



Research Paper

Random Rectangular Graphs: A Generalization of Random Geometric Graphs with Theoretical Analysis

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Abstract

Random Geometric Graphs (RGGs) constitute one of the most important probabilistic models for studying spatially embedded networks. However, the classical assumption that vertices are uniformly distributed within a unit square limits the applicability of these models to many real-world systems whose deployment regions are rectangular rather than square. This paper introduces a generalized mathematical framework for Random Rectangular Graphs (RRGs) by extending the classical Random Geometric Graph model to rectangles of constant area and arbitrary aspect ratio.

The proposed framework establishes explicit analytical relationships among the rectangle aspect ratio, communication radius, expected degree, edge probability, connectivity threshold, clustering coefficient, graph density, and average shortest-path length. Several new theoretical results are derived, including analytical bounds for the expected degree, asymptotic connectivity conditions, and geometric correction factors describing the influence of anisotropic domains on network topology. The analysis further demonstrates that the proposed model naturally recovers the classical Random Geometric Graph when the aspect ratio equals one and approaches one-dimensional geometric graphs as the rectangle becomes increasingly elongated.

Numerical simulations are presented to validate the theoretical analysis and illustrate the effect of geometric elongation on connectivity and communication efficiency. The simulation results agree closely with the derived theoretical expressions. The proposed model provides a unified probabilistic framework for studying geometric random graphs over anisotropic domains and offers practical applications in wireless sensor networks, transportation systems, environmental monitoring, pipeline surveillance, intelligent infrastructure, and spatial communication networks.

Keywords: Random Rectangular Graphs; Random Geometric Graphs; Spatial Networks; Random Graph Theory; Connectivity; Geometric Probability; Complex Networks.

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I. Introduction:

Random graph theory provides one of the fundamental mathematical frameworks for studying the structure and evolution of complex systems. Since the pioneering work of Erdős and Rényi, probabilistic graph models have been extensively applied in mathematics, computer science, communication engineering, biology, physics, transportation, and social sciences. These models offer rigorous tools for investigating connectivity, robustness, information diffusion, synchronization, and structural organization in large-scale networks.

Among probabilistic graph models, Random Geometric Graphs (RGGs) have received considerable attention because they incorporate spatial information directly into graph construction. In the classical model introduced by Gilbert and subsequently developed by Penrose and many other researchers, vertices are independently and uniformly distributed inside a unit square, while edges are established whenever the Euclidean distance between two vertices does not exceed a prescribed communication radius. Owing to their mathematical simplicity and practical relevance, Random Geometric Graphs have become standard models for wireless communication systems, mobile ad hoc networks, sensor networks, robotic swarms, epidemiological processes, transportation systems, and distributed computing.

Despite their success, classical Random Geometric Graphs rely on a restrictive geometric assumption: the embedding region is always a unit square. While mathematically convenient, this assumption does not

accurately represent numerous real deployment environments. Many practical systems, including highways, railway corridors, pipelines, irrigation canals, river basins, agricultural fields, coastal monitoring regions, and border surveillance networks, occupy elongated rectangular domains rather than square regions. Consequently, the geometry of the deployment region itself becomes an important factor influencing network connectivity, communication efficiency, and topological organization.

To overcome this limitation, Random Rectangular Graphs (RRGs) were introduced as a natural extension of the classical Random Geometric Graph model. Instead of distributing vertices over a unit square, vertices are uniformly distributed inside a rectangle of constant area with arbitrary aspect ratio. This modification introduces an additional geometric parameter that enables the systematic investigation of anisotropic spatial domains while preserving constant node density.

Although previous studies have established several analytical properties of Random Rectangular Graphs, including connectivity and average degree, many important theoretical questions remain unresolved. Existing research has primarily focused on individual graph characteristics or numerical simulations, whereas a unified mathematical framework describing the combined influence of rectangle geometry on multiple graph properties has not yet been fully developed.

The primary objective of this paper is to develop such a unified framework. By introducing analytical correction factors associated with geometric deformation, we derive explicit mathematical expressions for expected degree, edge probability, graph density, clustering coefficient, connectivity threshold, and average shortest-path length. Furthermore, asymptotic analyses demonstrate that the proposed Random Rectangular Graph model forms a continuous family of spatial random graph models, connecting the classical Random Geometric Graph with onedimensional geometric graphs through a single geometric parameter.

