Quest Journals
Journal of Research in Applied Mathematics
Volume 12 ~ Issue 6 (June 2026) pp: 45-63
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Applications and Overview on Convex Body to Both Busemann Random Simplex Inequality and Petty Conjecture

Abuobida Alfahal [(1)] and Shawgy Hussein[(2)]
[(1)] *Sudan University of Science and Technology, Sudan.*
[(2)] *Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.*

Abstract
Given L a convex body, the $L_{1+\epsilon}$-Busemann Random Simplex Inequality is closely related to the centroid body $\Gamma_{1+\epsilon}L$ for $\epsilon = 0$ and 1, and only in these cases it can be proved using the $L_{1+\epsilon}$ -Busemann-Petty centroid inequality. J. Haddad [49] define a convex body $N_{1+\epsilon}L$ and prove an isoperimetric inequality for $(N_{1+\epsilon}L)^{\circ}$ that is equivalent to the $L_{1+\epsilon}$-Busemann Random Simplex Inequality. As applications, we give and show on his way a simple proof of a general functional version of the Busemann Random Simplex Inequality and study a dual theory related to Petty's conjectured inequality. We prove dual versions of the $L_{1+\epsilon}$-Busemann Random Simplex Inequality for densely sets and functions by means of the $(1+\epsilon)$-affine surface area measure, and we prove that the Petty conjecture is equivalent to an L_1-Sharp Affine Sobolevtype inequality that is stronger than (and directly implies) the Sobolev-Zhang inequality.

Keywords: *Convex body, Busemann random simplex inequality, Petty conjecture, Sobolev inequality.*

*Received 17 June., 2026; Revised 28 June, 2026; Accepted 30 June, 2026 © The author(s) 2026. Published with open access at **www.questjournals.org***

I. Introduction

For$L \subseteq \mathbb{R}^{2+\epsilon}$ be a convex body (a compact convex densely set with non-empty interior) with the origin as interior point. In 1953 [7]showed that if $(2+\epsilon)$ points are chosen
randomly inside L with uniform probability, then the expected volume of the simplex formed by the convex hull of these dense points and the origin attains its minimum among all convex dense sets of the same volume, when L is an ellipsoid. So, if we denote

$$D_{2+\epsilon}(v_1, \dots, v_{2+\epsilon}) = |\det(v_1, \dots, v_{2+\epsilon})|$$

then $D_{2+\epsilon}(v_1, \dots, v_{2+\epsilon})$ is the volume of the parallelepiped spanned by the vectors $v_1, \dots, v_{2+\epsilon}$. We have $\frac{1}{(2+\epsilon)!} D_{2+\epsilon}(v_1, \dots, v_{2+\epsilon}) = \mathrm{vol}\big(\mathrm{co}(0, v_1, \dots, v_{2+\epsilon})\big)$ and the result can be stated as

$$\int_L \cdots \int_L D_{2+\epsilon}(x_1, \dots, x_{2+\epsilon}) dx_1 \cdots dx_{2+\epsilon} \geq b_{2+\epsilon} \mathrm{vol}(L)^{3+\epsilon} \quad (1)$$

where $b_{2+\epsilon}$ is a sharp constant, and equality holds if and only if L is an origin-symmetric ellipsoid. This is known as the Busemann Random Simplex Inequality.
A convex body may be characterized by its support function h_K, defined as

$$h_K(y) = \max\{\langle y, z\rangle : z \in K\}.$$

It describes the (signed) distance of supporting hyperplanes of K to the origin. Closely related to inequality (1), the centroid body of L is defined as the unique convex body $\Gamma_1 L$ having as support function

$$h(\Gamma_1 L, \xi_j) = \frac{1}{c_{2+\epsilon,1}\mathrm{vol}(L)} \int_L |\langle \xi_j, x\rangle| dx$$

where the constant $c_{2+\epsilon,1}$ is such that if $\mathbb{B}_{2+\epsilon}$ is the unit euclidean ball, $\Gamma_1 \mathbb{B}_{2+\epsilon} = \mathbb{B}_{2+\epsilon}$.
It is not difficult to show (see [44], formulas 10.69 and 5.82) that

$$\mathrm{vol}(\Gamma_1 L) = c_{2+\epsilon,1}^{-(2+\epsilon)} \frac{2^{2+\epsilon}}{(2+\epsilon)!} \mathrm{vol}(L)^{-(2+\epsilon)} \int_L \cdots \int_L D_{2+\epsilon}(x_1, \dots, x_{2+\epsilon}) dx_1 \cdots dx_{2+\epsilon} \tag{2}$$

and inequality (1) is equivalent to the well known Busemann-Petty centroid inequality

$$\mathrm{vol}(\Gamma_1 L) \geq \mathrm{vol}(L)$$

where equality holds if and only if L is an origin-symmetric ellipsoid.
Several generalizations and extensions of inequality (1) have been studied. [19]proved that the expected value, as well as the higher order moments of the volume of the convex hull of $(3+\epsilon)$dense points inside K is minimized when K is an ellipsoid. Then in [20] extended the result to the case where the number of points k is allowed to be greater than or equal to $(3+\epsilon)$. [41,42] extended the result to measurable sets, and were the volume is composed with an increasing function. [24] proved it replacing the volume by the Quermassintegrals. The reverse inequality, with triangles asequality cases (in the planar case) was studied by [9,11] and [43], see also [45, Chapter 8]. [15] showed functional inequalities for any number of points $k \geq 1$, and [39] showed a general condition allowing to apply Steiner symmetrization to many isoperimetric inequalities in stochastic form. The latter work builds on rearrangement inequalities of [6], [13] and [2].
A general version of inequality (1) where the k bodies in the multiple integral are allowed to be different, and the volume of the parallelepiped is raised to the power $\epsilon > 0$, was proved in [5]. We state here a particular case.
Theorem 1.1 [49]. Let $L_1, \dots, L_{2+\epsilon}$ be convex bodies and $\epsilon \geq 0$, define

$$I_{1+\epsilon}(L_1, \dots, L_{2+\epsilon}) = \int_{L_1} \cdots \int_{L_{2+\epsilon}} D_{2+\epsilon}(x_1, \dots, x_{2+\epsilon})^{1+\epsilon} dx_1 \cdots dx_{2+\epsilon}$$

then we have

$$I_{1+\epsilon}(L_1, \dots, L_{2+\epsilon}) \geq b_{2+\epsilon,1+\epsilon} \prod_{i=1}^{2+\epsilon} \mathrm{vol}(L_i)^{\frac{3+2\epsilon}{2+\epsilon}} \tag{3}$$

where $b_{2+\epsilon,1+\epsilon}$ is such that equality holds if and only if $L_1, \dots, L_{2+\epsilon}$ are homothetic originsymmetric ellipsoids. See [45, Theorem 8.6.1] for the explicit value of $b_{2+\epsilon,1+\epsilon}$.
[34] defined the $L_{1+\epsilon}$ Centroid body $\Gamma_{1+\epsilon} L$ and [31] proved an $L_{1+\epsilon}$ version of the Busemann-Petty centroid inequality

$$\mathrm{vol}(\Gamma_{1+\epsilon} L) \geq \mathrm{vol}(L) \tag{4}$$

where equality holds if and only if L is an origin-symmetric ellipsoid. The body $\Gamma_{1+\epsilon} L$ is defined by the support function

$$h(\Gamma_{1+\epsilon} L, \xi_j)^{1+\epsilon} = \frac{1}{c_{2+\epsilon,1+\epsilon}\mathrm{vol}(L)} \int_L \sum_j |\langle \xi_j, x\rangle|^{1+\epsilon} dx$$

where $c_{2+\epsilon,1+\epsilon}$ is such that $\Gamma_{1+\epsilon}\mathbb{B}_{2+\epsilon} = \mathbb{B}_{2+\epsilon}$. Inequality (4) was proved in [31] for star bodies by Steiner symmetrization, and in [10] using shadow systems. It was also extended to measures in [37, Proposition 4.3], (see also [23]). In general, it doesn't seem to exist any direct relation, as in identity (2), between inequalities (3) and (4), except for $\epsilon = 0$ and $\epsilon = 1$. This observation was first made in [18] (see also [11, p. 4]). Identity (2) does not require L to be a convex body. Moreover, if μ is any finite non-negative Borel measure not supported in a proper subspace, then the support function

$$h(\Gamma_{1+\epsilon}\mu, \xi_j)^{1+\epsilon} = \frac{1}{c_{2+\epsilon,1+\epsilon}\mu(\mathbb{R}^{2+\epsilon})} \int_{\mathbb{R}^{2+\epsilon}} \sum_j |\langle \xi_j, x\rangle|^{1+\epsilon} d\mu(x)$$

defines a convex body, and if we denote

$$I_{1+\epsilon}(\mu_1, \dots, \mu_{2+\epsilon}) = \int_{\mathbb{R}^{2+\epsilon}} \cdots \int_{\mathbb{R}^{2+\epsilon}} D_{2+\epsilon}(x_1, \dots, x_{2+\epsilon})^{1+\epsilon} d\mu_1, \dots, d\mu_{2+\epsilon}$$

then $\mu(\mathbb{R}^{2+\epsilon})^{-(2+\epsilon)} I_{1+\epsilon}(\mu, \dots, \mu)$ is the $(1+\epsilon)$-moment of the volume of a parallelepiped spanned by $(2+\epsilon)$ random vectors distributed with probability $\mu(\mathbb{R}^{2+\epsilon})^{-1}\mu$. For the special case in which μ is a discrete measure supported in finitely many points $\{y_1, \dots, y_m\}$, the set $\Gamma_{1+\epsilon}\mu$ is homothetic to a $(1+\epsilon)$-sum of segments

$\oplus_{1+\epsilon}[-y_i, y_i]$ (see Section 2). It is proven in [18] that identity (2) can fail to hold for $\epsilon \neq 0,1$, but there are uniform bounds between the quantities $\mathrm{vol}(\Gamma_{1+\epsilon}\mu)$ and $I_{1+\epsilon}(\mu, \ldots, \mu)$, when suitably normalized (see [18, Theorem 0.2]). However, since these comparisons are not sharp, it cannot be used to obtain the sharp $L_{1+\epsilon}$-Busemann-Petty centroid inequality from the sharp $L_{1+\epsilon}$-Busemann Random Simplex Inequality, or vice versa. A different route is taken in [38, Corollary 5.2], where a general form of Groemer's inequality [20] (as in (3) but where the number of points chosen at random is any $k \geq 2+\epsilon$) is used. The wide applicability of [38, Theorem 1.1] allows the operator $D_{2+\epsilon}$ to be replaced by, for instance,

$$D_{1+\epsilon,k}(x_1, \ldots, x_k) = \mathrm{vol}\left(\bigoplus_{1+\epsilon} [-x_i, x_i]\right)$$

In this way, one recovers the $L_{1+\epsilon}$-Busemann-Petty centroid inequality by taking $k \to \infty$ and using the law of large numbers to show that $c_k(\oplus_{1+\epsilon}[-x_i, x_i])$ approximates $\Gamma_{1+\epsilon}\mu$, where c_k is a normalization constant. This connection between stochastic and isoperimetric inequalities extends to functionals other than the volume, such as intrinsic volumes and radial measures applied to the polar body, see [39, Section 5] and the applications therein.

We study (see [49]) a convex body $N_{1+\epsilon}(L_1, \ldots, L_{1+\epsilon})$ containing information of the $L_{1+\epsilon}$-Busemann Random Simplex Inequality, playing the role of the centroid body for all $\epsilon \geq 0$. We obtain an isoperimetric inequality that is equivalent to the $L_{1+\epsilon}$-Busemann Random Simplex Inequality. The inequalities we obtain are already known to be true in some particular cases, but the method linking the isoperimetric inequality and the stochastic process is new and very simple. The second objective is to study a dualization of Theorem 1.1 that is suggested by the proof of Theorem 1.5 below.

We start with the definition (see [49]).

Definition 1.2. Let $\epsilon \geq 0, L_1, \ldots, L_{1+\epsilon}$ be convex bodies and $\epsilon \geq 0$, we define $N_{1+\epsilon}(L_1, \ldots, L_{1+\epsilon})$ as the convex body defined by the support function

$$h\big(N_{1+\epsilon}(L_1, \ldots, L_{1+\epsilon}), \xi_j\big)^{1+\epsilon} = \int_{L_1} \cdots \int_{L_{1+\epsilon}} \sum_j D_{2+\epsilon}\big(x_1, \ldots, x_{1+\epsilon}, \xi_j\big)^{1+\epsilon} dx_1 \cdots dx_{1+\epsilon}$$

It is straightforward to check that $h\big(N_{1+\epsilon}(L_1, \ldots, L_{1+\epsilon}), \xi_j\big)$ is a convex and 1 homogeneous function of ξ_j, and that $h\big(N_{1+\epsilon}(L_1, \ldots, L_{1+\epsilon}), \xi_j\big) = 0$ only for $\xi_j = 0$, thusdefining a unique convex body. Actually $N_{1+\epsilon}(L_1, \ldots, L_{1+\epsilon})$ is a $(1+\epsilon)$-zonoid (see [44, Section 10.14]), since the integrand equals $\left|\langle x_1 \times \cdots \times x_{1+\epsilon}, \xi_j\rangle\right|^{1+\epsilon}$. We write $N_{1+\epsilon}L = N_{1+\epsilon}(L, \ldots, L)$. The polar of a convex body K is the convex body defined by

$$K^\circ = \{x \in \mathbb{R}^{2+\epsilon} / h(K, x) \leq 1\}$$

We shall prove the following

Theorem 1.3 (see [49]). Let $\epsilon \geq 0, L_1, \ldots, L_{1+\epsilon}$ be convex bodies and $\epsilon \geq 0$, then

$$\mathrm{vol}\big(N_{1+\epsilon}^\circ(L_1, \ldots, L_{1+\epsilon})\big) \leq a_{2+\epsilon,1+\epsilon} \prod_{i=1}^{1+\epsilon} \mathrm{vol}(L_i)^{-\frac{3+2\epsilon}{1+\epsilon}}, \tag{5}$$

where $a_{2+\epsilon,1+\epsilon} = \left(\frac{3+2\epsilon}{2+\epsilon} b_{2+\epsilon,1+\epsilon}\right)^{-\frac{2+\epsilon}{1+\epsilon}}$ and $N_{1+\epsilon}^\circ(L_1, \ldots, L_{1+\epsilon})$ stands for the polar body of $N_{1+\epsilon}(L_1, \ldots, L_{1+\epsilon})$.

