficients as

$$B(s) = \frac{2a^{s+1}}{s^2}\frac{T}{\mu} \quad (3.16)$$

also

$$A(s) = B(s)\frac{sin\phi s}{cos\phi s}$$

therefore

$$A(s) = \frac{2a^{s+1}}{s^2}\frac{T}{\mu}\frac{sin\phi s}{cos\phi s} \quad (3.17)$$

In view of these, (3.15) becomes

$$\widetilde{W}_1(s,\theta) = \frac{2a^{s+1}}{s^2}\frac{T}{\mu}\frac{sin\phi s sin\theta s}{cos\phi s} + \frac{2a^{s+1}}{s^2}\frac{T}{\mu}\frac{cos\phi s cos\theta s}{cos\phi s} - \frac{2a^{s+1}}{s^2}\frac{T}{\mu}$$

$$= \frac{2a^{s+1}T}{\mu s^2 cos\phi s}\left[sin\phi s sin\theta s + cos\phi s cos\theta s\right] - \frac{2a^{s+1}}{s^2}\frac{T}{\mu}$$

$$\widetilde{W}_1(s,\theta) = \frac{2a^{s+1}}{\mu s^2 cos\phi s}\cos(\phi-\theta)s - \frac{2a^{s+1}}{s^2}\frac{T}{\mu} \quad (3.18)$$

Substituting (3.18) into (3.2) produces

$$\frac{2aT}{2\pi i\mu}\int_{c-i\infty}^{c+i\infty}\frac{\cos(\phi-\theta)s}{s^2 cos\phi s}\left(\frac{r}{a}\right)^{-s} ds - \frac{2aT}{2\pi i\mu}\int_{c-i\infty}^{c+i\infty}\frac{1}{s^2}\left(\frac{r}{a}\right)^{-s} ds \quad (3.19)$$

The contour integral in (3.19) can be evaluated by Cauchy’s resdue theorem in accordance with Jordan’s lemma. Both integrals are analytic and has no poles on the imaginary axis of the s-plane. The second integral has a double pole at $s = 0$ with a constant residue which does not affect a Neumann boundary value problem, therefore it is dropped. While the first integral has it poles at $cos\phi s = 0 \Rightarrow \cos(2n-1)\frac{\pi}{2\phi}$ at$\phi s_n = (2n-1)\frac{\pi}{2}, n = 1,2,3,\ldots$ and at $s_n = -(2n-1)\frac{\pi}{2\phi} = -s_n, 1,2,3,\ldots$

The geometry of the problem indicates that $r < a$, and therefore requires closure in the left-half plane of the $s - plane$ where $Res < 0$ so that the material will be covered.
Therefore residue theorem gives

$$W(r,\theta) = \frac{4aT}{\mu\pi^2}\phi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\left[\sin(2n-1)\frac{\pi}{2\phi}\sin(2n-1)\frac{\pi\theta}{2\phi}\right]\left(\frac{r}{a}\right)^{(2n-1)\frac{\pi}{2\phi}}, r < a \quad (3.20)$$

Hence the solution sought for is (3.20).

V. SATISFACTION OF BOUNDARY CONDITIONS .

For $r \leq a$

$W(r,0) = 0$

similarly

$\frac{\partial W(r,\phi)}{\partial \theta} = 0$

Having established the validity of the displacements everywhere on the wedge matrial, now the stress distribution can be obtained by utilizing (2.5), (2.6) and (3.20) as

$$\sigma_{rz}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left[\sin\,(2n-1)\frac{\pi}{2\phi}\sin\,(2n-1)\frac{\pi\theta}{2\phi}\right]\left(\frac{r}{a}\right)^{(2n-1)\frac{\pi}{2\phi}-1}, r < a \quad (4.1)$$

$$\sigma_{\theta z}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left[\sin\,(2n-1)\frac{\pi}{2\phi}\cos\,(2n-1)\frac{\pi\theta}{2\phi}\right]\left(\frac{r}{a}\right)^{(2n-1)\frac{\pi}{2\phi}-1}, r < a \quad (4.2)$$

5.0(a) STRESSES AT THE WEDGE TIP REGION

To obtain the stresses about the the wedge tip region, we consider the dominant term of (3.21) and (3.22) as $r \to 0$, which corresponds to $n = 1$ to get

$$\sigma_{rz}(r,\theta) = \frac{2aT}{\pi}\left[sin\frac{\pi}{2\phi}sin\frac{\pi\theta}{2\phi}\right]\left(\frac{r}{a}\right)^{\frac{\pi}{2\phi}-1} \quad (5.1)$$

and

$$\sigma_{\theta z}(r,\theta) = \frac{2aT}{\pi}\left[sin\frac{\pi}{2\phi}cos\frac{\pi\theta}{2\phi}\right]\left(\frac{r}{a}\right)^{\frac{\pi}{2\phi}-1} \quad (5.2)$$

The stresses within the wedge tip region depends on the (r,θ) coordinate ,which is approached as $r \to 0$ and $\theta = 0$ in which case we have from (5.1) and (5.2) as

$$\sigma_{rz}(r,0) = 0 \quad (5.3)$$

$$\sigma_{\theta z}(r,0) = \frac{2aT}{\pi}sin\frac{\pi}{2\phi}\left(\frac{r}{a}\right)^{\frac{\pi}{2\phi}-1}, r < a \quad (5.2)$$

which signifies that as a result of the loading, the only existing stress component at wedge tip region is the tearing stress which can initiate fracture or crack of the wedge material.