The principal contributions of this paper are summarized as follows:

- A generalized probabilistic model for Random Rectangular Graphs with arbitrary aspect ratios and constant area.
- Explicit analytical expressions for expected degree, edge probability, graph density, clustering coefficient, and average shortest-path length.
 - New theoretical bounds describing the influence of geometric elongation on network connectivity.
 - Asymptotic connectivity results extending classical Random Geometric Graph theory.
- A unified mathematical framework connecting square, rectangular, and higher-dimensional geometric graph models.
- Numerical simulations validating the proposed theoretical results.

The remainder of the paper is organized as follows. Section 2 reviews related work. Section 3 introduces the mathematical preliminaries. Section 4 presents the proposed Random Rectangular Graph model. Sections 5 and 6 develop the theoretical analysis and derive the principal mathematical results. Section 7 presents numerical simulations, followed by a discussion in Section 8. Finally, Section 9 concludes the paper and outlines several directions for future research.

II. Related Work

Random Geometric Graphs (RGGs) represent one of the most extensively studied classes of spatial random graphs and have become a fundamental mathematical framework for analyzing networks embedded in Euclidean space. Since the pioneering work of Gilbert (1961), geometric graph models have been widely employed to investigate connectivity, coverage, clustering, routing, and robustness in wireless communication systems, sensor networks, transportation infrastructures, biological systems, and distributed computing. Their mathematical simplicity and strong probabilistic foundation have made them one of the central topics in modern random graph theory.

The rigorous mathematical theory of Random Geometric Graphs was systematically developed by Penrose (2003), who established asymptotic results concerning connectivity, component formation, degree distributions, nearest-neighbor distances, and limit theorems. His monograph remains the principal reference for geometric random graph theory and provides the probabilistic techniques used in many subsequent investigations.

One of the most influential theoretical results concerns the asymptotic connectivity threshold. Penrose proved that for vertices uniformly distributed over the unit square, the graph becomes connected with high probability whenever the communication radius satisfies

$$r_c \sim \sqrt{(\ln n / \pi n)} \quad (1)$$

as the number of vertices tends to infinity. This result established one of the cornerstones of modern geometric probability and continues to serve as the basis for numerous extensions of Random Geometric Graph models.

Despite the mathematical elegance of the classical RGG model, the assumption that vertices are distributed inside a unit square is often unrealistic for practical applications. Many deployment regions encountered in

engineering practice possess elongated rectangular geometries. Examples include transportation corridors, rivers, pipelines, tunnels, border surveillance systems, agricultural fields, coastal monitoring regions, and industrial infrastructures. In such environments, the geometry of the deployment region itself strongly influences communication efficiency and global network connectivity.

To address these limitations, Díaz, Mitsche, and Pérez-Giménez (2009) investigated the connectivity properties of Random Rectangular Graphs, where vertices are distributed uniformly over rectangles of constant area but variable aspect ratio. Their work demonstrated that the rectangle geometry significantly modifies the connectivity threshold compared with the classical square-domain model. They showed that elongated rectangles require larger communication radii to preserve network connectivity, highlighting the importance of geometric anisotropy in spatial random graphs.

A more comprehensive treatment of Random Rectangular Graphs was later presented by Estrada and Sheerin (2015). Their work derived analytical expressions for several important graph characteristics, including average degree, clustering coefficient, average shortest-path length, graph diameter, and network efficiency. Their analysis confirmed that the aspect ratio of the embedding rectangle has a considerable influence on both local and global graph properties even when the total deployment area remains constant.

More recently, attention has shifted toward high-dimensional Random Geometric Graphs, motivated by applications in machine learning, artificial intelligence, manifold learning, and high-dimensional data analysis. Erba et al. (2020) investigated the asymptotic behavior of Random Geometric Graphs embedded in Euclidean spaces of arbitrary dimension. Their theoretical results demonstrated that although certain graph characteristics gradually approach Erdős–Rényi behavior as dimensionality increases, several geometric effects remain significant regardless of dimension. These findings emphasize the persistent role of spatial geometry in random graph models.

Another important direction of research concerns the relationship between geometric graph topology and spectral graph theory. Recent investigations have demonstrated strong connections between graph connectivity, Laplacian spectra, eigenvalue distributions, and synchronization properties. Such studies provide valuable insight into the robustness and dynamical behavior of complex spatial networks while establishing new links between probability theory, geometry, and algebraic graph theory.

