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Research Paper

Automorphism Groups of Finite Semigroups and Monoids: Symmetry, Invariance, and Structural Transformations

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Abstract
Automorphism groups provide a systematic means of studying symmetry within finite semigroups and monoids. An automorphism preserves multiplication while permuting the elements of the algebraic structure, and therefore retains all properties that can be defined intrinsically from the operation. This paper examines the construction, structural action, and computation of automorphism groups of finite semigroups and monoids. Particular attention is given to invariant subsets, Green's relations, idempotents, ideals, maximal subgroups, regular elements, index-period data, and the group of units. Inner automorphisms induced by units are distinguished from outer automorphisms, and their relationship with normalizers and centralizers is established. Important families are analysed, including finite groups, semilattices, left-zero and right-zero semigroups, rectangular bands, full transformation monoids, symmetric inverse monoids, and Rees matrix semigroups. The automorphism groups of full transformation and symmetric inverse monoids are shown to arise from permutations of their underlying sets. A structural procedure for computing automorphism groups from multiplication tables and transformation representations is also presented. Applications to semigroup classification, isomorphism testing, characteristic substructures, endomorphism monoids, finite automata, and symmetry reduction demonstrate that automorphism groups are central invariants in finite semigroup theory.

Keywords: *automorphism group, finite semigroup, finite monoid, inner automorphism, transformation monoid, Green's relations, Rees matrix semigroup, structural symmetry.*

I. Introduction

An automorphism of an algebraic structure is an isomorphism from the structure onto itself. It represents a symmetry that may change the names or positions of elements while leaving all algebraic relations unchanged. For groups, automorphism groups have long been used to identify characteristic subgroups, distinguish isomorphism types, and measure internal symmetry. In semigroup theory, the absence of cancellation and inverses makes the automorphism problem more varied because the multiplication operation may encode ideals, idempotents, nilpotent elements, partial symmetries, order relations, and group-like components simultaneously.

The structural theory developed by Clifford and Preston provides the standard algebraic foundation for studying homomorphisms, ideals, regularity, and transformations in semigroups [1]. Green's relations make this structure more explicit by classifying elements according to their generated left, right, and two-sided ideals [2]. Since an automorphism preserves multiplication, it necessarily preserves Green's relations and permutes their equivalence classes. Automorphism groups consequently act not only on the elements of a semigroup but also on its entire ideal and subgroup structure.

Finite semigroups are especially suitable for automorphism analysis. Their multiplication tables are finite relational objects, their elements possess finite index-period sequences, and their Green classes, idempotents, and maximal subgroups can be computed exactly. Standard treatments of finite and regular semigroups show that such invariants frequently divide a semigroup into structurally distinguishable subsets before any explicit automorphism search is undertaken [3].

The concept of an automorphism invariant was studied systematically by Barit, who emphasized that subsets and relations defined from semigroup multiplication must be preserved under every automorphism [4]. For transformation semigroups, a particularly strong phenomenon occurs: automorphisms are often induced by permutations of the underlying set. Sullivan established broad results showing that many natural transformation semigroups possess only such inner automorphisms [5].

More general theoretical descriptions have been developed through strong representations of semigroups in endomorphism monoids. These methods connect the automorphism group of a semigroup with the normalizer of its represented image and provide unified results for transformation and inverse semigroups [6]. Computationally, specialized algorithms now determine automorphism groups by exploiting Green's relations, simple components, transformation representations, and graph-automorphism techniques rather than examining all permutations of the underlying set [7].

The objectives of this paper are to explain the algebraic structure of automorphism groups of finite semigroups and monoids, identify the principal invariants preserved by their actions, classify automorphisms of several major finite families, and describe practical computational methods and applications.

II. Automorphisms of Semigroups and Monoids

Let S be a semigroup. A mapping

$$\phi: S \to S$$

is an automorphism if it is bijective and satisfies

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in S$.

The set of all automorphisms of S is denoted by

$$Aut(S).$$

Under composition of mappings, this set forms a group. The identity mapping is an automorphism, the composition of two automorphisms is an automorphism, and the inverse of an automorphism is again multiplicative.