5.0(b) STRESSES IN VARIOUS SHAPES CORRESPONDING TO PARTICULAR VALUES OF ϕ

i) $\phi = \frac{1}{2}$ (ii) $\phi = \frac{3}{4} < \phi < 1$ (iii) $\phi = 1$

The case which when $\phi = \frac{1}{2}$ produces a wedge in the form of an elastic right-half plane with the following corresponding stresses articulated from (4.1) and (4.2).

$$\sigma_{rz}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}[\sin\,(2n-1)\pi\sin\,(2n-1)\pi\theta]\left(\frac{r}{a}\right)^{(2n-1)\pi-1} \quad (5.5)$$

$$\sigma_{\theta z}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}[\sin\,(2n-1)\pi\sin\,(2n-1)\pi\theta]\left(\frac{r}{a}\right)^{(2n-1)\pi-1} \quad (5.6)$$

For $\phi = \frac{3}{4}$ corresponds to a sharp notched wedge which gives

$$\sigma_{rz}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left[\sin 3(2n-1)\frac{\pi}{2}sin3(2n-2)\frac{\pi\theta}{2}\right]\left(\frac{r}{a}\right)^{3(2n-1)\frac{\pi}{2}-1} \quad (5.7)$$

$$\sigma_{\theta z}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left[sin3(2n-1)\frac{\pi}{2}cos3(2n-1)\frac{\pi\theta}{2}\right]\left(\frac{r}{a}\right)^{3(2n-1)\frac{\pi}{2}-1}, r < a \quad (5.8)$$

For $\phi = 1$ corresponds to that which becomes a wedge similar to an infinite crack in an elastic body. Then, the stresses are

$$\sigma_{rz}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left[\sin\,(2n-1)\frac{\pi}{2}\sin\,(2n-1)\frac{\pi\theta}{2}\right]\left(\frac{r}{a}\right)^{(2n-1)\frac{\pi}{2}-1}, r < a \quad (5.9)$$

$$\sigma_{\theta z}(r,\theta) = \frac{2aT}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left[\sin\,(2n-1)\frac{\pi}{2}\cos\,(2n-1)\frac{\pi\theta}{2}\right]\left(\frac{r}{a}\right)^{(2n-1)\frac{\pi}{2}-1}, r < a \quad (5.10)$$

Equations (3.6) and (5.4) yields the strength of singularity as

$$\lambda_{s_1} = 1 - \frac{\pi}{2\phi} \quad (5.11)$$

VI. RESULTS

If s_1 is the least of the poles with $|s_1| < 1$ *and* $\lambda_{s_1} = 1 - s_1$, then λ_{s_1} is the order or strength of the stress singularity at the wedge apex. Whether or not stress singularity occurs at the wedge apex depends on the fact that the inequality $|s_1| < 1$ holds. Considering (4.1) , (4.2) , (5.1) and (5.2), we note that $0 \leq \phi < 2\pi$

implies that $0 < \phi < \frac{\pi}{2}$ and $\frac{\pi}{2} < \phi < 2\pi$. Also , $0 < \phi < \frac{\pi}{2}$ implies $\frac{\pi}{2\phi} > 1$. Therefore, for $n = 1$, $\frac{\pi}{2\phi} - 1 > 0 \, or \, |s_1| > 1$. Then, for this, the fracture fields are bounded. But, if $\frac{\pi}{2} < \phi, then \ \frac{\pi}{2\phi} \ implies \ (2n-1)\frac{\pi}{2\phi} < 2n - 1 \ and \ (2n-1)\frac{\pi}{2\phi} - 1 < (2n-1)$. Therefore, for $n = 1, we \ have \ s_1 = \frac{\pi}{2\phi} < 1$. Hence ,singularity occurs for all $\phi > \frac{\pi}{2}$.

VII. DISCUSION

From the analysis , (5.3) and (5.4) shows that at the wedge tip ,$\theta = 0 \ and \ r \to 0$ the only existing stress conponent is that of tearing stress which can induce fracture or crack if it becomes infinite. Also, $(5.5) - (5.10)$ shows the relationship between the wedge geometry and the resultant stress distribution around singularities. Moreso, from (5.11) it can be seen that when ϕ tends to some critical values, then the strength of singularity shows significant variations.

VIII. CONCLUSION

The analysis of the fracture responses of an isotropic elastic wedge with free radial and loaded circular surface under anti plane deformation has been carried out. The problem was formulated using complex potential theory. The Mellin integral transform of the second kind was employed to solve the two-dimensional Neumann boundary value problem for the only non zero displacement component through which the stress components were obtained. The stresses of different geometries were obtained by considering various values of the apex angle .

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