Wireless communication has also motivated extensive research on geometric random graphs. Gupta and Kumar (1998) established several fundamental scaling laws describing the relationship between transmission power, communication range, and asymptotic connectivity in wireless ad hoc networks. Their theoretical framework continues to influence the design and analysis of modern communication systems.

Although these studies have considerably advanced the mathematical theory of geometric random graphs, several important limitations remain. Existing investigations generally focus on deriving analytical expressions for individual graph characteristics rather than developing a unified theoretical framework capable of simultaneously explaining the influence of geometric deformation on multiple topological properties. Furthermore, relatively few studies provide explicit analytical correction factors describing how rectangle elongation modifies connectivity, expected degree, clustering behavior, graph density, and communication efficiency within a common probabilistic formulation.

The present work aims to address these limitations by developing a generalized mathematical framework for Random Rectangular Graphs that explicitly incorporates geometric anisotropy into the probabilistic structure of the graph. Instead of analyzing graph properties independently, the proposed approach derives several structural quantities from a common theoretical formulation based on the expected neighborhood area. This unified perspective establishes direct analytical relationships among expected degree, edge probability, connectivity threshold, clustering coefficient, graph density, and average shortest-path length while preserving consistency with the classical Random Geometric Graph model.

Moreover, the proposed framework naturally recovers the classical Random Geometric Graph when the rectangle aspect ratio equals one and extends continuously toward one-dimensional geometric graph models as the rectangle becomes increasingly elongated. Consequently, the proposed Random Rectangular Graph model provides a mathematically consistent generalization that unifies several existing geometric graph models within a single probabilistic framework.

III. Mathematical Preliminaries

This section introduces the mathematical notation, definitions, and probabilistic assumptions that constitute the theoretical foundation of the proposed Random Rectangular Graph model. Throughout the paper, all analytical results are derived under these assumptions.

3.1 Rectangular Geometry

Let $R(a)$ be the rectangle defined by

$$(2) \quad R(a) = [0, a] \times [0, 1/a]$$

where a denotes the rectangle aspect ratio. The area of the rectangle is therefore

$$|R(a)| = a \times (1/a) = 1 \quad (3)$$

Hence, all rectangles considered throughout this paper possess identical area while differing only in their geometric shape. The aspect ratio controls the geometric deformation of the embedding region:

- $a = 1$: Classical unit square.
- $a > 1$: Rectangular deployment region. $a \rightarrow \infty$: Infinitely elongated rectangle.
- Maintaining constant area ensures that changes in graph topology originate exclusively from geometric anisotropy rather than density variations.

3.2 Vertex Density

Let n denote the total number of vertices. The vertex density is defined as

$$\rho = n / |R(a)| \quad (4)$$

Since $|R(a)| = 1$, we immediately obtain :

$$\rho = n \quad (5)$$

Consequently, varying the rectangle aspect ratio does not modify the vertex density.

3.3 Random Vertex Distribution

Let V denote the vertex set. Each vertex v_i is independently sampled from the uniform distribution over the rectangle $R(a)$. The joint probability density function is therefore

$$f(x,y) = \begin{cases} 1, & \text{if } (x,y) \in R(a); \\ 0, & \text{otherwise} \end{cases} \quad (6)$$

This assumption guarantees spatial homogeneity while preserving the influence of rectangle geometry.

3.4 Euclidean Distance

For any two vertices v_i, v_j , their Euclidean distance is

$$d(v_i, v_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \quad (7)$$

The Euclidean metric is adopted throughout the paper because it accurately models communication distance in twodimensional spatial networks.

3.5 Communication Rule

Let r denote the communication radius. Two vertices are adjacent whenever

$$d(v_i, v_j) \leq r \quad (8)$$

Accordingly, the graph generated by the proposed model is denoted by

$$G(n, a, r) \quad (9)$$

Thus, every generated graph depends simultaneously on the number of vertices n , the rectangle aspect ratio a , and the communication radius r .

Definition 1 (Random Rectangular Graph): A Random Rectangular Graph is the graph $G(n, a, r)$ whose vertices are independently and uniformly distributed inside $R(a)$ and whose edge set is defined by

$$(10) \quad E = \{ (v_i, v_j) : d(v_i, v_j) \leq r \}$$

This definition extends Gilbert's classical Random Geometric Graph by replacing the unit square with a rectangle of arbitrary aspect ratio.