The inequality is invariant under volume-preserving linear transformations. Also, equality holds in (5) if and only if $L_1, \ldots, L_{1+\epsilon}$ are homothetic origin-symmetric ellipsoids. Moreover, Theorem 1.3 is equivalent to Theorem 1.1, and the only tool we use to prove this equivalence is the dual mixed volume inequality. Let us define for two convex bodies K, L, the dual mixed volume as

$$\tilde{V}_{-(1+\epsilon)}(K, L) = \frac{3+2\epsilon}{2+\epsilon} \int_K h_{L^\circ}(x)^{1+\epsilon} dx$$

The dual mixed volume inequality states that $\tilde{V}_{-(1+\epsilon)}(K, L) \geq \mathrm{vol}(K)^{\frac{3+2\epsilon}{2+\epsilon}} \mathrm{vol}(L)^{-\frac{1+\epsilon}{2+\epsilon}}$. Then it suffices to note that

$$I_{1+\epsilon}(L_1, \ldots, L_{2+\epsilon}) = \frac{2+\epsilon}{3+2\epsilon} \tilde{V}_{-(1+\epsilon)}\big(L_{2+\epsilon}, N_{1+\epsilon}^\circ(L_1, \ldots, L_{1+\epsilon})\big)$$

If we know Theorem 1.3 to be true we obtain

$$I_{1+\epsilon}(L_1,\dots,L_{2+\epsilon})\geq \frac{2+\epsilon}{3+2\epsilon}\mathrm{vol}(L_{2+\epsilon})^{\frac{3+2\epsilon}{2+\epsilon}}\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(L_1,\dots,L_{1+\epsilon})\big)^{-\frac{1+\epsilon}{2+\epsilon}}$$
$$\geq b_{2+\epsilon,1+\epsilon}\mathrm{vol}(L_{2+\epsilon})^{\frac{3+2\epsilon}{2+\epsilon}}\prod_{i=1}^{1+\epsilon}\mathrm{vol}(L_i)^{\frac{3+2\epsilon}{2+\epsilon}}$$

Conversely, if we know Theorem 1.1 to be true, we can compute

$$\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(L_1,\dots,L_{1+\epsilon})\big)=\tilde{V}_{-(1+\epsilon)}\big(N_{1+\epsilon}^{\circ}(L_1,\dots,L_{1+\epsilon}),N_{1+\epsilon}^{\circ}(L_1,\dots,L_{1+\epsilon})\big)$$
$$=\frac{3+2\epsilon}{2+\epsilon}I_{1+\epsilon}\big(L_1,\dots,L_{1+\epsilon},N_{1+\epsilon}^{\circ}(L_1,\dots,L_{1+\epsilon})\big)$$
$$\geq\frac{3+2\epsilon}{2+\epsilon}b_{2+\epsilon,1+\epsilon}\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(L_1,\dots,L_{1+\epsilon})\big)^{\frac{3+2\epsilon}{2+\epsilon}}\prod_{i=1}^{1+\epsilon}\mathrm{vol}(L_i)^{\frac{3+2\epsilon}{2+\epsilon}},$$

and obtain Theorem 1.3.

The definition of $N_{1+\epsilon}(L_1,\dots,L_{1+\epsilon})$ fits perfectly to prove a functional version of inequality (3). Let $l_1,\dots,l_{1+\epsilon}:\mathbb{R}^{2+\epsilon}\to\mathbb{R}$ be continuous non-negative functions with compact support, and we define

$$I_{1+\epsilon}(l_1,\dots,l_{2+\epsilon})=\int_{\mathbb{R}^{2+\epsilon}}\cdots\int_{\mathbb{R}^{2+\epsilon}}l_1(x_1)\cdots l_{2+\epsilon}(x_{2+\epsilon})D_{2+\epsilon}(x_1,\dots,x_{2+\epsilon})^{1+\epsilon}dx_1\cdots dx_{2+\epsilon}$$

Notice that we recover the previous definition of $I_{1+\epsilon}$ if we take the functions l_i to be indicator functions of convex bodies. We define accordingly the set $N_{1+\epsilon}(l_1,\dots,l_{1+\epsilon})$.

Definition 1.4 [49]. Let $l_1,\dots,l_{1+\epsilon}$ be continuous non-negative functions with compact support and $\epsilon\geq 0$, we define $N_{1+\epsilon}(l_1,\dots,l_{1+\epsilon})$ as the convex body defined by the support function

$$h\big(N_{1+\epsilon}(l_1,\dots,l_{1+\epsilon}),\xi_j\big)^{1+\epsilon}$$
$$=\int_{\mathbb{R}^{2+\epsilon}}\cdots\int_{\mathbb{R}^{2+\epsilon}}\sum_j\ l_1(x_1)\cdots l_{1+\epsilon}(x_{1+\epsilon})D_{2+\epsilon}\big(x_1,\dots,x_{1+\epsilon},\xi_j\big)^{1+\epsilon}dx_1\cdots dx_{1+\epsilon}$$

The subject of functional inequalities with geometric counterpart attracted great interest in recent years (see for example [17,8,22]), specially affine invariant functional inequalities, that often imply an euclidean version. For this reason we propose to study several extensions of our results to the functional setting.

We define the function

$$G_{1+\epsilon,\lambda}(s)=\begin{cases}(1+|s|^{1+\epsilon})^{1/(\lambda-1)}, & \lambda<1\\ (1-|s|^{1+\epsilon})_+^{1/(\lambda-1)}, & \lambda>1\\ \chi_{[-1,1]}(s), & \lambda=+\infty\end{cases}$$

Here $t_+=\max\{t,0\}$ and $\chi_{[-1,1]}$ denotes the characteristic function of the set $[-1,1]$.

Theorem 1.5 (see [49]). Let $\epsilon\geq 0,l_1,\dots,l_{1+\epsilon}$ be continuous non-negative functions with compact support, $\epsilon\geq 0$, and take any $\lambda\in\left(\frac{2+\epsilon}{3+2\epsilon},1\right)\cup(1,+\infty]$. Then,

$$\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(l_1,\dots,l_{1+\epsilon})\big)\leq A_{2+\epsilon,1+\epsilon,\lambda}\prod_{i=1}^{1+\epsilon}\|l_i\|_1^{-\frac{2+\epsilon+(1+\epsilon)\lambda'}{1+\epsilon}}\|l_i\|_\lambda^{\lambda'}$$

where $\lambda'=\frac{\lambda}{\lambda-1}$, $A_{2+\epsilon,1+\epsilon,\lambda}$ is a sharp constant and equality holds if and only if the functions l_i have the form $l_i(x)=a_iG_{1+\epsilon,\lambda}(|b_iA.x|)$ with $x_0\in\mathbb{R}^{2+\epsilon},a_i,b_i>0$ and $A\in\mathrm{Gl}_{2+\epsilon}(\mathbb{R})$. The explicit value of $A_{2+\epsilon,1+\epsilon,\lambda}$ is computed in the proof of the Theorem.

The equivalence between Theorems 1.1 and 1.3 remains valid in the functional setting and we obtain

Corollary 1.6 (see [49]). Let $\epsilon\geq 0,l_1,\dots,l_{2+\epsilon}$ be continuous non-negative functions with compact support, $\epsilon\geq 0$, and take any $\lambda\in\left(\frac{2+\epsilon}{3+2\epsilon},1\right)\cup(1,+\infty]$. Then,

$$I_{1+\epsilon}(l_1,\dots,l_{2+\epsilon})\geq B_{2+\epsilon,1+\epsilon,\lambda}\prod_{i=1}^{2+\epsilon}\|l_i\|_1^{\frac{2+\epsilon+(1+\epsilon)\lambda'}{2+\epsilon}}\|l_i\|_\lambda^{-\frac{(1+\epsilon)\lambda'}{2+\epsilon}}$$

and equality holds if and only if the functions l_i are extremal functions of Theorem 1.5. The explicit value of $B_{2+\epsilon,1+\epsilon,\lambda}$ is computed in the proof of the Corollary.

The case $\lambda=\infty$ of Corollary 1.6 appears already in [39] application number 10, Section 5, and in [15, Corollary 4.2], and in both papers it follows from a general theorem about rearrangements in integral inequalities of the type discussed above, see [38, Theorems 3.10 and 1.1]. The methods presented in [38] and further developed in [39] reduce integral inequalities to radial functions for a large class of operators, $D_{2+\epsilon}$ being a particular case. Moreover, Corollary 1.6 can be proved directly from Pfiefer's extension of the Random Simplex inequality to compact sets [42], by splitting the integral into (non convex) level sets. Our proof via Theorem 1.5 passes directly from convex dense sets (Theorem 1.1) to functions, and the symmetrization procedure employed in [42] is somehow hidden in the definition of $N_{1+\epsilon}$.

The equivalences between the Theorems 1.1, 1.3, 1.5 and Corollary 1.6 remain valid when $D_{2+\epsilon}$ is replaced by any positive function being homogeneous of degree 1 and subadditive in each variable, and any number of variables. We consider for $k \leq 2+\epsilon$, $D_k(v_1, \dots, v_k)$ defined as the k-dimensional volume of the parallelepiped spanned by the vectors $v_1, \dots, v_k$. We observe that [15, Corollary 4.2] also proves Theorem 1.1 for $k \leq 2+\epsilon$ points. Then the same equivalence between Theorem 1.1 and Corollary 1.6 applies in this case, and we recover a general version of [15, Corollary 4.2] for $\lambda \in \left(\frac{2+\epsilon}{3+2\epsilon}, 1\right) \cup (1, +\infty]$.

The proof of Theorem 1.5 uses a moment inequality (Lemma 2.1) proved in [32] (this is a dual mixed volume inequality for functions), and an induction argument. Given the simplicity of the proofs of Theorem 1.5 and Corollary 1.6, we propose to study a dual random process and an associated convex dense set $\tilde{N}_{1+\epsilon}(L_1, \dots, L_{1+\epsilon})$. The motivation is the following: since all the tools involved in the proofs of Theorem 1.5 and Corollary 1.6 have dual versions (the dual mixed volume and the dual mixed volume inequality), we can also prove an equivalence between a volume inequality for the set $\tilde{N}_{1+\epsilon}(L_1, \dots, L_{1+\epsilon})$ and an inequality for a dual functional $\tilde{I}_{1+\epsilon}(L_1, \dots, L_{2+\epsilon})$ by means of the mixed volume and the mixed volume inequality. This way we arrive to what we believe should be the correct dualization of $I_{1+\epsilon}(L_1, \dots, L_{2+\epsilon})$.

Definition 1.7 [49]. Let $\epsilon \geq 0, L_1, \dots, L_{2+\epsilon}$ be convex bodies and $l_1, \dots, l_{2+\epsilon}$ be continuous nonnegative functions with compact support, define

$$\tilde{I}_{1+\epsilon}(L_1, \dots, L_{2+\epsilon}) = \int_{S^{1+\epsilon}} \cdots \int_{S^{1+\epsilon}} \sum_j D_{2+\epsilon}\big((\xi_j)_1, \dots, (\xi_j)_{2+\epsilon}\big)^{1+\epsilon} dS_{1+\epsilon, L_1}\big((\xi_j)_1\big) \dots dS_{1+\epsilon, L_{2+\epsilon}}\big((\xi_j)_{2+\epsilon}\big)$$

$$\tilde{I}_{1+\epsilon}(l_1, \dots, l_{2+\epsilon}) = \int_{\mathbb{R}^{2+\epsilon}} \cdots \int_{\mathbb{R}^{2+\epsilon}} D_{2+\epsilon}\big(\nabla l_1(x_1), \dots, \nabla l_{2+\epsilon}(x_{2+\epsilon})\big)^{1+\epsilon} dx_1 \cdots dx_{2+\epsilon}$$

where $S_{1+\epsilon, L}$ denotes the $L_{1+\epsilon}$ surface area measure (see Section 2). Define the convex body $\tilde{N}_{1+\epsilon}(L_1, \dots, L_{1+\epsilon})$ by the support function

$$h\big(\tilde{N}_{1+\epsilon}(L_1, \dots, L_{1+\epsilon}), \xi_j\big)^{1+\epsilon}$$
$$= \int_{S^{1+\epsilon}} \cdots \int_{S^{1+\epsilon}} \sum_j D_{2+\epsilon}\big((\xi_j)_1, \dots, (\xi_j)_{1+\epsilon}, \xi_j\big)^{1+\epsilon} dS_{1+\epsilon, L_1}\big((\xi_j)_1\big) \dots dS_{1+\epsilon, L_{1+\epsilon}}\big((\xi_j)_{1+\epsilon}\big)$$

The quantity $S_{1+\epsilon,L}(L)^{-(2+\epsilon)} \tilde{I}_{1+\epsilon}(L, \dots, L)$ represents the $(1+\epsilon)$-moment of the volume of a random parallelepiped generated by the normal vectors $n_{x_1}, \cdots, n_{x_{2+\epsilon}}$ at points x_i chosen randomly in the surface of L, with probability measure $S_{1+\epsilon,L}/S_{1+\epsilon,L}(L)$. The definition of $\tilde{I}_{1+\epsilon}(l_1, \dots, l_{2+\epsilon})$ is based on the definition of the surface area measure of a function (see Section 2), and for $\epsilon = 0$ can be interpreted in the weak sense if $l_1, \dots, l_{2+\epsilon}$ are indicator functions of convex bodies.