If S is finite, then every automorphism is a permutation of its elements. Thus,

$$Aut(S) \leq Sym(S),$$

where $Sym(S)$ is the full symmetric group on the set S. Consequently,

$$|Aut(S)| \mid |S|!.$$

The automorphism group is generally much smaller than the full symmetric group because its permutations must preserve every product in the multiplication table.

For a monoid M with identity 1, every automorphism fixes the identity. Indeed, for every $x \in M$,

$$\phi(1)\phi(x) = \phi(1x) = \phi(x)$$

and

$$\phi(x)\phi(1) = \phi(x1) = \phi(x).$$

Since ϕ is surjective, $\phi(1)$ acts as an identity on every element of M. By uniqueness of the identity,

$$\phi(1) = 1.$$

Similarly, if a semigroup has a zero element 0, then

$$\phi(0) = 0,$$

because the zero is uniquely characterized by

$$0x = x0 = 0$$

for every $x \in S$.

III. Structural Invariants Preserved by Automorphisms

3.1 Idempotents and Regular Elements

Let

$$E(S) = \{e \in S: e^2 = e\}$$

be the set of idempotents of S. If $e \in E(S)$, then

$$\phi(e)^2 = \phi(e^2) = \phi(e).$$

Therefore,

$$\phi\big(E(S)\big) = E(S).$$

The set of idempotents is consequently a characteristic subset of every semigroup.

Regularity is also preserved. If a is regular, there exists $x \in S$ such that

$$axa = a.$$

Applying ϕ gives

$$\phi(a)\phi(x)\phi(a) = \phi(a).$$

Thus $\phi(a)$ is regular. Automorphisms therefore permute regular elements and preserve the regular and nonregular parts of a finite semigroup.

In an inverse semigroup, the inverse of an element is unique. Hence,

$$\phi(a^{-1}) = \phi(a)^{-1}.$$

The natural partial order on an inverse semigroup is defined by

$$a \leq b \Leftrightarrow a = eb$$

for some idempotent e. Since automorphisms preserve multiplication and idempotents, they are also order automorphisms of this natural partial order.

3.2 Green's Relations

For $a, b \in S$, Green's relations are defined through principal ideals:

$$a\,L\,b \Leftrightarrow S^1a = S^1b,$$

$$a\,R\,b \Leftrightarrow aS^1 = bS^1,$$

and

$$a\,J\,b \Leftrightarrow S^1aS^1 = S^1bS^1.$$

An automorphism satisfies

$$\phi(S^1a) = S^1\phi(a),$$

$$\phi(aS^1) = \phi(a)S^1,$$

and

$$\phi(S^1aS^1) = S^1\phi(a)S^1.$$

It follows that

$$a\,L\,b \Leftrightarrow \phi(a)\,L\,\phi(b),$$

with analogous equivalences for R, H, D, and J.

Every automorphism therefore induces permutations of the Green classes. These induced permutations preserve class sizes, the incidence of L- and R-classes, the partial order of J-classes, and the regularity of each class.

If an H-class H_e contains an idempotent e, then it is a maximal subgroup. An automorphism maps it to

$$\phi(H_e) = H_{\phi(e)},$$

and its restriction defines a group isomorphism

$$H_e \cong H_{\phi(e)}.$$

Hence automorphisms may permute maximal subgroups only when those subgroups have compatible group structures and occupy equivalent structural positions.

3.3 Ideals and Minimal Components

If I is an ideal of S, then

$$SI \cup IS \subseteq I.$$

Applying an automorphism shows that $\phi(I)$ is also an ideal. Thus the automorphism group acts on the lattice of ideals of S.

A finite semigroup has a unique minimal ideal, usually denoted by $K(S)$. Since automorphisms preserve ideals and inclusion,

$$\phi(K(S)) = K(S).$$

The minimal ideal is therefore characteristic. If it is completely simple or completely 0-simple, its group components, Green classes, and sandwich-matrix structure impose strong restrictions on the automorphism group of the whole semigroup.