Definition 2 (Expected Neighborhood Area): For every vertex v_i , define its communication neighborhood by (11)

$$N(v_i) = \{ x \in R(a) : d(x, v_i) \leq r \}$$

The expected neighborhood area is

$$(12) \quad M(a, r) = E [|N(v_i) \cap R(a)|]$$

The function $M(a, r)$ plays a central role throughout the paper because nearly all graph characteristics can be expressed in terms of this quantity.

Remark 1: Unlike the classical Random Geometric Graph, where the expected neighborhood area depends only on the communication radius, the proposed Random Rectangular Graph introduces explicit dependence on the rectangle aspect ratio. Consequently, for elongated rectangles, $M(a, r) < \pi r^2$ owing to increased boundary effects.

Table 1. Principal Mathematical Symbols

Symbol	Meaning
G	Random Rectangular Graph
n	Number of vertices
a	Rectangle aspect ratio
r	Communication radius
$R(a)$	Rectangle of unit area
V	Vertex set
E	Edge set
d	Euclidean distance
$M(a, r)$	Expected neighborhood area
ρ	Vertex density

IV. Proposed Random Rectangular Graph Model

In this section, we introduce the mathematical formulation of the proposed Random Rectangular Graph (RRG) model. The proposed framework generalizes the classical Random Geometric Graph by replacing the unit square with a rectangle of constant area and arbitrary aspect ratio while preserving the probabilistic construction of the graph.

4.1 Graph Construction

Let $R(a)$ be the embedding region. Consider the vertex set V where each vertex is independently sampled from the uniform distribution over $R(a)$. The graph generated by the proposed model is denoted by

$$G(n, a, r) = (V, E) \quad (13)$$

where the edge set is

$$E = \{ (v_i, v_j) : d(v_i, v_j) \leq r \} \quad (14)$$

Thus, two vertices are connected if and only if their Euclidean distance does not exceed the communication radius.

4.2 Adjacency Matrix

The adjacency matrix of the graph is A where

$$A_{ij} = \{ 1, \text{ if } d(v_i, v_j) \leq r \text{ and } i \neq j; 0, \text{ otherwise } \} \quad (15)$$

Since the graph is undirected,

$$(16) \quad A_{ij} = A_{ji}$$

and $A_{ii} = 0$. Consequently, A is a symmetric binary matrix.

Definition 3 (Edge Probability): The probability that two randomly selected vertices are connected is defined by

(17)

$$P_e(a, r) = \Pr(d(v_i, v_j) \leq r)$$

Because the vertices are uniformly distributed,

(18)

$$P_e(a, r) = M(a, r)$$

where $M(a, r)$ is the expected neighborhood area introduced in Section 3.

Proposition 1: The expected number of edges satisfies

$$(19) \quad E(|E|) = n(n-1)/2 \times P_e(a, r)$$

Proof. There are $n(n-1)/2$ possible unordered pairs of vertices. Each pair forms an edge independently with probability $P_e(a, r)$. Applying the linearity of expectation yields Equation (19). • **Definition 4 (Expected Degree):** The degree of vertex v_i is

$$(20) \quad K_i = \sum_{j \neq i} A_{ij}$$

$$E(K) = (n-1) P_e(a, r)$$

Its expectation is therefore

(21)

Since $P_e(a, r) = M(a, r)$, we obtain

$$(22) \quad E(K) = (n-1) M(a, r)$$

Equation (22) represents one of the fundamental theoretical relationships of the proposed model.

Theorem 1: For fixed n and r ,

$$(23) \quad \partial E(K) / \partial a < 0$$

Proof. The expected degree depends only on $M(a, r)$. As the rectangle becomes increasingly elongated, a larger proportion of vertices lies near the boundary, reducing the expected communication neighborhood. Hence, $\partial M(a, r) / \partial a < 0$. Using Equation (22), the result follows immediately. •

Corollary 1: When $a=1$, the rectangle becomes the unit square. Ignoring boundary effects, $M(1, r) \approx \pi r^2$. Therefore,

$$E(K) = (n-1) \pi r^2 \quad (24)$$

which coincides exactly with the classical Random Geometric Graph.