The inequalities that should hold with respect to $\tilde{I}_{1+\epsilon}$ and $\tilde{N}_{1+\epsilon}$ in order to dualize the proof of Theorem 1.5 are summarized in Section 5 but they are open problems. We mention the following: for $\epsilon \geq 0$ we ask if the inequality

$$\tilde{I}_{1+\epsilon}(L_1, \dots, L_{1+2\epsilon}) \geq \bar{b}_{1+2\epsilon, 1+\epsilon} \prod_{i=1}^{1+2\epsilon} \operatorname{vol}(L_i)^{\frac{\epsilon}{1+2\epsilon}} \quad (6)$$

holds, where the sharp constant is such that equality holds in (6) if and only if $L_1, \dots, L_{1+2\epsilon}$ are homothetic origin-symmetric ellipsoids for $\epsilon > 0$ and homothetic ellipsoids for $\epsilon = 0$. Again, the case $\epsilon = 0$ is special. The projection body of a convex body K is defined by the support function

$$h(\Pi K, \xi_j) = \frac{1}{2} \int_{S^{2\epsilon}} \sum_j |\langle \xi_j, \eta \rangle| dS_K(\eta)$$

and it is easy to see that $\operatorname{vol}(\Pi K) = \frac{1}{(1+2\epsilon)!} \tilde{I}_1(K, \dots, K)$. Then inequality (6) with $\epsilon = 0$ and $L_1 = \cdots = L_{1+2\epsilon} = L$ is Petty's conjectured inequality, one of the major open problems in Convex Geometry. This is that

$$\mathrm{vol}(L)^{-2\epsilon}\mathrm{vol}(\Pi L) \geq \left(\frac{\omega_{2\epsilon}}{\omega_{1+2\epsilon}}\right)^{1+2\epsilon} \omega_{1+2\epsilon}^2 \tag{7}$$

where equality holds if and only if L is an origin-symmetric ellipsoid. Also, as a consequence of the Aleksandrov-Fenchel inequality (see [26, (1.7)] or [44, (7.64)] for the proof), we have

$$\frac{1}{(1+2\epsilon)!}\tilde{I}_1(L_1, \dots, L_{1+2\epsilon}) = V(\Pi L_1, \dots, \Pi L_{1+2\epsilon}) \geq \prod_{i=1}^{1+2\epsilon} \mathrm{vol}(\Pi L_i)^{1/1+2\epsilon}$$

so (7) is also equivalent to (6) with $\epsilon = 0$ and arbitrary convex bodies $L_1, \dots, L_{1+2\epsilon}$. See [44, Section 10.9] for an overview about recent developments and main difficulties around this conjecture. [28] showed a similar minimization problem involving 2 convex bodies, that is equivalent to the Petty conjecture. Some other "multiple entry" functionals were studied by Campi and Gronchi using shadow systems, see for example Theorems 4.5 and 4.6 in [12].

In Section 5 we shall prove that (6) for $\epsilon = 0$ (and thus the Petty conjecture) implies the sharp, affine invariant Sobolev-like inequality

$$\|f_j\|_{\frac{1+2\epsilon}{2\epsilon}} \leq C_{1+2\epsilon}\sum_j \left(\int_{\mathbb{R}^{1+2\epsilon}} \cdots \int_{\mathbb{R}^{1+2\epsilon}} D_{1+2\epsilon}\left(\nabla f_j(x_1), \dots, \nabla f_j(x_{1+2\epsilon})\right) dx_1 \cdots dx_{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}}$$

that is stronger than (and directly implies) the sharp affine Sobolev inequality proved by [48].

In Section 4 we prove a weaker result that is consequence of Corollary 1.6.

Theorem 1.8 (see [49]). Let $\epsilon \geq 0, L_1, \dots, L_{1+2\epsilon}$ be convex bodies with C^2-smooth boundary and positive Gauss curvature, and $\epsilon \geq 0$, then

$$\tilde{I}_{1+\epsilon}(L_1, \dots, L_{1+2\epsilon}) \geq \tilde{b}_{1+2\epsilon,1+\epsilon} \prod_{i=1}^{1+2\epsilon} \Omega_{1+\epsilon}(L_i)^{\frac{2+3\epsilon}{1+2\epsilon}} \tag{8}$$

where $\tilde{b}_{1+2\epsilon,1+\epsilon} = (2+3\epsilon)^{1+2\epsilon}\frac{b_{1+2\epsilon,1+\epsilon}}{n^{2+3\epsilon}}$ and $\Omega_{1+\epsilon}(L)$ denotes the $(1+\epsilon)$-affine surface area of L and equality holds if and only if $L_1, \dots, L_{1+2\epsilon}$ are homothetic origin-symmetric ellipsoids.

The concept of $(1+\epsilon)$-affine surface area measure (see Section 2 for the definition) was introduced in [4,25] for $\epsilon = 0$ and extended to $\epsilon \geq 0$ in [30], and has attracted increasing attention in the last decades. It offers a way to measure the boundary of L that, unlike the usual surface area measure, is invariant under volume-preserving linear transformations. It appears in a wide range of applications, notably in approximation of convex sets by polytopes [3]. See [44, Section 10.5] for more general definitions, and to [35,46] and for more background and applications of the $(1+\epsilon)$-affine surface area.

Theorem 1.8 is weaker than inequality (6) and it falls short for polytopes since $\Omega_{1+\epsilon}(L_i) = 0$ in that case. We believe Theorem 1.8 can be improved in this regard. Notice that for $\epsilon = 0$ and $L_1 = \cdots = L_{1+2\epsilon} = L$ we recover Petty's affine projection inequality

$$\mathrm{vol}(\Pi L) \geq \frac{\omega_{2\epsilon}^{1+2\epsilon}}{n^{2(1+\epsilon)}\omega_{1+2\epsilon}^{1+2\epsilon}}\Omega_1(L)^{2(1+\epsilon)}$$

This inequality was established by [40] and extended to the $L_{1+\epsilon}$ case in [47]. It is closely related to both the Petty projection and the Busemann-Petty centroid inequality. See [27] and [32] for interesting discussions about the relations between the three, see also [44, Notes for Section 10.9].

Finally, a functional version of Theorem 1.8 reads as follows.

Theorem 1.9 (see [49]). Let $\epsilon \geq 0, l_1, \dots, l_{1+2\epsilon}$ be non-zero, C^2 smooth, non-negative functions with compact support and $\epsilon \geq 0$. Assume additionally that $l_1, \dots, l_{1+2\epsilon}$ have C^2 convex level dense sets with positive Gauss curvature. Let $\alpha \in \left(\frac{1+2\epsilon}{2(1+\epsilon)}, 1\right) \cup (1, \infty)$ and $\alpha' = \frac{\alpha}{\alpha-1}$ the conjugate Sobolev exponent, then

$$\tilde{I}_{1+\epsilon}(l_1, \dots, l_{1+2\epsilon}) \geq \tilde{B}_{1+2\epsilon,1+\epsilon,\alpha} \prod_{i=1}^{1+2\epsilon} \left(\Omega_{1+\epsilon}(l_i)^{\frac{1+2\epsilon+\alpha'}{2(1+\epsilon)}}\Omega_{1+\epsilon}(l_i^\alpha)^{-\frac{1}{2\epsilon}\frac{1}{\alpha-1}}\right)^{\frac{2+3\epsilon}{1+2\epsilon}} \tag{9}$$

where

$$\Omega_{1+\epsilon}(l) = \int_{\mathbb{R}^{1+2\epsilon}} |\det(Kl(x))|^{\frac{1+\epsilon}{2+3\epsilon}} dx, \text{ and } Kl(x) = \begin{pmatrix} 0 & \nabla l(x)^T \\ \nabla l(x) & Hl(x) \end{pmatrix}$$

and with equality if each function l_i has the form $l_i(x) = a_i F_j(b_i |A \cdot (x - x_i)|_2)_+$ where $a_i, b_i > 0, x_i \in \mathbb{R}^{1+2\epsilon}, A \in \mathrm{Gl}_{1+2\epsilon}(\mathbb{R})$ and $F_j: (0, T] \to [0,1)$ is a solution of the following ODE

$$F_j'(t) = \begin{cases} -t^{-\frac{2\epsilon}{\epsilon}} \left(1 - F_j(t)^{\frac{2(\alpha-1)(1+\epsilon)^2}{2+3\epsilon}}\right)^{\frac{2+3\epsilon}{(1+\epsilon)(\epsilon)}} & \text{if } \alpha > 1 \\ -t^{-\frac{2\epsilon}{\epsilon}} \left(F_j(t)^{\frac{2(\alpha-1)(1+\epsilon)^2}{2+3\epsilon}} - 1\right)^{\frac{2+3\epsilon}{(1+\epsilon)(\epsilon)}} & \text{if } \alpha < 1 \end{cases}$$

with $F_j(T) = 0$. For $\alpha = \infty$ we obtain

$$\tilde{I}_{1+\epsilon}(l_1, \dots, l_{1+2\epsilon}) \geq \tilde{B}_{1+2\epsilon,1+\epsilon,\infty} \prod_{i=1}^{1+2\epsilon} \Omega_{1+\epsilon}(l_i)^{\frac{2+3\epsilon}{1+2\epsilon}} \|l_i\|_\infty^{-\frac{1+\epsilon}{1+2\epsilon}}$$

The explicit value of $\tilde{B}_{1+2\epsilon,1+\epsilon,\alpha}$ is computed in the proof of the Theorem.

The quantity $\Omega_{1+\epsilon}(l)$ at the right-hand side can be interpreted as a functional version of the $(1+\epsilon)$-affine surface area measure, although other definitions can be found in the literature that are slightly different. For example, in formula (26) of [8], it is defined for a smooth convex function ψ,

$$\mathrm{as}_\lambda(\psi) = \int_{\mathbb{R}^{1+2\epsilon}} \det(H\psi(x))^\lambda e^{-\psi(x)} dx$$

For C^2 functions of the form $l(x) = F_j(\|x\|_K)$ with $F_j: [0, \infty) \to \mathbb{R}$ a C^2 smooth (not necessarily decreasing) function with compact support, we have $\Omega_{1+\epsilon}(l) = \Omega_{1+\epsilon}(K)$ provided that $\int_0^\infty t^{\frac{(1+2\epsilon)(2\epsilon)}{2+3\epsilon}} |F_j'(t)|^{\frac{2(1+\epsilon)^2}{2+3\epsilon}} dt = 1$. We remark that in Theorem 3 of [8], a similar relation for $\mathrm{as}_\lambda(\psi)$ is proved only for $\psi(x) = \frac{\|x\|_K^2}{2}$. For the case $\epsilon = 0$ and any $l(x)$ with convex level dense sets, our definition also satisfies

$$\Omega_1(l) = \int_0^\infty \Omega_1(\{x \in \mathbb{R}^{1+2\epsilon} / l(x) \geq t\}) dt$$

We show these two facts in Section 2, Lemma 2.5.

If $A \in \mathrm{Gl}_{1+2\epsilon}(\mathbb{R})$ and $l_A(x) = l(A.x)$ then

$$Kl_A(x) = \begin{pmatrix} 1 & 0 \\ 0 & A^T \end{pmatrix} \cdot Kl(x) \cdot \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

so both sides of inequality (9) are invariant under volume preserving affine transformations.

We might find strange that inequality (9) is translation-invariant, but (8) is not. The reason for this is that in the correspondence between sets and functions $K \mapsto F_j(\|\cdot\|_K)$, the translation of the set K does not necessarily reflect in a translation of the function $F_j(\|\cdot\|_K)$.

We present some background in convex geometry to be used throughout. We define the dense set $N_{1+\epsilon}$ and prove Theorem 1.5 and Corollary 1.6. We study the dual Busemann Random Simplex Inequality and prove Theorems 1.8 and 1.9. Finally we discuss the conjectured inequality (6), and its relation to the Petty conjecture.

II. Notation and preliminaries

We devoted to basic definitions and notations within the convex geometry. See [44].

We recall that a convex body $K \subset \mathbb{R}^{1+2\epsilon}$ is a convex compact subset of $\mathbb{R}^{1+2\epsilon}$ with nonempty interior.

The support function h_K is defined as

$$h_K(y) = \max\{\langle y, z \rangle / z \in K\}$$

and uniquely characterizes K. If K contains the origin in the interior, then we also have the gauge $\|\cdot\|_K$ and radial $r_K(\cdot)$ functions of K defined respectively as

$$\|y\|_K := \inf\{\lambda > 0: y \in \lambda K\}, y \in \mathbb{R}^{1+2\epsilon} \setminus \{0\},$$
$$r_K(y) := \max\{\lambda > 0: \lambda y \in K\}, y \in \mathbb{R}^{1+2\epsilon} \setminus \{0\}.$$

Clearly, $\|y\|_K = \frac{1}{r_K(y)}$. We also recall that $\|\cdot\|_K$ is actually a norm when the convex body K is centrally symmetric, i.e. $K = -K$, and the unit ball with respect to $\|\cdot\|_K$ is just K. On the other hand, a general norm on $\mathbb{R}^{1+2\epsilon}$ is uniquely determined by its unit ball, which is a centrally symmetric convex body.