3.4 Powers, Index, and Period

For an element a of a finite semigroup, the sequence

$$a, a^2, a^3, \ldots$$

is eventually periodic. There exist least positive integers m and r such that

$$a^{m+r} = a^m.$$

The integer m is the index and r is the period of a. Since

$$\phi(a^k) = \phi(a)^k,$$

the elements a and $\phi(a)$ have the same index and period.

The ordered pair

$$(ind(a), per(a))$$

is therefore an automorphism invariant. In practical calculations, elements with distinct index-period pairs cannot lie in the same orbit of the automorphism group.

Other useful invariants include the sizes of principal ideals, the number of left and right identities associated with an element, the number of solutions to equations such as

$$xy = a,$$

and the cardinalities of centralizers.

IV. Inner Automorphisms of Monoids

Let M be a monoid and let $U(M)$ denote its group of units. For every $u \in U(M)$, define

$$\iota_u(x) = uxu^{-1}.$$

Then

$$\iota_u(xy) = uxyu^{-1} = uxu^{-1}uyu^{-1},$$

so

$$\iota_u(xy) = \iota_u(x)\iota_u(y).$$

Thus ι_u is an automorphism of M. Such automorphisms are called inner automorphisms.

The assignment

$$\Theta: U(M) \to Aut(M)$$

defined by

$$\Theta(u) = \iota_u$$

is a group homomorphism. Its kernel consists of the units commuting with every element of M:

$$ker\Theta = C_{U(M)}(M) = \{u \in U(M): ux = xu \ for \ all \ x \in M\}.$$

Consequently,

$$Inn(M) \cong \frac{U(M)}{C_{U(M)}(M)}.$$

When M is a group G, this reduces to the familiar formula

$$Inn(G) \cong \frac{G}{Z(G)}.$$

An automorphism not belonging to $Inn(M)$ is called an outer automorphism. Unlike the group case, the term “inner” for transformation semigroups is also commonly used for automorphisms induced by conjugation by a permutation of the underlying set, even when that permutation does not belong to the semigroup itself.

V. Transformation Semigroups and Normalizers

Let X be a finite set and let T_X denote the full transformation monoid on X. Suppose that

$$S \leq T_X.$$

The normalizer of S in the symmetric group $Sym(X)$ is

$$N_{Sym(X)}(S) = \{\pi \in Sym(X): \pi^{-1} S \pi = S\}.$$

Every π in this normalizer induces an automorphism

$$\alpha \mapsto \pi^{-1} \alpha \pi.$$

The kernel of this conjugation action is the centralizer

$$C_{Sym(X)}(S) = \{\pi \in Sym(X): \pi\alpha = \alpha\pi \ for \ all \ \alpha \in S\}.$$

Therefore, the group of automorphisms induced by permutations of X is isomorphic to

$$\frac{N_{Sym(X)}(S)}{C_{Sym(X)}(S)}.$$

For many transformation semigroups, every automorphism arises in this manner. Fitzpatrick and Symons showed that suitable large families of transformations force automorphisms to be induced by permutations [8]. Levi subsequently proved broad innerness results for normal transformation semigroups and identified conditions under which all automorphisms are obtained by conjugation [9]. For finite transformation semigroups with sufficiently large inner automorphism groups, this phenomenon is especially rigid [10].

VI. Automorphisms of Classical Finite Monoids

6.1 Full Transformation Monoids

Let

$$X_n = \{1, 2, \dots, n\}$$

and let T_n be the monoid of all functions from X_n to itself. Its group of units is the symmetric group S_n.

For each $x \in X_n$, let c_x denote the constant transformation with image $\{x\}$. The constant transformations form the minimal ideal of T_n, and this ideal is preserved by every automorphism.

Suppose that $\phi \in Aut(T_n)$. Since ϕ permutes the constant transformations, there is a permutation $\pi \in S_n$ such that

$$\phi(c_x) = c_{x\pi}.$$

Using the identity

$$c_x\alpha = c_{x\alpha},$$

one obtains

$$\phi(c_x\alpha) = \phi(c_x)\phi(\alpha).$$

Therefore,

$$c_{(x\alpha)\pi} = c_{(x\pi)\phi(\alpha)}.$$

It follows that

$$(x\alpha)\pi = (x\pi)\phi(\alpha)$$

for every x, and hence

$$\phi(\alpha) = \pi^{-1}\alpha\pi.$$

Thus every automorphism of T_n is inner in the transformation-semigroup sense, and

$$Aut(T_n) \cong S_n.$$

The result illustrates how a small characteristic subset—the constant maps—can determine the action of an automorphism on the whole monoid [5].