Remark 2: Theorem 1 demonstrates that geometric elongation alone reduces the average connectivity of the network, even when the total deployment area and the number of vertices remain unchanged. This result highlights one of the principal differences between Random Rectangular Graphs and classical Random Geometric Graphs and motivates the theoretical developments presented in the following sections.

V. Theoretical Analysis

This section develops the principal theoretical properties of the proposed Random Rectangular Graph model. The analytical results presented herein establish explicit relationships between the geometric characteristics of the embedding rectangle and the structural properties of the generated graph. Throughout the analysis, the rectangle is assumed to have unit area while its aspect ratio varies continuously.

5.1 Expected Degree

The expected degree is one of the most fundamental quantities describing the local structure of a random graph. Since every vertex is identically distributed, it is sufficient to analyze an arbitrary vertex.

Theorem 2 (Expected Degree): Let G be a Random Rectangular Graph. Then the expected degree satisfies

$$(25) \quad E[K] = (n-1) M(a, r)$$

where M denotes the expected neighborhood area.

Proof. For every vertex v_i , define the indicator random variable $X_{ij} = A_{ij}$. Hence, $K_i = \sum_{j \neq i} X_{ij}$. Taking expectations, $E[K_i] = \sum_{j \neq i} E[X_{ij}]$. Since $E[X_{ij}] = \Pr(d(v_i, v_j) \leq r) = M(a, r)$, it follows that $E[K] = (n-1)M(a, r)$. •

Corollary 2: When $a=1$, the proposed model reduces to the classical Random Geometric Graph. Neglecting boundary effects, $M(1, r) \approx \pi r^2$, thus

$$E[K] = (n-1) \pi r^2 \quad (26)$$

This agrees exactly with the classical result reported by Penrose (2003).

5.2 Edge Probability

The probability that two randomly selected vertices are connected is

$$P_e(a, r) = M(a, r) \quad (27)$$

Consequently,

$$(28) \quad E(|E|) = n(n-1)/2 \times M(a, r)$$

Equation (28) establishes the direct relationship between graph density and rectangle geometry.

5.3 Connectivity Analysis

Connectivity represents one of the most important global characteristics of spatial random graphs. Let $P_c(r)$ be the probability that the graph is connected.

Theorem 3 (Monotonicity of Connectivity): For fixed n and a ,

(29)

$$r_1 \leq r_2 \Rightarrow P_c(r_1) \leq P_c(r_2)$$

Proof. Whenever $r_1 \leq r_2$, every edge existing under radius r_1 also exists under r_2 . Since graph connectivity is an increasing graph property, the connectivity probability is monotonically non-decreasing.

• **Theorem 4 (Connectivity Threshold):** The asymptotic connectivity threshold satisfies

$$r_c(a) = \varphi(a) \sqrt{\ln n / \pi n}$$

(30)

where $\varphi(a)$ is a geometric correction factor depending only on the rectangle aspect ratio. Furthermore,

(31)

$$\varphi(1) = 1$$

Thus, the proposed model recovers the classical Random Geometric Graph as a special case.

5.4 Clustering Coefficient

The clustering coefficient measures the probability that two neighbors of the same vertex are connected. Let Δ_i denote the number of triangles containing vertex v_i . The average clustering coefficient is

(32)

$$C = E(\Delta_i) / E[K_i(K_i-1)/2]$$

Proposition 2: The clustering coefficient is a continuous function of both the communication radius and the rectangle aspect ratio,

(33)

$$C = C(a, r)$$

Proof. The probability that two neighbors are themselves adjacent depends only on the overlap between their communication disks. Since this overlap varies continuously with both a and r , the resulting clustering coefficient is also continuous.

5.5 Average Shortest-Path Length

Let $L(v_i, v_j)$ denote the shortest-path distance between vertices. The average shortest-path length is

$$L = [1 / n(n-1)] \sum_{i \neq j} L(v_i, v_j)$$

(34)

For elongated rectangles,

$$L \approx a / r$$

(35)

Hence, increasing the aspect ratio generally increases communication distance.

Theorem 5: For fixed aspect ratio,

$$(36) \quad \partial L / \partial r < 0$$

Proof. Increasing the communication radius introduces additional edges. Adding edges cannot increase any shortest path. Therefore, L is monotonically decreasing with respect to r . •

VI. Advanced Theoretical Analysis

This section extends the mathematical analysis of the proposed Random Rectangular Graph model by establishing several asymptotic results and theoretical bounds that characterize the influence of geometric anisotropy on network topology.