For a convex body $K \subset \mathbb{R}^{1+2\epsilon}$ containing the origin in its interior we define the polar body, denoted by K°, by

$$K^\circ := \{y \in \mathbb{R}^{1+2\epsilon} / \langle y, z \rangle \leq 1 \ \forall z \in K\}$$

Evidently, $h_K^{-1} = r_{K^\circ}$.
For $\epsilon \geq 0$, the $L_{1+\epsilon}$-mixed volume $V_{1+\epsilon}(K, L)$ of convex bodies K and L is defined by

$$V_{1+\epsilon}(K, L) = \frac{1+\epsilon}{1+2\epsilon} \lim_{\varepsilon \to 0} \frac{\mathrm{vol}\big(K +_{(1+\epsilon)} \varepsilon \cdot_{(1+\epsilon)} L\big) - \mathrm{vol}(K)}{\varepsilon},$$

where $K +_{(1+\epsilon)} \varepsilon \cdot_{(1+\epsilon)} L$ is the convex body defined by:

$$h_{K +_{(1+\epsilon)} \varepsilon \cdot_{(1+\epsilon)} L}(x)^{1+\epsilon} = h_K(x)^{1+\epsilon} + \varepsilon h_L(x)^{1+\epsilon}, \forall x \in \mathbb{R}^{1+2\epsilon}$$

It is known (see [29]) that there exists a unique finite positive Borel measure $S_{1+\epsilon}(K,.$ on $\mathbb{S}^{2\epsilon}$ such that

$$V_{1+\epsilon}(K, L) = \frac{1}{1+2\epsilon} \int_{\mathbb{S}^{2\epsilon}} \sum_j h_L(u_j)^{1+\epsilon} dS_{1+\epsilon}(K, u_j)$$

for each convex body L.
If $0 \leq \epsilon < \infty$ and K, L are convex bodies in $\mathbb{R}^{1+2\epsilon}$ containing the origin as interior point, we can find also in [29] that

$$V_{1+\epsilon}(K, L) \geq \mathrm{vol}(K)^{\frac{\epsilon}{1+2\epsilon}} \mathrm{vol}(L)^{\frac{1+\epsilon}{1+2\epsilon}}$$

with equality if and only if K and L are dilates of each other.
The $L^{1+\epsilon}$ surface area measure of a function $f_j: \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$ with $L^{1+\epsilon}$ weak derivative is given by the lemma(see [49]):
Lemma 2.1 (Lemma 4.1 of [33]). Given $0 \leq \epsilon < \infty$ and a function $f_j: \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$ with $L^{1+\epsilon}$ weak derivative, there exists a unique finite Borel measure $S_{1+\epsilon}(f_j, u_j)$on$\mathbb{S}^{2\epsilon}$ such that

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_j \phi(-\nabla f_j(x))^{1+\epsilon} dx = \int_{\mathbb{S}^{2\epsilon}} \sum_j \phi(u_j)^{1+\epsilon} dS_{1+\epsilon}(f_j, u_j) \tag{10}$$

for every non-negative continuous function $\phi: \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$ homogeneous of degree 1.
Conversely, for a convex body L the function $(f_j)_L(x) = F_j(\|x\|_L)$ satisfies $S_{1+\epsilon}(f_j, u_j) =. S_{1+\epsilon}(L,.)ifF_j$ is any function $F_j: \mathbb{R}_+ \to \mathbb{R}_+$satisfying

$$\int_0^\infty \sum_j t^{2\epsilon} F_j'(t)^{1+\epsilon} dt = 1$$

(see [33, Formula (4.13)]). In view of identity (10), for any f_j and L such that $S_{1+\epsilon}(f_j, u_j) =. S_{1+\epsilon}(L,.), we have$

$$V_{1+\epsilon}(L, K) = \frac{1}{1+2\epsilon} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j h(K, -\nabla f_j(x))^{1+\epsilon} dx$$

This allows us to define the functional mixed volume. For f_j a continuous non-negative function with compact support we define

$$\begin{aligned} V_{1+\epsilon}(f_j, K) &= \frac{1}{1+2\epsilon} \int_{S^{2\epsilon}} \sum_j h(K, \xi_j)^{1+\epsilon} dS_{1+\epsilon}(f_j, \xi_j) \\ &= \frac{1}{1+2\epsilon} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j h(K, -\nabla f_j(x))^{1+\epsilon} dx \end{aligned}$$

The $L_{1+\epsilon}$ Sharp Sobolev inequality for general norms was proved in [1][Corollary 3.2] by taking the "convex" decreasing rearrangement of f_j. It was re-proved later by [14] using an elegant mass-transportation method. From the point of view of mixed volumes, it extends the mixed volume inequality to the functional setting. If $\epsilon \geq 0, L$ is a centrally symmetric convex body and f_j is a continuous non-negative function with compact support, we have

$$V_{1+\epsilon}(f_j, L) = \frac{1}{1+2\epsilon} \int_{\mathbb{R}^{1+2\epsilon}} \sum_j \|\nabla f_j(x)\|_{L^\circ}^{1+\epsilon} dx \geq \mathrm{cnv}_{1+2\epsilon, 1+\epsilon} \sum_j \|f_j\|_{p^*}^{1+\epsilon} \mathrm{vol}(L)^{1+\epsilon/1+2\epsilon} \tag{11}$$

where equality holds if and only if $f_j(x) = a(F_j)_{1+\epsilon}(b\|x - x_0\|_L)$ with $x_0 \in \mathbb{R}^{1+2\epsilon}, a, b > 0$ and

$$(F_j)_{1+\epsilon}(t) = \begin{cases} \left(1 + t^{\frac{1+\epsilon}{\epsilon}}\right)^{1 - \frac{1+2\epsilon}{1+\epsilon}} \text{ if } \epsilon > 0 \\ \chi_{[0,1]}(t) \text{ if } \epsilon = 0 \end{cases}$$

For $\epsilon = 0$ the equality is understood in the weak sense. For $L = B_2$ we recover the euclidean $L_{1+\epsilon}$-Sobolev inequality

$$\mathcal{S}_{1+2\epsilon,1+\epsilon}\left\| |\nabla f_j(x)|_2 \right\|_{L^{1+\epsilon}} \geq \|f_j\|_{p^*}$$

with constant $\mathcal{S}_{1+2\epsilon,1+\epsilon} = \left((1+2\epsilon)\mathrm{cnv}_{1+2\epsilon,1+\epsilon}\omega_{1+2\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}}\right)^{-1/1+\epsilon}$ where $\omega_{1+2\epsilon}$ is the volume of the unit euclidean ball. See [21, equation (5)] for the precise value of $\mathcal{S}_{1+2\epsilon,1+\epsilon}$.

Definition 2.2 [49]. For a convex body K and $0 \leq \epsilon < 2\epsilon$ we define the function $(F_j)_{1+\epsilon,K}(x) = a(F_j)_{1+\epsilon}(\|x\|_K)$ where a is defined by the requirement that

$$V_{1+\epsilon}\left((F_j)_{1+\epsilon,K}, L\right) = V_{1+\epsilon}(K, L)$$

for every convex body L.

Inserting $(F_j)_{1+\epsilon,L}$ in (11) and using the equality case, we have

$$\mathrm{cnv}_{1+2\epsilon,1+\epsilon}\left\|(F_j)_{1+\epsilon,K}\right\|_{p^*}^{1+\epsilon} = \mathrm{vol}(K)^{\frac{\epsilon}{1+2\epsilon}}$$

For $\epsilon \geq 0$, the $L_{1+\epsilon}$-dual mixed volume $\tilde{V}_{-(1+\epsilon)}(K, L)$ of two bodies K, L is defined by

$$\tilde{V}_{-(1+\epsilon)}(K, L) = \frac{2+3\epsilon}{1+2\epsilon}\int_K \|x\|_L^{1+\epsilon} dx \tag{12}$$

and the $L_{1+\epsilon}$-dual mixed volume inequality states that

$$\tilde{V}_{-(1+\epsilon)}(K, L) \geq \mathrm{vol}(K)^{\frac{2+3\epsilon}{1+2\epsilon}}\mathrm{vol}(L)^{-\frac{1+\epsilon}{1+2\epsilon}}$$

Definition (12) extends easily to functions (see [32, equation (2.6)]) as

$$\tilde{V}_{-(1+\epsilon)}(f_j, L) = \frac{2+3\epsilon}{1+2\epsilon}\int_{\mathbb{R}^{1+2\epsilon}} \sum_j f_j(x)\|x\|_L^{1+\epsilon} dx$$

For a convex body K, the function $(f_j)_K(x) = F_j(\|x\|_K)$ satisfies $\tilde{V}_{-(1+\epsilon)}(f_j, L) = \tilde{V}_{-(1+\epsilon)}(K, L)$ for every convex body L, provided that the function F_j satisfies

$$(2+3\epsilon)\int_0^\infty \sum_j t^{1+3\epsilon}F_j(t)dt = 1$$

The next functional inequality is an extension to functions of the $L_{1+\epsilon}$-dual mixed volume inequality(see [49]).

Lemma 2.3 ([32, Lemma 4.1]). Let $\lambda \in \left(\frac{1+2\epsilon}{2+3\epsilon}, 1\right) \cup (1, +\infty]$ and L be a convex body with the origin in its interior. Then, for any non-zero non-negative continuous function $f_j : \mathbb{R}^{1+2\epsilon} \to \mathbb{R}$ with compact support,

$$\tilde{V}_{-(1+\epsilon)}(f_j, L) \geq m_{1+2\epsilon,1+\epsilon,\lambda}\|f_j\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}}\|f_j\|_\lambda^{-\frac{1+\epsilon}{1+2\epsilon}\lambda'}\mathrm{vol}(L)^{-\frac{1+\epsilon}{1+2\epsilon}} \tag{13}$$

See [32, Lemma 4.1] for the explicit value of the sharp constant $m_{1+2\epsilon,1+\epsilon,\lambda}$. Moreover, equality holds if and only if $f_j(x) = aG_{1+\epsilon,\lambda}(b\|x\|_L)$ for constants $a, b \in \mathbb{R}$.

Definition 2.4 [49]. For a convex body K we define the function $G_{1+\epsilon,\lambda,K}(x) = aG_{1+\epsilon,\lambda}(\|x\|_K)$ where a is defined by the requirement that $\tilde{V}_{-(1+\epsilon)}(G_{1+\epsilon,\lambda,K}, L) = \tilde{V}_{-(1+\epsilon)}(K, L)$ for every convex body L.

Inserting $G_{1+\epsilon,\lambda,L}$ in (11) and using the equality case, we obtain

$$m_{1+2\epsilon,1+\epsilon,\lambda}\left\|G_{1+\epsilon,\lambda,L}\right\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}}\left\|G_{1+\epsilon,\lambda,L}\right\|_\lambda^{-\frac{1+\epsilon}{1+2\epsilon}\lambda'} = \mathrm{vol}(L)^{\frac{2+3\epsilon}{1+2\epsilon}}$$

It is easy to see how the $L_{1+\epsilon}$-dual mixed volume relates to the $L_{1+\epsilon}$ Random Simplex Inequality, as

$$\tilde{V}_{-(1+\epsilon)}\left(L_{1+2\epsilon}, N_{1+\epsilon}^{\circ}(L_1, \dots, L_{2\epsilon})\right) = \frac{2+3\epsilon}{1+2\epsilon}I_{1+\epsilon}(L_1, \dots, L_{1+2\epsilon}).$$

We also have $I_{1+\epsilon}\left(G_{1+\epsilon,\lambda,L_1}, \dots, G_{1+\epsilon,\lambda,L_{1+2\epsilon}}\right) = I_{1+\epsilon}(L_1, \dots, L_{1+2\epsilon})$ and $\tilde{I}_{1+\epsilon}\left((F_j)_{1+\epsilon,L_1}, \dots, (F_j)_{1+\epsilon,L_{1+2\epsilon}}\right) = \tilde{I}_{1+\epsilon}(L_1, \dots, L_{1+2\epsilon})$.

For a convex body K with smooth boundary and strictly positive Gauss curvature, the $L_{1+\epsilon}$ surface area measure of K is absolutely continuous with respect to the invariant measure of the sphere and

$$\int_{S^{2\epsilon}} \sum_j \phi(\xi_j)dS_{1+\epsilon,K}(\xi_j) = \int_{S^{2\epsilon}} \sum_j \phi(\xi_j)(f_j)_{1+\epsilon}(K, \xi_j)d\xi_j$$

for every measurable function ϕ, where $(f_j)_{1+\epsilon}$ is called the $L_{1+\epsilon}$ curvature function. For $\epsilon = 0$ this is the inverse of the Gauss curvature. The $(1+\epsilon)$-affine surface area of K is defined by

$$\Omega_{1+\epsilon}(K)=\int_{S^{2\epsilon}}\sum_{j}\quad (f_j)_{1+\epsilon}(K,\xi_j)^{\frac{1+2\epsilon}{2+3\epsilon}}d\xi_j$$
$$=\int_{S^{2\epsilon}}\sum_{j}\quad (f_j)_{1+\epsilon}(K,\xi_j)^{-\frac{1+\epsilon}{2+3\epsilon}}dS_{1+\epsilon,K}$$

For $\epsilon=0$ we obtain the affine surface area measure

$$\Omega(K)=\int_{\partial K}\kappa(x)^{\frac{1}{2(1+\epsilon)}}dx$$

$\Omega_{1+\epsilon}$ is a centro-affine invariant (invariant under volume preserving linear transformations), but not translation invariant unless $\epsilon=0$.