6.2 Symmetric Inverse Monoids

The symmetric inverse monoid I_n consists of all partial bijections of X_n. Its idempotents are precisely the partial identity mappings

$$1_A: A \to A,$$

where $A \subseteq X_n$.

These idempotents form a Boolean semilattice isomorphic to

$$(P(X_n),\cap).$$

The atoms of this semilattice are the singleton identities

$$1_{\{x\}}.$$

Every automorphism must permute these atoms and therefore induces a permutation $\pi \in S_n$. The action of a partial bijection on singleton idempotents determines the partial bijection itself. It follows that every automorphism is conjugation by a permutation of X_n, giving

$$Aut(I_n) \cong S_n.$$

The innerness of automorphisms of symmetric inverse semigroups originates in the classical work of Liber [11]. The structural theory of T_n, I_n, and related partial transformation monoids is developed systematically by Ganyushkin and Mazorchuk [12].

6.3 Finite Groups

A group considered as a monoid has only one idempotent, namely its identity. A semigroup automorphism of a group fixes this identity and preserves inverses because

$$\phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = 1.$$

Hence its semigroup automorphisms are exactly its group automorphisms:

$$Aut_{Sgp}(G) = Aut_{Grp}(G).$$

For a cyclic group C_n,

$$Aut(C_n) \cong U(n),$$

where $U(n)$ is the multiplicative group of units modulo n. Therefore,

$$|Aut(C_n)| = \varphi(n),$$

where φ is Euler's totient function.

6.4 Semilattices

Let E be a finite meet-semilattice, regarded as a semigroup under

$$xy = x \wedge y.$$

The partial order is recovered from multiplication by

$$x \leq y \Leftrightarrow xy = x.$$

Consequently, semigroup automorphisms of E are exactly its order automorphisms.

For a finite chain, every order automorphism fixes each element, and therefore its automorphism group is trivial. By contrast, the Boolean semilattice

$$(P(X_n),\cap)$$

has automorphism group S_n, since every automorphism is determined by a permutation of its atoms.

6.5 Left-Zero, Right-Zero, and Null Semigroups

In a left-zero semigroup,

$$xy = x$$

for all x, y. Every permutation preserves this multiplication. Hence a left-zero semigroup of order n has

$$Aut(S) \cong S_n.$$

The same conclusion holds for a right-zero semigroup, where

$$xy = y.$$

Consider a finite null semigroup

$$N = \{0, a_1, \dots, a_m\}$$

with

$$xy = 0$$

for all $x, y \in N$. Every automorphism fixes 0, while the remaining elements may be permuted arbitrarily. Therefore,

$$Aut(N) \cong S_m.$$

These examples show that very weak multiplication laws can produce large automorphism groups.

6.6 Rectangular Bands

A rectangular band has the form

$$B = I \times \Lambda$$

with multiplication

$$(i, \lambda)(j, \mu) = (i, \mu).$$

Let $\sigma \in Sym(I)$ and $\tau \in Sym(\Lambda)$. Then

$$(i, \lambda) \mapsto \big(\sigma(i), \tau(\lambda)\big)$$

is an automorphism. Conversely, Green's R-classes are indexed by I, while the L-classes are indexed by Λ. Every automorphism must independently permute these classes. Thus,

$$Aut(B) \cong Sym(I) \times Sym(\Lambda).$$

VII. Automorphisms of Rees Matrix Semigroups

Completely simple semigroups are represented by Rees matrix semigroups, following Rees's structure theorem [13]. Let

$$S = M[G; I, \Lambda; P],$$

where G is a group and

$$P = (p_{\lambda i})$$

is a sandwich matrix over G. Multiplication is defined by

$$(i, g, \lambda)(j, h, \mu) = \left(i, gp_{\lambda j}h, \mu\right).$$

An automorphism must permute the R-classes indexed by I, permute the L-classes indexed by Λ, and induce an automorphism of the maximal subgroup G.