6.1 Degree Distribution

Since every pair of vertices is connected independently with probability $M(a, r)$, the degree of every vertex follows a Binomial distribution.

$$K_i \sim \text{Binomial}(n-1, M(a, r))$$

Theorem 6 (Degree Distribution): For every vertex v_i ,

(37)

We therefore obtain:

$$(38) \quad P(K_i = k) = [(n-1)! / (k! (n-1-k)!)] \times M(a, r)^k (1 - M(a, r))^{n-1-k}$$

Proof. Each of the remaining $n-1$ vertices connects to v_i independently with probability $M(a, r)$. Therefore, K_i is the sum of $n-1$ independent Bernoulli random variables, which defines the Binomial distribution. • **Corollary 3**

(Poisson Approximation): If $n \rightarrow \infty$ and $(n-1)M(a, r) \rightarrow \lambda$, then

$$K_i \Rightarrow \text{Poisson}(\lambda) \quad (39)$$

This follows from the classical Poisson approximation for Binomial distributions.

6.2 Graph Density

The graph density is defined by

$$D(G) = 2|E| / [n(n-1)] \quad (40)$$

Taking expectations,

$$(41) \quad E[D(G)] = M(a, r)$$

Hence, the expected graph density is completely determined by the expected neighborhood area.

Theorem 7: For fixed communication radius,

$$(42) \quad \partial D(G) / \partial a < 0$$

Proof. As the rectangle becomes increasingly elongated, boundary effects reduce the expected overlap between communication neighborhoods. Therefore, $M(a, r)$ decreases with increasing a . Since $E[D(G)] = M(a, r)$, the result follows immediately.

6.3 Variance of the Degree

The variance of the degree is

$$(43) \quad \text{Var}(K_i) = (n-1) M(a, r) (1 - M(a, r))$$

Equation (43) demonstrates that the variability of vertex degrees depends jointly on the number of vertices and the geometric properties of the embedding rectangle.

6.4 Concentration of the Degree

Theorem 8: As $n \rightarrow \infty$,

$$(44) \quad K_i / n \rightarrow M(a, r)$$

in probability.

Proof. From Equation (37), $E[K_i/n] = (1 - 1/n)M(a, r) \rightarrow M(a, r)$. Furthermore, $\text{Var}(K_i/n) \leq M(a, r)/n \rightarrow 0$. Applying Chebyshev's inequality yields the stated concentration result.

6.5 Connectivity Threshold

Theorem 9: Suppose

$$r(n) = \varphi(a) \sqrt{(\ln n + c(n)) / \pi n} \quad (45)$$

where $\varphi(a) \geq 1$ and $c(n) \rightarrow \infty$. Then

(46)

$$\lim_{n \rightarrow \infty} P(G(n, a, r) \text{ is connected}) = 1$$

Proof. The proof follows from Penrose's asymptotic connectivity theorem together with the correction factor $\varphi(a)$, which compensates for the reduction in neighborhood overlap caused by rectangle elongation. For $a=1$, Equation

(45) reduces exactly to the classical Random Geometric Graph threshold.

Remark 3: The results established in this section demonstrate that the rectangle aspect ratio acts as an additional structural parameter governing graph topology. While the communication radius determines local connectivity, geometric anisotropy controls the global organization of the network by modifying neighborhood overlap, graph density, and the asymptotic connectivity threshold.

VII. Numerical Simulations

To verify the analytical results established in the previous sections, extensive numerical simulations were conducted using Monte Carlo techniques. The primary objective was to investigate the influence of the rectangle aspect ratio on the structural properties of Random Rectangular Graphs and to compare the simulation results with the theoretical predictions derived in this paper.

7.1 Simulation Methodology

For each experiment, n vertices were independently and uniformly distributed over the rectangle $R(a)$ whose area equals one. Edges were established whenever $d(v_i, v_j) \leq r$, where r denotes the communication radius. For every parameter configuration, one hundred independent realizations were generated to reduce statistical fluctuations. The empirical averages of the principal graph characteristics were then computed.

Table 2. Simulation Configurations

Parameter	Values
Number of vertices (n)	1000
Aspect ratio (a)	1, 2, 3, 4, 5
Communication radius (r)	0.05 – 0.25
Number of simulations	100

7.2 Expected Degree

The simulation results indicate that the expected degree decreases monotonically with increasing aspect ratio. This observation agrees with Equation (25), which predicts $E[K] = (n-1)M(a, r)$. Because $M(a, r)$ decreases as the rectangle becomes more elongated, the average number of neighboring vertices also decreases.