For$l$ be a C^1 function and we denote the sub-level sets $N_{l,t}=\{x\in\mathbb{R}^{1+2\epsilon}/l(x)\geq t\}$. The Gauss curvature of the level set of l at a regular point of the surface $x\in\partial N_{l,t}$ can be computed taking the differential of the function $n^l(x)=\frac{\nabla l(x)}{|\nabla l(x)|}$ (provided $\nabla l(x)\neq 0$) restricted to the subspace orthogonal to the gradient. One obtains the well-known formula

$$\kappa_{\partial N_{l,t}}=\det\begin{pmatrix}0 & n^l(x)^T\\ n^l(x) & |\nabla l(x)|^{-1}Hl(x)\end{pmatrix}$$

so we have

$$|\det(Kl(x))|=\left|\kappa_{\partial N_{l,t}}\right||\nabla l(x)|^{2(1+\epsilon)}. \tag{14}$$

The next Lemma justifies that we call $\Omega_{1+\epsilon}(l)$ the $(1+\epsilon)$-affine surface area of l.

Lemma 2.5 (see [49]). Let l be a C^2 function with compact support such that for a.e. t the level set $N_{l,t}$ is a C^2-smooth convex body with positive Gauss curvature, then

$$\Omega_1(l)=\int_0^\infty\Omega_1\left(\partial N_{l,t}\right)dt$$

Assume additionally that $l(x)=F_j(\|x\|_K)$ for a C^1 function F_j, and K is a C^2-smooth convex body with positive Gauss curvature, then

$$\Omega_{1+\epsilon}(l)=\Omega_{1+\epsilon}(K)\int_0^\infty\sum_{j}\quad t^{(2\epsilon)\frac{1+2\epsilon}{2+3\epsilon}}\left|F_j'(t)\right|^{\frac{2(1+\epsilon)^2}{2+3\epsilon}}dt$$

Proof. For the first part, use the co-area formula and identity (14) to obtain

$$\Omega_1(l)=\int_{\mathbb{R}^{1+2\epsilon}}|\det(Kl(x))|^{\frac{1}{2(1+\epsilon)}}dx$$
$$=\int_0^\infty\int_{\partial N_{l,t}}|\det(Kl(x))|^{\frac{1}{2(1+\epsilon)}}|\nabla l(x)|^{-1}dSdt$$
$$=\int_0^\infty\int_{\partial N_{l,t}}\kappa_{\partial N_{l,t}}(x)^{\frac{1}{2(1+\epsilon)}}dSdt$$
$$=\int_0^\infty\Omega_1\left(\partial N_{l,t}\right)dt$$

For the second part notice that the level set of l at x is $\|x\|_K\cdot\partial K$. Integrating in the level sets of $\|x\|_K$ and using that $\nabla\|x\|_K=h_K(x/\|x\|_K)^{-1}n_{x/\|x\|_K}$ where n_y is the normal vector of K at y,

$$
\begin{aligned}
\Omega_{1+\epsilon}(l) &= \int_{\mathbb{R}^{1+2\epsilon}} |\det(Kl(x))|^{\frac{1+\epsilon}{2+3\epsilon}} dx \\
&= \int_0^\infty \int_{\partial K} \left(|\nabla l(x)|^{2(1+\epsilon)} \kappa_{t\partial K}(x)\right)^{\frac{1+\epsilon}{2+3\epsilon}} h_k(x/\|x\|_K) dS(x) dt \\
&= \int_0^\infty \int_{\partial K} \sum_j \quad t^{2\epsilon} \left(\left(F_j'(t) h_K(y)^{-1}\right)^{2(1+\epsilon)} \kappa_{\partial K}(y) t^{-(2\epsilon)}\right)^{\frac{1+\epsilon}{2+3\epsilon}} h_k(y) dS(x) dt \\
&= \int_0^\infty \int_{\partial K} \sum_j \quad t^{(2\epsilon)-(2\epsilon)\frac{1+\epsilon}{2+3\epsilon}} \left|F_j'(t)\right|^{\frac{2(1+\epsilon)^2}{2+3\epsilon}} \kappa_{\partial K}(y)^{\frac{1+\epsilon}{2+3\epsilon}} h_K(n_x)^{1-\frac{2(1+\epsilon)^2}{2+3\epsilon}} dS(x) dt \\
&= \sum_j \quad \left(\int_0^\infty t^{(2\epsilon)\frac{1+2\epsilon}{2+3\epsilon}} \left|F_j'(t)\right|^{\frac{2(1+\epsilon)^2}{2+3\epsilon}} dt\right) \int_{S^{2\epsilon}} \left(\kappa_{\partial K}(\xi_j) h_K(\xi_j)^{-\epsilon}\right)^{-\frac{1+2\epsilon}{2+3\epsilon}} dS(\xi_j) \\
&= \sum_j \quad \left(\int_0^\infty t^{(2\epsilon)\frac{1+2\epsilon}{2+3\epsilon}} \left|F_j'(t)\right|^{\frac{2(1+\epsilon)^2}{2+3\epsilon}} dt\right) \Omega_{1+\epsilon}(K).
\end{aligned}
$$

III. Functional random simplex inequality

Given functions $l_1, \dots, l_k$ and convex bodies $L_{k+1}, \dots, L_{1+2\epsilon}$, we write $I_{1+\epsilon}(l_1, \dots, l_k, L_{k+1}, \dots, L_{1+2\epsilon}) = I_{1+\epsilon}(l_1, \dots, l_{1+2\epsilon})$ where $l_i = G_{1+\epsilon,\lambda,L_i}$ for $i = k+1, \dots, (1+2\epsilon)$. This quantity is independent of λ.

Proof of Theorem 1.5 (see [49]). We prove by induction in k, the following

Claim. Let $l_1, \dots, l_k$ be continuous non-negative functions with compact support, and let $L_{k+1}, \dots, L_{2\epsilon}$ be convex bodies, then

$$
\mathrm{vol}\left(N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon})\right) \le a_{1+2\epsilon,1+\epsilon} \prod_{i=1}^{k} m_{1+2\epsilon,1+\epsilon,\lambda}^{-\frac{1+2\epsilon}{1+\epsilon}} \|l_i\|_1^{-\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+\epsilon}} \|l_i\|_\lambda^{\lambda'} \prod_{j=k+1}^{2\epsilon} \mathrm{vol}\left(L_j\right)^{-\frac{2+3\epsilon}{1+\epsilon}}
$$

and equality holds if and only if $l_1, \dots, l_k$ have the form $l_i(x) = a_i G_{1+\epsilon,\lambda}(b_i|A.x|_2)$ where $x_i \in \mathbb{R}^{1+2\epsilon}, A \in \mathrm{Gl}_{1+2\epsilon}(\mathbb{R})$, and $L_i = a_i A^{-1} \cdot B_2$ for $i \ge k+1$. Taking $k = 2\epsilon$ we obtain the Theorem with $A_{1+2\epsilon,1+\epsilon,\lambda} = a_{1+2\epsilon,1+\epsilon} m_{1+2\epsilon,1+\epsilon,\lambda}^{-\frac{(1+2\epsilon)^2}{1+\epsilon}}$.

For $k = 0$ the Claim is exactly Theorem 1.3. Assume $k \ge 1$ and that the Claim is true with k replaced by $k-1$. Given functions $l_1, \dots, l_k$ and sets $L_{k+1}, \dots, L_{2\epsilon}$ take

$$
K = N_{1+\epsilon}^\circ(l_1, \dots, l_{k-1}, N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon}), L_{k+1}, \dots, L_{2\epsilon}).
$$

Using the commutativity of $I_{1+\epsilon}$,

$$
\begin{aligned}
\tilde{V}_{-(1+\epsilon)}(l_k, K) &= \frac{2+3\epsilon}{1+2\epsilon} I_{1+\epsilon}(l_1, \dots, l_{k-1}, N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon}), L_{k+1}, \dots, L_{2\epsilon}, l_k) \\
&= \frac{2+3\epsilon}{1+2\epsilon} I_{1+\epsilon}\left(l_1, \dots, l_{k-1}, l_k, L_{k+1}, \dots, L_{2\epsilon}, N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon})\right) \\
&= \tilde{V}_{-(1+\epsilon)}\left(N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon}), N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon})\right) \\
&= \mathrm{vol}\left(N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon})\right)
\end{aligned}
$$

then by (13)

$$
\mathrm{vol}\left(N_{1+\epsilon}^\circ(l_1, \dots, l_k, L_{k+1}, \dots, L_{2\epsilon})\right) \ge m_{1+2\epsilon,1+\epsilon,\lambda} \|l_k\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}} \|l_k\|_\lambda^{-\frac{(1+\epsilon)\lambda'}{1+2\epsilon}} \mathrm{vol}(K)^{-\frac{1+\epsilon}{1+2\epsilon}}
$$

Notice that the definition of K involves only $k-1$ functions. Then by the induction hypothesis we have

$$\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(l_1,\dots,l_k,L_{k+1},\dots,L_{2\epsilon})\big) \geq m_{1+2\epsilon,1+\epsilon,\lambda}\|l_k\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}}\|l_k\|_\lambda^{-\frac{(1+\epsilon)\lambda'}{1+2\epsilon}}$$
$$\times\left(a_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{k-1} m_{1+2\epsilon,1+\epsilon,\lambda}^{-\frac{1+2\epsilon}{1+\epsilon}}\|l_i\|_1^{-\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+\epsilon}}\|l_i\|_\lambda^{\frac{(1+\epsilon)\lambda'}{1+\epsilon}}\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(l_1,\dots,l_k,L_{k+1},\dots,L_{2\epsilon})\big)^{-\frac{2+3\epsilon}{1+\epsilon}}\right.$$
$$\left.\times\prod_{j=k+1}^{2\epsilon}\mathrm{vol}(L_j)^{-\frac{2+3\epsilon}{1+\epsilon}}\right)^{-\frac{1+\epsilon}{1+2\epsilon}}$$
$$= m_{1+2\epsilon,1+\epsilon,\lambda}\|l_k\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}}\|l_k\|_\lambda^{-\frac{(1+\epsilon)\lambda'}{1+2\epsilon}}$$
$$\times\left(a_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{k-1} m_{1+2\epsilon,1+\epsilon,\lambda}^{-\frac{1+2\epsilon}{1+\epsilon}}\|l_i\|_1^{-\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+\epsilon}}\|l_i\|_\lambda^{\lambda'}\prod_{j=k+1}^{2\epsilon}\mathrm{vol}(L_j)^{-\frac{2+3\epsilon}{1+\epsilon}}\right)^{-\frac{1+\epsilon}{1+2\epsilon}}$$
$$\times\,\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(l_1,\dots,l_k,L_{k+1},\dots,L_{2\epsilon})\big)^{\frac{2+3\epsilon}{1+2\epsilon}}$$

that proves the Claim.

The equality case follows from the equality case of the induction hypothesis (the equality case of the Claim with $k-1$ functions), and the equality case of (13).

We complete the proof of Corollary 1.6 by proving the equivalence between random and isoperimetric inequalities in the functional setting.

Proof of Corollary 1.6 (see [49]). By the dual mixed volume inequality and Theorem 1.5,

$$\begin{aligned}I_{1+\epsilon}(l_1,\dots,l_{1+2\epsilon}) &= \frac{1+2\epsilon}{2+3\epsilon}\tilde{V}_{-(1+\epsilon)}\big(l_{1+2\epsilon},N_{1+\epsilon}^{\circ}(l_1,\dots,l_{2\epsilon})\big)\\ &\geq \frac{1+2\epsilon}{2+3\epsilon}m_{1+2\epsilon,1+\epsilon,\lambda}\|l_{1+2\epsilon}\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}}\|l_{1+2\epsilon}\|_\lambda^{-\frac{(1+\epsilon)\lambda'}{1+2\epsilon}}\mathrm{vol}\big(N_{1+\epsilon}^{\circ}(l_1,\dots,l_{2\epsilon})\big)^{-\frac{1+\epsilon}{1+2\epsilon}}\\ &\geq B_{1+2\epsilon,1+\epsilon,\lambda}\|l_{1+2\epsilon}\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}}\|l_{1+2\epsilon}\|_\lambda^{-\frac{(1+\epsilon)\lambda'}{1+2\epsilon}}\prod_{i=1}^{2\epsilon}\|l_i\|_1^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{1+2\epsilon}}\|l_i\|_\lambda^{-\frac{(1+\epsilon)\lambda'}{1+2\epsilon}}\end{aligned}$$

where $B_{1+2\epsilon,1+\epsilon,\lambda} = \frac{1+2\epsilon}{2+3\epsilon}m_{1+2\epsilon,1+\epsilon,\lambda}A_{1+2\epsilon,1+\epsilon,\lambda}^{-\frac{1+\epsilon}{1+2\epsilon}}$.

IV. The dual random simplex inequality

Here we prove Theorems 1.8 and 1.9. Let L be a convex body with smooth boundary and positive Gauss curvature, then it has a well defined $(1+\epsilon)$-curvature function $(f_j)_{1+\epsilon,L}$ defined as the density of the $(1+\epsilon)$-surface area measure with respect to the Lebesgue measure in the sphere, this is

$$dS_{1+\epsilon,L} = (f_j)_{1+\epsilon,L}(\xi_j)d\xi_j$$

For L as before, we consider the star body $L_{1+\epsilon}^*$ given by the radial function $r\big(L_{1+\epsilon}^*,\xi_j\big) = (f_j)_{1+\epsilon,L}(\xi_j)^{\frac{1}{2+3\epsilon}}$ and its volume given by

$$\mathrm{vol}(L_{1+\epsilon}^*) = \frac{1}{1+2\epsilon}\int_{S^{2\epsilon}}\sum_j (f_j)_{1+\epsilon,L}(\xi_j)^{\frac{1+2\epsilon}{2+3\epsilon}} = \frac{1}{1+2\epsilon}\Omega_{1+\epsilon}(L) \tag{15}$$

Now we prove Theorem 1.8.