More precisely, compatible automorphisms are described by:

- a group automorphism $\theta \in Aut(G)$;
- permutations $\sigma \in Sym(I)$ and $\tau \in Sym(\Lambda)$;
- elements $u_i, v_\lambda \in G$;

satisfying

$$p_{\tau(\lambda),\sigma(i)} = v_\lambda \theta(p_{\lambda i}) u_i$$

for all $i \in I$ and $\lambda \in \Lambda$.

The associated automorphism has the form

$$(i, g, \lambda) \mapsto \left(\sigma(i), u_i^{-1}\theta(g)v_\lambda^{-1}, \tau(\lambda)\right).$$

Thus the automorphism group is controlled by the stabilizer of the sandwich matrix under group automorphisms, row and column permutations, and group-valued row-column transformations.

This description demonstrates that the automorphism group of a completely simple semigroup combines three types of symmetry:

1. internal symmetries of its maximal subgroup;
2. permutations of structurally equivalent left and right ideals;
3. symmetries of the sandwich matrix connecting the group components.

VIII. Orbits and Stabilizers

The automorphism group acts on the underlying set of a finite semigroup by

$$(\phi, a) \mapsto \phi(a).$$

The orbit of $a \in S$ is

$$Orb(a) = \{\phi(a) : \phi \in Aut(S)\}.$$

Elements in the same orbit possess identical automorphism invariants. In particular, they have the same index, period, Green-class sizes, regularity status, principal-ideal sizes, and local subgroup type.

The stabilizer of a is

$$Stab(a) = \{\phi \in Aut(S) : \phi(a) = a\}.$$

By the orbit-stabilizer theorem,

$$|Orb(a)| = \frac{|Aut(S)|}{|Stab(a)|}.$$

Automorphism orbits provide a more refined partition than many elementary invariants. Two elements may have identical Green and index-period data but still belong to different automorphism orbits because they occupy inequivalent positions in the complete multiplication table.

IX. Computational Determination

Let $S = s_1, \ldots, s_n$ be a finite semigroup with multiplication table. A direct algorithm could examine all (n!) permutations and retain those satisfying

$$\phi(s_i s_j) = \phi(s_i)\phi(s_j).$$

This method becomes infeasible even for moderately large n.

A more effective procedure begins by partitioning the elements according to readily computable invariants:

1. identity, zero, and idempotent status;
2. regularity and inverse counts;
3. index-period pairs;
4. sizes of principal left, right, and two-sided ideals;
5. Green-class types and sizes;
6. numbers of factorizations;
7. centralizer sizes;
8. maximal subgroup isomorphism types.

An automorphism must preserve every block of this refined partition. Backtracking can then be restricted to permutations within compatible blocks.

The multiplication table can also be represented as a finite relational structure with ternary relation

$$R = \{(a, b, c) \in S^3 : ab = c\}.$$

The automorphism group of the semigroup is exactly the automorphism group of the relational structure

$$(S, R).$$

By converting this ternary relation into a coloured graph, finite graph-automorphism algorithms can be applied.

Araújo, Bünau, Mitchell, and Neunhöffer developed a specialized method for calculating automorphism groups of finite transformation semigroups [7]. Their approach uses structural information and a dedicated procedure for finite simple semigroups. It was applied to classify automorphism groups of all semigroups of order at most seven and to investigate automorphism groups arising from other finite algebraic constructions.

X. Examples

Finite semigroup or monoid	Order	Automorphism group
Left-zero semigroup L_n	n	S_n
Right-zero semigroup R_n	n	S_n
Null semigroup with m nonzero elements	m + 1	S_m
Chain semilattice of order n	n	Trivial group
Boolean semilattice on n atoms	2^n	S_n
Rectangular band I × Λ	|I||Λ|	Sym(I) × Sym(Λ)
Full transformation monoid T_n	n^n	S_n
Symmetric inverse monoid I_n	Σ (n choose r)^{2}r!, $0 \le r \le n$	S_n
Cyclic group C_n	n	U(n)

The table illustrates that the size of a semigroup does not directly determine the size of its automorphism group. A chain semilattice is rigid, whereas a left-zero semigroup of the same order admits every permutation as an automorphism.