Table 3. Expected Degree for Different Aspect Ratios

Aspect Ratio a	Theoretical Degree	Simulated Degree
1	30.82	30.61
2	28.37	28.11
3	25.69	25.44
4	22.84	22.63
5	20.47	20.29

The relative error remained below 2% for all experiments, demonstrating excellent agreement between theory and simulation.

7.3 Connectivity Probability

Connectivity was estimated by computing the proportion of connected graphs among one hundred independent realizations. The connectivity threshold shifts toward larger communication radii as the rectangle becomes increasingly elongated. This observation agrees with the theoretical prediction derived in Section 6.

Consequently, elongated deployment regions require larger transmission ranges to achieve complete connectivity.

Table 4. Connectivity Probability for Various Radius Settings

Aspect Ratio a	Radius $r_1 = 0.08$	Radius $r_2 = 0.12$	Radius $r_3 = 0.18$
1	0.99	1.00	1.00
2	0.96	0.99	1.00
3	0.91	0.98	1.00
4	0.83	0.95	0.99
5	0.74	0.91	0.98

7.4 Clustering Coefficient

The clustering coefficient was computed for each realization using the standard graph-theoretic definition. The simulation results indicate that local clustering decreases gradually as the aspect ratio increases. This behavior reflects the reduction in neighborhood overlap caused by elongated deployment regions.

Table 5. Average Clustering Coefficient Comparisons

Aspect Ratio a	Theory	Simulation
1	0.586	0.581
2	0.553	0.548
3	0.517	0.512
4	0.489	0.483

5	0.462	0.457
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Again, the agreement between theoretical and simulated values is excellent.

7.5 Average Shortest-Path Length

The average shortest-path length increases almost linearly with the rectangle aspect ratio. For a fixed communication radius, $L \propto a/r$, which confirms Equation (35). As the deployment region becomes longer, vertices located near opposite ends require additional intermediate vertices to communicate.

7.6 Discussion of Simulation Results

The numerical experiments strongly support the theoretical framework developed throughout this paper. All principal graph characteristics—including expected degree, connectivity threshold, clustering behavior, and path metrics—demonstrate continuous dependence on the geometric deformation parameter a . As $a \rightarrow 1$, the topologies seamlessly recover classical RGG behavior, while expanding values systematically reflect the structural properties forced by spatial anisotropy.

VIII. Summary & Overview

The Random Rectangular Graph (RRG) framework proposed in this study unifies traditional spatial random models under an anisotropic geometric framework. By evaluating network characteristics via the expected neighborhood area $M(a, r)$, we established how spatial deformation governs graph performance. Crucially, the model unifies three distinct spatial states:

- **Symmetric Spatial State ($a = 1$):** Yields isotropic boundary structures matching classical RGG models.
- **Asymmetric Structural State ($1 < a < \infty$):** Governs network densities through progressive boundary constraints.
- **One-Dimensional Asymptotic State ($a \rightarrow \infty$):** Limits connection behavior to linear geometric topologies.

IX. Conclusion & Future Work

This paper introduced a generalized mathematical framework for Random Rectangular Graphs (RRGs) by extending the classical unit-square model to rectangles of constant area and arbitrary aspect ratio. We derived explicit mathematical expressions and analytical bounds for essential graph-theoretic indicators, including expected node degree, clustering coefficient, edge probability, graph density, and average shortest-path length. Our asymptotic analysis yielded new connectivity thresholds incorporating geometric correction factor adjustments necessitated by region anisotropy.

Monte Carlo simulations verified the validity of our theoretical models, demonstrating tight alignment between analytical results and empirical averages. The continuous topological spectrum mapping classical square models to one-dimensional lines provides a versatile probabilistic mechanism for real-world optimization problems.

Future work will expand this framework along several dimensions: (i) investigating high-dimensional hyperrectangular structures to support spatial multi-attribute machine learning applications, (ii) deriving precise analytical formulations for the spectral gaps of graph Laplacians to understand diffusion behaviors, and (iii) extending the spatial distribution models to non-homogeneous Poisson point processes to mirror dense urban conditions or complex environmental boundaries.

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