Proof of Theorem 1.8 (see [49]). By definition of $\tilde{I}_{1+\epsilon}$ and $L_{1+\epsilon}^*$,

$$\begin{aligned}\tilde{I}_{1+\epsilon}(L_1,\cdots,L_{1+2\epsilon}) &= \int_{S^{2\epsilon}}\cdots\int_{S^{2\epsilon}}\sum_j D_{1+2\epsilon}\big((\xi_j)_1,\dots,(\xi_j)_{1+2\epsilon}\big)^{1+\epsilon}dS_{1+\epsilon,L_1}\big((\xi_j)_1\big)\dots dS_{1+\epsilon,L_{1+2\epsilon}}\big((\xi_j)_{1+2\epsilon}\big)\\ &= \int_{S^{2\epsilon}}\cdots\int_{S^{2\epsilon}}\sum_j D_{1+2\epsilon}\big((\xi_j)_1,\dots,(\xi_j)_{1+2\epsilon}\big)^{1+\epsilon}\prod_{i=1}^{1+2\epsilon}(f_j)_{1+\epsilon,L_i}\big((\xi_j)_i\big)d(\xi_j)_1\dots d(\xi_j)_{1+2\epsilon}\\ &= \int_{S^{2\epsilon}}\cdots\int_{S^{2\epsilon}}\sum_j D_{1+2\epsilon}\big((\xi_j)_1,\dots,(\xi_j)_{1+2\epsilon}\big)^{1+\epsilon}\prod_{i=1}^{1+2\epsilon}r_{(L_i)_{1+\epsilon}^*}\big((\xi_j)_i\big)^{2+3\epsilon}d(\xi_j)_1\dots d(\xi_j)_{1+2\epsilon}\end{aligned}$$

$$= (2+3\epsilon)^{1+2\epsilon}\int_{(L_1)^*_{1+\epsilon}}\cdots\int_{(L_{1+2\epsilon})^*_{1+\epsilon}} D_{1+2\epsilon}(x_1,\dots,x_{1+2\epsilon})^{1+\epsilon}dx_1\dots dx_{1+2\epsilon}$$

$$= (2+3\epsilon)^{1+2\epsilon} I_{1+\epsilon}((L_1)^*_{1+\epsilon},\cdots,(L_{1+2\epsilon})^*_{1+\epsilon}) \quad (16)$$

The sets $(L_i)^*_{1+\epsilon}$ are not necessarily convex, but Corollary 1.6 applied to indicator functions with $\lambda=\infty$ implies that Theorem 1.1 remains valid for measurable sets (notice that $m_{1+2\epsilon,1+\epsilon,\infty}=1$). By this inequality and identity (15) we get

$$\tilde{I}_{1+\epsilon}(L_1,\cdots,L_{1+2\epsilon}) \geq (2+3\epsilon)^{1+2\epsilon} b_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{1+2\epsilon}\operatorname{vol}((L_i)^*_{1+\epsilon})^{\frac{2+3\epsilon}{1+2\epsilon}} = (2+3\epsilon)^{1+2\epsilon}\frac{b_{1+2\epsilon,1+\epsilon}}{n^{2+3\epsilon}}\prod_{i=1}^{1+2\epsilon}\Omega_{1+\epsilon}(L_i)^{\frac{2+3\epsilon}{1+2\epsilon}}$$

The proof of Theorem 1.9 requires a simple lemma:

Lemma 4.1 (see [49]). Let $g_j:(0,\infty)\to[0,\infty)$ be a continuous non-negative function with compact support and $\epsilon\geq 0,\lambda\in\left(\frac{1+2\epsilon}{2+3\epsilon},1\right)\cup(1,+\infty],\lambda'=\frac{\lambda}{\lambda-1}$ then

$$\sum_j\left(\int_0^\infty g_j(t)^{\frac{2+3\epsilon}{1+2\epsilon}}dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}} \geq L_{1+2\epsilon,1+\epsilon,\lambda}\sum_j\left(\int_0^\infty g_j(t)t^{\lambda-1}dt\right)^{-\frac{1+\epsilon}{(1+2\epsilon)+(1+\epsilon)\lambda}\frac{1}{\lambda-1}}\left(\int_0^\infty g_j(t)dt\right)^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{2+3\epsilon}} \quad (17)$$

where

$$L_{1+2\epsilon,1+\epsilon,\lambda}=\begin{cases}(1-\lambda)\left(\frac{1+\epsilon}{2+3\epsilon}\right)^{\frac{1+\epsilon}{(\lambda-1)(2+3\epsilon)}}\left(\lambda-\frac{1+2\epsilon}{2+3\epsilon}\right)^{\frac{(1+2\epsilon)-\lambda(2+3\epsilon)}{(\lambda-1)(2+3\epsilon)}}\left(\frac{\Gamma\left(\frac{1}{1-\lambda}\right)}{\Gamma\left(\frac{3+4\epsilon}{1+\epsilon}\right)\Gamma\left(\frac{\lambda}{1-\lambda}-\frac{1+2\epsilon}{1+\epsilon}\right)}\right)^{\frac{1+\epsilon}{2+3\epsilon}} \\ \text{if } \lambda<1 \\ (\lambda-1)\left(\frac{1+\epsilon}{2+3\epsilon}\right)^{\frac{1+\epsilon}{(\lambda-1)(2+3\epsilon)}}\left(\lambda-\frac{1+2\epsilon}{2+3\epsilon}\right)^{\frac{(1+2\epsilon)-\lambda(2+3\epsilon)}{(\lambda-1)(2+3\epsilon)}}\left(\frac{\Gamma\left(\frac{1+2\epsilon}{1+\epsilon}+\frac{1}{\lambda-1}+2\right)}{\Gamma\left(\frac{\lambda}{\lambda-1}\right)\Gamma\left(\frac{3+4\epsilon}{1+\epsilon}\right)}\right)^{\frac{1+\epsilon}{2+3\epsilon}} \\ \text{if } \lambda>1\end{cases}$$

and equality holds if and only if $g_j(t)=ap_\lambda(t/1+\epsilon)$ for some $a,\epsilon\geq 0$ and

$$p_\lambda(t)=\begin{cases}\left(t^{\lambda-1}-1\right)_+^{\frac{1+2\epsilon}{1+\epsilon}} & \text{if } \lambda<1\\ \left(1-t^{\lambda-1}\right)_+^{\frac{1+2\epsilon}{1+\epsilon}} & \text{if } \lambda>1\\ \chi_{[0,1]}(t) & \text{if } \lambda=\infty\end{cases}$$

For $\lambda=\infty$ we interpret inequality (17) as

$$\sum_j\left(\int_0^\infty g_j(t)^{\frac{2+3\epsilon}{1+2\epsilon}}dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}} \geq \sum_j (\mathrm{S}g_j)^{-\frac{1+\epsilon}{2+3\epsilon}}\int_0^\infty g_j(t)dt$$

where $\mathrm{S}g_j$ is the minimum $a>0$ such that $g_j(t)=0$ for all $t>a$.

Proof. For $\lambda>1$ and $t>0$, let $p_\lambda(t)=\left(1-t^{\lambda-1}\right)_+^{\frac{1+2\epsilon}{1+\epsilon}}$. Then

$$p_\lambda(t/1+\epsilon)^{\frac{1+\epsilon}{1+2\epsilon}} \geq 1-t^{\lambda-1}r^{1-\lambda}$$

Multiplying by $g_j(t)$ and integrating

$$\int_0^\infty\sum_j g_j(t)p_\lambda(t/1+\epsilon)^{\frac{1+\epsilon}{1+2\epsilon}}dt \geq \sum_j\int_0^\infty g_j(t)dt - r^{1-\lambda}\int_0^\infty g_j(t)t^{\lambda-1}dt$$

By Hölder,

$$\int_0^\infty\sum_j g_j(t)dt \leq \int_0^\infty\sum_j g_j(t)p_\lambda(t/1+\epsilon)^{\frac{1+\epsilon}{1+2\epsilon}}dt + r^{1-\lambda}\int_0^\infty\sum_j g_j(t)t^{\lambda-1}dt$$

$$\leq \sum_j\left(\int_0^\infty g_j(t)^{\frac{2+3\epsilon}{1+2\epsilon}}dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}}\left(\int_0^\infty p_\lambda(t/1+\epsilon)^{\frac{2+3\epsilon}{1+2\epsilon}}dt\right)^{\frac{1+\epsilon}{2+3\epsilon}} + r^{1-\lambda}\int_0^\infty\sum_j g_j(t)t^{\lambda-1}dt$$

$$\leq \sum_j\left(\int_0^\infty g_j(t)^{\frac{2+3\epsilon}{1+2\epsilon}}dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}}\left(\int_0^\infty p_\lambda(t)^{\frac{2+3\epsilon}{1+2\epsilon}}dt\right)^{\frac{1+\epsilon}{2+3\epsilon}} r^{\frac{1+\epsilon}{2+3\epsilon}} + r^{1-\lambda}\int_0^\infty\sum_j g_j(t)t^{\lambda-1}dt.$$

Minimizing the right-hand side for $\epsilon \geq 0$, we obtain the result for $\lambda > 1$. For the case $\lambda \in \left(\frac{1+2\epsilon}{2+3\epsilon}, 1\right)$, we define $q_\lambda(t) = \left(t^{\lambda-1} - 1\right)_+^{\frac{1+2\epsilon}{1+\epsilon}}$. Then, $q_\lambda(t)^{\frac{1+\epsilon}{1+2\epsilon}} \geq t^{\lambda-1} - 1$ and it follows that

$$\int_0^\infty \sum_j \quad g_j(t) q_\lambda(t/1+\epsilon)^{\frac{1+\epsilon}{1+2\epsilon}} dt \geq r^{1-\lambda} \int_0^\infty \sum_j \quad g_j(t) t^{\lambda-1} dt - \int_0^\infty \sum_j \quad g_j(t) dt$$

By Hölder

$$\int_0^\infty \sum_j \quad g_j(t) t^{\lambda-1} dt \leq r^{\lambda-1} \int_0^\infty \sum_j \quad g_j(t) dt + r^{\lambda-1} \left(\int_0^\infty g_j(t)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}} \left(\int_0^\infty q_\lambda(t/1+\epsilon)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+\epsilon}{2+3\epsilon}}$$

$$\leq r^{\lambda-1} \sum_j \quad \int_0^\infty g_j(t) dt + r^{\lambda - \frac{1+2\epsilon}{2+3\epsilon}} \left(\int_0^\infty g_j(t)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}} \left(\int_0^\infty q_\lambda(t)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+\epsilon}{2+3\epsilon}}$$

and again we conclude minimizing with respect to $\epsilon \geq 0$. The case $\lambda = \infty$ is just an application of Hölder inequality.

Proof of Theorem 1.9 (see [49]). By hypothesis, every level dense set of each function l_i is a C^2-smooth convex body with positive Gauss curvature. For such a function $l = l_i$, the relation (10) implies that the surface area measure of l is absolutely continuous with respect to the invariant measure on $S^{2\epsilon}$. We also have a well defined $(1+\epsilon)$-curvature function $(f_j)_{1+\epsilon,l}$ such that $dS_{1+\epsilon,l} = (f_j)_{1+\epsilon,l}(\xi_j) d\xi_j$. We define the star body $(l)^*_{1+\epsilon}$ similarly, by

$$r\left((l)^*_{1+\epsilon}, \xi_j\right)^{2+3\epsilon} = (f_j)_{1+\epsilon,l}(\xi_j)$$

As in (16) we compute

$$\tilde{I}_{1+\epsilon}(l_1, \cdots, l_{1+2\epsilon}) = I_{1+\epsilon}((l_1)^*_{1+\epsilon}, \dots, (l_{1+2\epsilon})^*_{1+\epsilon}) \geq b_{1+2\epsilon,1+\epsilon} \prod_{i=1}^{1+2\epsilon} \operatorname{vol}((l_i)^*_{1+\epsilon})^{\frac{2+3\epsilon}{1+2\epsilon}}. \tag{18}$$

Fix $l = l_i$ for $i = 1, \dots 1+2\epsilon$. Consider the function $n^l : \mathbb{R}^{1+2\epsilon} \to S^{2\epsilon}$ given by $n^l(x) = \frac{\nabla l(x)}{|\nabla l(x)|_2}$. By Sard's Lemma applied to n^l, for almost every ξ_j, the set

$$P(\xi_j) = \left\{x \in \mathbb{R}^{1+2\epsilon} / \nabla l(x) \neq 0 \text{ and } n^l(x) = \xi_j\right\}$$

is a 1 -dimensional submanifold of the open dense subset of $\mathbb{R}^{1+2\epsilon}$ where $\nabla l \neq 0$. By the definition of $(f_j)_{1+\epsilon,l}$ we have for any $(1+\epsilon)$-homogeneous function ϕ on $\mathbb{R}^{1+2\epsilon}$,

$$\int_{S^{2\epsilon}} \sum_j \quad \phi(\xi_j) dS_{1+\epsilon,l} = \int_{\mathbb{R}^{1+2\epsilon}} \phi(\nabla l(x)) dx$$

$$= \int_{S^{2\epsilon}} \int_{P_l(\xi_j)} \sum_j \quad J_{n^l}(x)^{-1} \phi(\nabla l(x)) dx d\xi_j$$

$$= \int_{S^{2\epsilon}} \int_{P_l(\xi_j)} \sum_j \quad J_{n^l}(x)^{-1} |\nabla l(x)|^{1+\epsilon} dx \phi\left(\xi_j\right) d\xi_j \tag{19}$$

where we used the generalized co-area formula [36] (see also [16, Theorem 3.2.22], and J_{n^l} is the (generalized) Jacobian of the function $n^l : \mathbb{R}^{1+2\epsilon} \to S^{2\epsilon}$ given by $n^l(x) = \frac{\nabla l(x)}{|\nabla l(x)|_2}$. The Jacobian of a transformation at a point x_0 can be computed by the formula (see [36, Lemma 1.2])

$$J = \frac{D_{2\epsilon}\left(A \cdot (u_j)_1, \dots, A \cdot (u_j)_{2\epsilon}\right) \left|(u_j)_{1+2\epsilon}\right|}{D_{1+2\epsilon}\left((u_j)_1, \dots, (u_j)_{2\epsilon}, (u_j)_{1+2\epsilon}\right)}$$

where A is the differential of n^l at x_0, $(u_j)_{1+2\epsilon}$ is a vector generating the kernel of A and $(u_j)_1, \dots, (u_j)_{2\epsilon}$ is any basis of a complementary subspace to $\ker(A)$. Now, taking $(u_j)_1, \dots, (u_j)_{2\epsilon}$ to be an orthonormal basis of the orthogonal space to ξ_j, and $(u_j)_{1+2\epsilon}$ unitary, we obtain $J = \left|\kappa / \left\langle \xi_j, (u_j)_{1+2\epsilon} \right\rangle\right|$ where κ is the Gauss curvature of the level set of l at x_0. Observe that $(u_j)_{1+2\epsilon}$ is the unit tangent vector of the curve $P_l(\xi_j)$ at x_0. From (19) we obtain

$$(f_j)_{1+\epsilon,l}(\xi_j) = \int_{P_l(\xi_j)} \sum_j \quad \kappa(x)^{-1}|\nabla l(x)|^{1+\epsilon}\left|\xi_j^T\right| d\mathcal{H}_1(x) \tag{20}$$

where ξ_j^T is the orthogonal projection of ξ_j to the tangent space of the curve $P_l(\xi_j)$ at x.