XI. Applications

11.1 Isomorphism Classification

The automorphism group is an isomorphism invariant. If

$$S \cong T,$$

then

$$Aut(S) \cong Aut(T).$$

Nonisomorphic automorphism groups therefore prove that two semigroups are nonisomorphic, although isomorphic automorphism groups do not necessarily imply isomorphic semigroups.

If one isomorphism

$$f: S \to T$$

is known, then every other isomorphism from S to T is obtained by composing f with an automorphism of S. Hence,

$$Isom(S,T) = \{f \circ \alpha : \alpha \in Aut(S)\}.$$

The number of isomorphisms between two isomorphic finite semigroups is therefore

$$|Aut(S)|.$$

11.2 Characteristic Substructures

A subset $A \subseteq S$ is characteristic if

$$\phi(A) = A$$

for every $\phi \in Aut(S)$. The identity, zero, idempotents, units, minimal ideal, regular elements, and center are characteristic.

Characteristic substructures are valuable because they are independent of a particular labeling of the semigroup. They may be used to construct canonical decompositions and to restrict possible isomorphisms.

11.3 Endomorphism Monoids of Relational Structures

Let A be a finite graph, ordered set, relational system, or algebraic structure. Its endomorphisms form a monoid

$$End(A).$$

Automorphisms of A act on this monoid by conjugation. Under appropriate hypotheses, the complete automorphism group of the endomorphism monoid can be recovered through a normalizer in the symmetric group. Araújo and Konieczny developed this method for endomorphism monoids of relational systems [14].

Related investigations of restricted transformation monoids show that order, partitions, equivalence relations, and other combinatorial structures constrain the normalizing permutations and hence the automorphism group [15].

11.4 Finite Automata and Transition Monoids

The transformations induced by input words of a finite deterministic automaton form its transition monoid. Relabeling the states by a permutation conjugates the transition transformations. The automorphism group of the transition monoid therefore identifies multiplication-preserving state symmetries and may reduce the number of structurally distinct transformations requiring analysis.

11.5 Symmetry Reduction

Suppose a family of subsemigroups, generating sets, congruences, or representations is being enumerated. The automorphism group acts on this family. Objects in the same orbit are structurally equivalent and need not be considered separately.

Burnside's lemma gives the number of equivalence classes:

$$N = \frac{1}{|Aut(S)|} \sum_{\phi \in Aut(S)} |Fix(\phi)|.$$

This principle is useful in computational classification, where symmetry reduction can substantially decrease the number of candidate structures.

XII. Conclusion

The automorphism group of a finite semigroup or monoid records the complete collection of multiplication-preserving symmetries of the structure. Its action preserves the identity, zero, idempotents, regular elements, index-period data, ideals, Green's relations, maximal subgroups, and the group of units. These invariant features provide both theoretical information and practical restrictions for automorphism computation.

The examples considered reveal several distinct patterns. Weak multiplication structures, such as left-zero and right-zero semigroups, may admit full symmetric groups as their automorphism groups. Ordered structures such as finite chains may be rigid. Full transformation and symmetric inverse monoids have automorphism groups isomorphic to the symmetric group on the underlying set because their automorphisms are induced by conjugating transformations with permutations. Completely simple semigroups display a more composite form of symmetry involving maximal-subgroup automorphisms, permutations of Green classes, and stabilizers of sandwich matrices.

Computationally, automorphism determination is transformed from an unrestricted permutation problem into a structured search using algebraic invariants, Green classes, characteristic components, and graph-automorphism methods. These techniques support isomorphism testing, classification, enumeration, and symmetry reduction.

Automorphism groups therefore occupy a central position in finite semigroup theory. They connect algebraic structure with group actions, transformation theory, combinatorics, computation, and the study of endomorphism monoids, providing a precise mathematical framework for symmetry, invariance, and structural transformation.

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