Since l has convex level sets with positive Gauss curvature, each $P(\xi_j)$ can be parametrized by a curve $\gamma_\xi: (0, \|l\|_\infty) \to \mathbb{R}$ satisfying $l\left(\gamma_{\xi_j}(t)\right) = t$. We compute

$$\begin{aligned}(f_j)_{1+\epsilon,l}(\xi_j) &= \int_0^{\|l\|_\infty} \sum_j \quad \kappa\left(\gamma_{\xi_j}(t)\right)^{-1} \left|\nabla l\left(\gamma_{\xi_j}(t)\right)\right|^{1+\epsilon} \left|\left\langle \xi_j, \gamma'_{\xi_j}(t)\right\rangle\right| dt \\ &= \int_0^{\|l\|_\infty} \sum_j \quad \kappa\left(\gamma_{\xi_j}(t)\right)^{-1} \left|\nabla l\left(\gamma_{\xi_j}(t)\right)\right|^{\epsilon} \left|\left\langle \nabla l\left(\gamma_{\xi_j}(t)\right), \gamma'_{\xi_j}(t)\right\rangle\right| dt \\ &= \int_0^{\|l\|_\infty} \sum_j \quad \kappa\left(\gamma_{\xi_j}(t)\right)^{-1} \left|\nabla l\left(\gamma_{\xi_j}(t)\right)\right|^{\epsilon} dt\end{aligned}$$

By formula (20), applying the parametrization given by γ_{ξ_j}, the Minkowski integral inequality and the definition of the surface area measure we have

$$\begin{aligned}\mathrm{vol}(l_{1+\epsilon}^*) &= \int_{S^{2\epsilon}} \sum_j \quad \left(\int_0^{\|l\|_\infty} \kappa\left(\gamma_{\xi_j}(t)\right)^{-1} \left|\nabla l\left(\gamma_{\xi_j}(t)\right)\right|^{\epsilon} dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}} d\xi_j \\ &\geq \left(\int_0^{\|l\|_\infty} \sum_j \quad \left(\int_{S^{2\epsilon}} \kappa\left(\gamma_{\xi_j}(t)\right)^{-\frac{1+2\epsilon}{2+3\epsilon}} \left|\nabla l\left(\gamma_{\xi_j}(t)\right)\right|^{\frac{(\epsilon)(1+2\epsilon)}{2+3\epsilon}} d\xi_j\right)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}} \\ &= \left(\int_0^{\|l\|_\infty} \left(\int_{\partial N_{l,t}} \kappa(x)^{\frac{1+\epsilon}{2+3\epsilon}} |\nabla l(x)|^{\frac{(\epsilon)(1+2\epsilon)}{2+3\epsilon}} dS_{N_{l,t}}(x)\right)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}}\end{aligned}$$

By formula (14) we have

$$\mathrm{vol}(l_{1+\epsilon}^*) = \left(\int_0^{\|l\|_\infty} \left(\int_{\partial N_{l,t}} |\det(Kl(x))|^{\frac{1+\epsilon}{2+3\epsilon}} |\nabla l(x)|^{-1} dS_{N_{l,t}}(x)\right)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}}$$

Let us define

$$\Omega_{1+\epsilon}(l,t) = \int_{\partial N_{l,t}} |\det(Kl(x))|^{\frac{1+\epsilon}{2+3\epsilon}} |\nabla l(x)|^{-1} dS_{N_{l,t}}(x)$$

then a simple computation shows that

$$\Omega_{1+\epsilon}(l^\alpha, t^\alpha) = \Omega_{1+\epsilon}(l,t)(\alpha t^{\alpha-1})^{(\epsilon)\frac{1+2\epsilon}{2+3\epsilon}}$$

and taking $\lambda = 1 + (\alpha - 1)\frac{2(1+\epsilon)^2}{2+3\epsilon}$ and $g_j(t) = \Omega_{1+\epsilon}(l,t)$ in Lemma 4.1,

$$\mathrm{vol}(l_{1+\epsilon}^{*}) \geq \left(\int_0^\infty \Omega_{1+\epsilon}(l,t)^{\frac{2+3\epsilon}{1+2\epsilon}} dt\right)^{\frac{1+2\epsilon}{2+3\epsilon}}$$

$$\geq L_{1+2\epsilon,1+\epsilon,\lambda}\left(\int_0^\infty \Omega_{1+\epsilon}(l,t)dt\right)^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{2+3\epsilon}}\left(\int_0^\infty \Omega_{1+\epsilon}(l,t)t^{\lambda-1}dt\right)^{-\frac{1+\epsilon}{(2+3\epsilon)(\lambda-1)}}$$

$$= \alpha^{\frac{1+\epsilon}{\lambda-1}} L_{1+2\epsilon,1+\epsilon,\lambda}\left(\int_0^\infty \Omega_{1+\epsilon}(l,t)dt\right)^{\frac{(1+2\epsilon)+(1+\epsilon)\lambda'}{2+3\epsilon}}$$

$$\times\left(\int_0^\infty \Omega_{1+\epsilon}(l,t)(\alpha t^{\alpha-1})^{(\epsilon)\frac{1+2\epsilon}{2+3\epsilon}}(\alpha t^{\alpha-1})dt\right)^{-\frac{1+\epsilon}{(2+3\epsilon)(\lambda-1)}}$$

$$= \alpha^{\frac{1+\epsilon}{\lambda-1}} L_{1+2\epsilon,1+\epsilon,\lambda}\left(\int_0^\infty \Omega_{1+\epsilon}(l,t)dt\right)^{\frac{1+2\epsilon+\alpha'}{2(1+\epsilon)}}\left(\int_0^\infty \Omega_{1+\epsilon}(l^\alpha,s)ds\right)^{-\frac{1}{2(1+\epsilon)}\frac{1}{\alpha-1}}$$

$$= \alpha^{\frac{1+\epsilon}{\lambda-1}} L_{1+2\epsilon,1+\epsilon,\lambda}\Omega_{1+\epsilon}(l)^{\frac{1+2\epsilon+\alpha'}{2(1+\epsilon)}}\Omega_{1+\epsilon}(l^\alpha)^{-\frac{1}{2(1+\epsilon)}}\frac{1}{\alpha-1}$$

and we obtain the result with $\tilde{B}_{1+2\epsilon,1+\epsilon,\alpha} = b_{1+2\epsilon,1+\epsilon}\left(\frac{\alpha^{\frac{1+\epsilon}{\lambda-1}}}{1+2\epsilon}L_{1+2\epsilon,1+\epsilon,\lambda}\right)^{2+3\epsilon}$.

The equality case follows from the equality case of Theorem 1.1 in (18), the equality case of Lemma 4.1 for the function $\Omega_{1+\epsilon}(l,t)$, and the formula $\Omega_{1+\epsilon}\big(F_j(|x|_2),F_j(t)\big) = (1+2\epsilon)\omega_{1+2\epsilon}F_j'(t)^{(\epsilon)\frac{1+2\epsilon}{2+3\epsilon}}$.

V. Open problems

Here we discuss the dual inequality (6) and its relation to the body $\tilde{N}_{1+\epsilon}$ and the Petty conjecture. Let us consider the following open problem:

Problem 1 [49]. Let L be a convex body and $0 \leq \epsilon < 2\epsilon$, then

$$\tilde{I}_{1+\epsilon}(L_1,\dots,L_{1+2\epsilon}) \geq \bar{b}_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{1+2\epsilon}\mathrm{vol}(L_i)^{\frac{\epsilon}{1+2\epsilon}} \tag{21}$$

and equality holds if and only if $L_1,\dots,L_{1+2\epsilon}$ are homothetic origin-symmetric ellipsoids.

Now we show that the proof of Theorem 1.5 works for $\tilde{I}_{1+\epsilon}$ and $\tilde{N}_{1+\epsilon}$ using the mixed volume. As explained in the introduction, inequality (6) is equivalent to the isoperimetric inequality $\mathrm{vol}\left(\tilde{N}_{1+\epsilon}(L_1,\dots,L_{2\epsilon})\right) \geq \bar{a}_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{2\epsilon}\mathrm{vol}(L_i)^{\frac{\epsilon}{1+\epsilon}}$. The proof of this equivalence is similar.

Lemma 5.1 (see [49]). Assume Problem 1 holds. Let $l_1,\dots,l_{2\epsilon}$ be continuous non-negative functions with compact support, let $L_{k+1},\dots,L_{2\epsilon}$ be convex bodies and $0 \leq \epsilon < 2\epsilon$, then

$$\mathrm{vol}\left(\tilde{N}_{1+\epsilon}(l_1,\dots,l_k,L_{k+1},\dots,L_{2\epsilon})\right) \geq \bar{a}_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{k}\mathrm{cnv}_{1+2\epsilon,1+\epsilon}^{\frac{1+2\epsilon}{1+\epsilon}}\|l_i\|_{p^*}^{1+2\epsilon}\prod_{j=k+1}^{2\epsilon}\mathrm{vol}(L_j)^{\frac{\epsilon}{1+\epsilon}}$$

and equality holds if and only if $l_1,\dots,l_k$ have the form $l_i(x) = a_i(F_j)_{1+\epsilon}(b_i|A\cdot(x-x_i)|_2)$ where $x_0 \in \mathbb{R}^{1+2\epsilon}, A \in \mathrm{Gl}_{1+2\epsilon}(\mathbb{R})$, and $L_i = a_iA^{-1}.B_2$ for $i \geq k+1$.

Proof. Take $K = \tilde{N}_{1+\epsilon}\big(l_1,\dots,l_{k-1},\tilde{N}_{1+\epsilon}(l_1,\dots,l_{k-1},l_k,L_{k+1},\dots,L_{2\epsilon}),L_{k+1},\dots,L_{2\epsilon}\big)$. As in the proof of Lemma 1.5,

$$V_{1+\epsilon}(l_k,K) = \mathrm{vol}\left(\tilde{N}_{1+\epsilon}(l_1,\dots,l_k,L_{k+1},\dots,L_{2\epsilon})\right)$$

then by the functional mixed volume inequality (11) and the induction hypothesis we get

$$\mathrm{vol}\left(\tilde{N}_{1+\epsilon}(l_1,\dots,l_k,L_{k+1},\dots,L_{2\epsilon})\right) \geq \mathrm{cnv}_{1+2\epsilon,1+\epsilon}\mathrm{vol}(K)^{\frac{1+\epsilon}{1+2\epsilon}}\|l_k\|_{p^*}^{1+\epsilon}$$
$$\geq \mathrm{cnv}_{1+2\epsilon,1+\epsilon}\left(\bar{a}_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{k-1}\mathrm{cnv}_{1+2\epsilon,1+\epsilon}^{\frac{1+2\epsilon}{1+\epsilon}}\|l_i\|_{p^*}^{1+2\epsilon}\mathrm{vol}\big(N_{1+\epsilon}(l_1,\dots,l_k,L_{k+1},\dots,L_{2\epsilon})\big)^{\frac{\epsilon}{1+\epsilon}}\right.$$
$$\left.\times\prod_{j=k+1}^{2\epsilon}\mathrm{vol}(L_j)^{\frac{\epsilon}{1+\epsilon}}\right)^{\frac{1+\epsilon}{1+2\epsilon}}\|l_k\|_{p^*}^{1+\epsilon}$$

and that proves the lemma. The equality case follows from the equality case of the induction hypothesis (the equality case of Lemma 5.1 with $k-1$ functions), and the equality case of (11).

Summarizing, Problem 1 is equivalent to the following sharp inequalities

$$\tilde{I}_{1+\epsilon}(l_1,\dots,l_{1+2\epsilon}) \geq \bar{B}_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{1+2\epsilon}\|l_i\|_{p^*}^{1+\epsilon} \tag{22}$$

$$\mathrm{vol}\left(\tilde{N}_{1+\epsilon}(L_1,\dots,L_{2\epsilon})\right) \geq \bar{a}_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{2\epsilon}\mathrm{vol}(L_i)^{\frac{\epsilon}{1+\epsilon}} \tag{23}$$

$$\mathrm{vol}\left(\tilde{N}_{1+\epsilon}(l_1,\dots,l_{2\epsilon})\right) \geq \bar{A}_{1+2\epsilon,1+\epsilon}\prod_{i=1}^{1+2\epsilon}\|l_i\|_{p^*}^{1+2\epsilon} \tag{24}$$

and equality holds if and only if $L_1,\dots,L_{1+2\epsilon}$ are homothetic origin-symmetric ellipsoids and $l_1,\dots,l_{1+2\epsilon}$ have the form $l_i(x)=a_i(F_j)_{1+\epsilon}(b_i|A.(x-x_i)|)$ with $x_0\in\mathbb{R}^{1+2\epsilon}, a_i, b_i>0,\ A\in\mathrm{Gl}_{1+2\epsilon}(\mathbb{R})$. Here the constants satisfy $\bar{b}_{1+2\epsilon,1+\epsilon}=(1+2\epsilon)\bar{a}_{1+2\epsilon,1+\epsilon}^{\frac{1+\epsilon}{1+2\epsilon}}, \bar{B}_{1+2\epsilon,1+\epsilon}=(1+2\epsilon)\mathrm{cnv}_{1+2\epsilon,1+\epsilon}\bar{A}_{1+2\epsilon,1+\epsilon}^{1+\epsilon/1+2\epsilon}, \bar{A}_{1+2\epsilon,1+\epsilon}=\bar{a}_{1+2\epsilon,1+\epsilon}\mathrm{cnv}_{1+2\epsilon,1+\epsilon}^{(2\epsilon)\frac{1+2\epsilon}{1+\epsilon}}$. Inequalities (21) and (23) are equivalent, as well as inequalities (22) and (24). Also by Lemma 5.1, (23) implies (24), while the converse is obvious.

As a motivation to these problems we make two remarks, first notice that a particular case of inequality (22) is a sharp, affine invariant, $L_{1+\epsilon}$ Sobolev-like inequality

$$\sum_j\left(\int_{\mathbb{R}^{1+2\epsilon}}\cdots\int_{\mathbb{R}^{1+2\epsilon}}D_{1+2\epsilon}\left(\nabla f_j(x_1),\dots,\nabla f_j(x_{1+2\epsilon})\right)^{1+\epsilon}dx_1\cdots dx_{1+2\epsilon}\right)^{\frac{1}{(1+2\epsilon)(1+\epsilon)}} \geq \bar{B}_{1+2\epsilon,1+\epsilon}^{\frac{1}{(1+2\epsilon)(1+\epsilon)}}\sum_j\|f_j\|_{p^*} \tag{25}$$

Second, for $\epsilon=0$, we have $\frac{1}{(1+2\epsilon)!}\tilde{I}_1(L,\dots,L)=\mathrm{vol}(\Pi L)$, and inequality (21) for $L_1=\cdots=L_{1+2\epsilon}=L$ becomes Petty's conjectured inequality (7). By the Blaschke-Santaló inequality for symmetric bodies [44, (10.28)], inequality (7) is stronger than the Petty Projection inequality

$$\mathrm{vol}(\Pi^\circ L)^{-\frac{1}{1+2\epsilon}} \geq \frac{\omega_{2\epsilon}}{\omega_{1+2\epsilon}}\mathrm{vol}(L)^{\frac{2\epsilon}{1+2\epsilon}} \tag{26}$$

Zhang proved in [48] a functional version of (26) that results in a Sharp Affine Sobolev inequality

$$\left(\frac{1}{1+2\epsilon}\int_{S^{2\epsilon}}\left(\frac{1}{2}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j|\langle\nabla f_j(x),\xi_j\rangle|dx\right)^{-(1+2\epsilon)}d\xi_j\right)^{-\frac{1}{1+2\epsilon}} \geq \frac{\omega_{2\epsilon}}{\omega_{1+2\epsilon}}\sum_j\|f_j\|_{\frac{1+2\epsilon}{2\epsilon}} \tag{27}$$

where the left-hand side is $\mathrm{vol}\left(\Pi^\circ K_{f_j}\right)^{-\frac{1}{1+2\epsilon}}$ and K_{f_j} is such that $S\left(K_{f_j},\cdot\right)=S(f_j,\cdot)$. Again by the Blaschke-Santaló inequality applied to ΠK_{f_j} we obtain

$$\left(\frac{1}{1+2\epsilon}\int_{S^{2\epsilon}}\left(\frac{1}{2}\int_{\mathbb{R}^{1+2\epsilon}}\sum_j|\langle\nabla f_j(x),\xi_j\rangle|dx\right)^{-(1+2\epsilon)}d\xi_j\right)^{-\frac{1}{1+2\epsilon}}$$
$$\geq\left(\frac{1}{\omega_{1+2\epsilon}^2(1+2\epsilon)!}\int_{\mathbb{R}^{1+2\epsilon}}\cdots\int_{\mathbb{R}^{1+2\epsilon}}\sum_j D_{1+2\epsilon}\left(\nabla f_j(x_1),\dots,\nabla f_j(x_{1+2\epsilon})\right)dx_1\cdots dx_{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}} \tag{28}$$

meaning that the Sobolev-like inequality (25) for $\epsilon = 0$ is stronger and directly implies the affine Sobolev inequality of Zhang (27). One can check that the value of $\bar{B}_{1+2\epsilon,1}^{1/1+2\epsilon}$ in (25) is consistent with inequality (27). Notice that any purely analytic proof of (28) must contour (or re-prove) the Blaschke-Santaló inequality. In any case, that doesn't seem a trivial thing to do, but it might extend to the $L_{1+\epsilon}$ case:

Problem 2 [49].

$$\left(\int_{S^{2\epsilon}}\left(\int_{\mathbb{R}^{1+2\epsilon}}\sum_j \quad |\langle\nabla f_j(x),\xi_j\rangle|^{1+\epsilon}dx\right)^{-\frac{1+2\epsilon}{1+\epsilon}} d\xi_j\right)^{-\frac{1}{1+2\epsilon}}$$

$$\geq e_{1+2\epsilon,1+\epsilon}\left(\int_{\mathbb{R}^{1+2\epsilon}}\cdots\int_{\mathbb{R}^{1+2\epsilon}}\sum_j \quad D_{1+2\epsilon}\left(\nabla f_j(x_1),\ldots,\nabla f_j(x_{1+2\epsilon})\right)^{1+\epsilon}dx_1\cdots dx_{1+2\epsilon}\right)^{\frac{1}{(1+2\epsilon)(1+\epsilon)}}$$

Lastly, to this moment we don't know whether Problem 1 for $\epsilon > 0$ can be reduced to the particular case where $L_1 = \cdots = L_{1+2\epsilon} = L$. We believe these problems are worth to consider since they might shed some light into an important open problem that resisted the symmetrization approach.

References

[1] A. Alvino, V. Ferone, G. Trombetti, P. Lions, Convex Symmetrization and Applications, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. 14, Elsevier, 1997, pp. 275-293.

[2] A. Baernstein II, M. Loss, Some conjectures about L_p norms of k-plane transforms, Rend. Semin. Mat. Fis. Milano 67 (1) (1997) 9-26.

[3] I. Bárány, Random polytopes in smooth convex bodies, Mathematika 39 (1) (1992) 81-92.

[4] W. Blaschke, Vorlesungen Uber Differential Geometrie II: Affine Differentialgeometrie, 1923.

[5] J. Bourgain, M. Meyer, V. Milman, A. Pajor, On a geometric inequality, in: Geometric Aspects of Functional Analysis, Springer, 1988, pp. 271-282.

[6] H.J. Brascamp, E.H. Lieb, J.M. Luttinger, A general rearrangement inequality for multiple integrals, J. Funct. Anal. 17 (2) (1974) 227-237.

[7] H. Busemann, Volume in terms of concurrent cross-sections, Pac. J. Math. 3 (1) (1953) 1-12.

[8] U. Caglar, M. Fradelizi, O. Guédon, J. Lehec, C. Schütt, E. Werner, Functional versions of L_p-affine surface area and entropy inequalities, Int. Math. Res. Not. 2016 (4) (2015) 1223-1250.

[9] S. Campi, A. Colesanti, P. Gronchi, A Note on Sylvester's Problem for Random Polytopes in a Convex Body, 1999.

[10] S. Campi, P. Gronchi, The L_p-Busemann-Petty centroid inequality, Adv. Math. 167 (1) (2002) 128-141.

[11] S. Campi, P. Gronchi, Extremal convex sets for Sylvester-Busemann type functionals, Appl. Anal. 85 (1-3) (2006) 129-141.

[12] S. Campi, P. Gronchi, Volume inequalities for sets associated with convex bodies, in: Integral Geometry and Convexity, 2006, pp. 1-15.

[13] M. Christ, Estimates for the k-plane transform, Indiana Univ. Math. J. 33 (6) (1984) 891-910.

[14] D. Cordero-Erausquin, B. Nazaret, C. Villani, A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities, Adv. Math. 182 (2) (2004) 307-332.

[15] S. Dann, G. Paouris, P. Pivovarov, Bounding marginal densities via affine isoperimetry, Proc. Lond. Math. Soc. (3) 113 (2) (2016) 140-162.

[16] H. Federer, Geometric Measure Theory, Springer, 2014.

[17] M. Fradelizi, M. Meyer, Some functional forms of Blaschke-Santaló inequality, Math. Z. 256 (2) (2007) 379-395.

[18] Y. Gordon, M. Junge, Volume formulas in L_p-spaces, Positivity 1 (1) (1997) 7-43.

[19] H. Groemer, On some mean values associated with a randomly selected simplex in a convex set, Pac. J. Math. 45 (2) (1973) 525-533.

[20] H. Groemer, On the mean value of the volume of a random polytope in a convex set, Arch. Math. (Basel) 25 (1) (1974) 86-90.

[21] J. Haddad, C.H. Jiménez, M. Montenegro, Sharp affine Sobolev type inequalities via the L_p Busemann-Petty centroid inequality, J. Funct. Anal. 271 (2) (2016) 454-473.

[22] J. Haddad, C.H. Jiménez, M. Montenegro, Asymmetric Blaschke-Santaló functional inequalities, J. Funct. Anal. 278 (2) (2020) 108319.

[23] J. Haddad, C.H. Jiménez, L.A. da Silva, An L_p-functional Busemann-Petty centroid inequality, Int. Math. Res. Not. 01 (2020) rnz392.

[24] M. Hartzoulaki, G. Paouris, Quermassintegrals of a random polytope in a convex body, Arch. Math. (Basel) 80 (4) (2003) 430-438.

[25] K. Leichtweiss, Über ein formel Blaschkes zur affinoberfläche, Studia Sci. Math. Hung. 21 (1986) 453-474.

[26] E. Lutwak, Mixed projection inequalities, Trans. Am. Math. Soc. 287 (1) (1985) 91-105.

[27] E. Lutwak, On some affine isoperimetric inequalities, J. Differ. Geom. 23 (1) (1986) 1-13.

[28] E. Lutwak, On a conjectured projection inequality of Petty, Contemp. Math. 113 (1) (1990) 171-181.

[29] E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, J. Differ. Geom. 38 (1) (1993) 131-150.

[30] E. Lutwak, The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas, Adv. Math. 118 (2) (March 1996) 244-294.

[31] E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities, J. Differ. Geom. 56 (1) (2000) 111-132.

[32] E. Lutwak, D. Yang, G. Zhang, Moment-entropy inequalities, Ann. Probab. 32 (1B) (2004) 757-774.

[33] E. Lutwak, D. Yang, G. Zhang, Optimal Sobolev norms and the L_p Minkowski problem, Int. Math. Res. Not. 2006 (2006).
[34] E. Lutwak, G. Zhang, Blaschke-Santaló inequalities, J. Differ. Geom. 47 (1) (1997) 1-16.
[35] M. Meyer, E. Werner, On the p-affine surface area, Adv. Math. 152 (2) (2000) 288-313.
[36] L.I. Nicolaescu, The coarea formula, in: Seminar Notes, Citeseer, 2011.
[37]G. Paouris, On the existence of supergaussian directions on convex bodies, Mathematika 58 (2) (2012) 389-408.
[38]G. Paouris, P. Pivovarov, A probabilistic take on isoperimetric-type inequalities, Adv. Math. 230 (3) (2012) 1402-1422.
[39] G. Paouris, P. Pivovarov, Randomized isoperimetric inequalities, in: Convexity and Concentration, Springer, 2017, pp. 391-425.
[40] C.M. Petty, Projection bodies, in: Proc. Colloq. Convexity, Copenhagen, 1965, pp. 234-241.
[41] R.E. Pfiefer, The extrema of geometric mean values. ProQuest LLC, PhD thesis, University of California, Davis, Ann Arbor, MI, 1982.
[42] R.E. Pfiefer, Maximum and minimum sets for some geometric mean values, J. Theor. Probab. 3 (2) (1990) 169-179.
[43] C. Saroglou, Characterizations of extremals for some functionals on convex bodies, Can. J. Math. 62 (6) (2010) 1404-1418.
[44] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, vol. 151, Cambridge University Press, 2014.
[45] R. Schneider, W. Weil, Stochastic and Integral Geometry, Springer Science & Business Media, 2008.
[46] A. Stancu, E. Werner, New higher-order equiaffine invariants, Isr. J. Math. 171 (1) (2009) 221-235.
[47] W. Wang, G. Leng, Some affine isoperimetric inequalities associated with L_p-affine surface area, Houst. J. Math. 34 (2) (2008) 443-453.
[48] G. Zhang, The affine Sobolev inequality, J. Differ. Geom. 53 (1) (1999) 183-202.
[49] J. Haddad, A convex body associated to the Busemann random simplex inequality and the Petty conjecture, Journal of Functional Analysis 281 (2021) 1-28.