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Research Paper

Discuss Statistical Convergence on Collatz Sequence

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Abstract

One of the famous unsolved problems was proposed by *Lothar Collatz* in the 1930s. It is a conjecture. Such problem is called as *Collatz Conjecture* or $3x + 1$ *problem*, etc.,. In this paper, we discuss the convergence of the *Collatz* sequence, which is the trajectory sequence of the *Collatz* function. For the convergence discussion, we use statistical convergence and *little o* operation. Finally, we conclude that our trajectory sequence converges partially, and we will find the limit.

Keywords: Collatz conjecture, Collatz sequence, Asymptotic density, Statistical convergent

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I. Introduction

One of the mathematicians in 1930's is *Lothar Collatz*, he introduced a function as $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$

$$f(x) = \begin{cases} \frac{x}{2}, & x \equiv 0(mod2) \\ 3x + 1, & x \equiv 1(mod2) \end{cases}$$

It is a recursive function. It reaches the value 1 when the function is repeatedly used by someone on positive integers. Many authors use the problem stated that

$$g(x) = \begin{cases} \frac{x}{2}\,, & x \equiv 0(mod2) \\ \frac{3x+1}{2}\,, & x \equiv 1(mod2) \end{cases}$$

The trajectory set or forward orbit of the Collatz sequence is $\{x, g^1(x), g^2(x), \cdots\}$

The stopping time $\sigma(x)$ of x is the least i such that $g^i(x) < x$ and is ∞ if no i exists. The total stopping time $\sigma_\infty(x)$ is the least i such that x iterates to 1 under i iterations of the function f. That is,

$$\sigma_\infty(x) := inf\{i | g^i(x) = 1\}$$

The scaled total stopping time or gamma value $\gamma(m)$ is given by $\gamma(m) := \frac{\sigma_\infty(m)}{\log m}$

The height $h(m)$ is the least k for which the Collatz function $f(x)$has $f^k(m) = 1$. It is also given by

$$h(m) := \sigma_\infty(m) + d(m),$$

where $d(m)$ counts the number of iterates $g^k(m) \equiv 1(mod2)$ for $0 \leq k < \sigma_\infty(m)$. Finally, the function $\pi_a(x)$ counts the number of n with $|n| \leq x$ that contain a in their forward orbit under T.

Many of the authors in [1, 2, 3, 4, 5, 6] are trying to give some ideas about how to solve that function. But this function is not yet proven. Some people used software like Oliva and Python to find the convergence of the sequence produced by the $Collatz\ function$, but till now only been proven up to 2.95×10^{20}. The largest number verified for the Collatz conjecture until now is $2^{10^5} - 1$.

There are various types of density in mathematics with definitions. Asymptotic density is the most commonly used density, which does not exist for all sequences. Asymptotic density is also referred to as Arithmetic density or Natural density. Other densities have been developed to fill those gaps made by asymptotic density. In [7, 8, 9, 10], various densities are given such as Relative density, Logarithmic density, Generalized densities that is Exponential density, Multiplicative density, Uniform density, Schnirelmann density, Divisor density, Lacunary density, Mean density, Abel density, Poisson density, logarithmic block density, Φ-density. Statistical convergence is derived by using asymptotic density. After that, statistical convergent developed by using other densities.

In the second volume of Bachmann's treatise on number theory [11], the symbol $o(x)$ appeared. the symbols $O(x)$ (sometimes called "big-O") and $o(x)$ (sometimes called "little-o") are known as the Landau symbols. It used to express the asymptotic behavior of a given function. And then many authors like Dimitris Koukoulopoulos[12], use the little o to propose their work.

Statistical convergence, while introduced nearly fifty years ago, has only recently become an area of active research. Different mathematicians studied properties of statistical convergence and applied this concept in various areas. In [13], I. J. Schoenberg gave some basic properties of Statistical convergence and also studied the concept as a summability method. Also, in [14], a convergent property using statistical convergence.

In this paper, we will check whether the trajectory set is convergent or not. Once it is convergent, then we find the limit. The convergence of the sequence will be discussed using statistical convergence. In section 3, some needed definitions and theorems are presented, and in section 4, we discuss the function and sequence of $Collatz's$ and the cardinality of the trajectory set. Convergence of the sequence and limits of the sequences are discussed in sections 5 and 6, respectively. Finally, in section 7, we conclude our discussion with make decision about the $Collatz$ function.

2 Origin and Previous works of Collatz conjecture

The problem disappears for some years only to reappear at some other moment in the hands of another mathematician. The fact that it was not easy to locate the "true" origins of the problem is supported by the observation that the very same problem is known under different names: Hasse's algorithm, the Syracuse problem, Kakutani's problem, Ulam's problem, and sometimes it is even referred to as the Hailstone problem. The last name is a reference to the behavior of the sequence of $g^i(n)$. It tends to move upwards and downwards much in the way that hailstones hit the ground and bounce back up again.

Many mathematicians like Jeffrey C Lagarias, T. Kannan, A.O.L. Atkin, Dhananjay P Mehendale, etc., have proposed many works on the Collatz conjecture. Some people like to give some results to prove the conjecture as well as convince the readers. But some people use this Collatz function, and they create some form of results for their work.

For proving the Collatz conjecture, some people like to work with the help of graph theory. By graph theory, they use eccentric concept such as directed graphs. It gives the root of the function and helps to find the cycle and so on. Someone was inspired by the concept in graph theory, and then they created content on fractals for the Collatz conjecture. Certain people use the binomial representation to solve such conjectures. A few people take an interest in finding the stopping time of the trajectory set. It helps to deal with the convergence of the Collatz trajectory set. Also, it gives the convergent value of the Collatz sequence. In topological space, a small number of people give some results on the Collatz sequence with the help of period of the set. Some people work with the help of inverse functions, stochastic approach, asymptotic density, concepts in sequence, etc.,.

A certain number of people combine the Collatz function and some other concepts in mathematics and have given some results on that. Some people modify the Collatz conjecture and frame some results.

3 Preliminaries

Definition 1. *The sequence x_n is statistically convergent to the number L provided that for each $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k < n : |x_k - L| > \epsilon\}| = 0,$$

In this case we write $st - lim = L$.

Definition 2. *If for every positive constant ϵ there exists a constant x_0 such that $|f(x)| \leq \epsilon g(x)\ \ \forall x \geq x_0$ then $f(x) = o(g(x))$ as $x \to \infty$.*

Theorem 1. *If x is a sequence such that $st - lim x_k = L$ and $\Delta x_k = o(\frac{1}{k})$, then $\lim x_k = L$*

Note that Δx_k is the forward difference of the elements.

4 *Collatz* Function and sequence

Let us define the function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$

$$f(x) = \begin{cases} \frac{x}{2}, & x \equiv 0(mod2) \\ 3x+1, & x \equiv 1(mod2) \end{cases}$$

It is an original form of the *Collatz* conjecture. It takes more iterations to make a full trajectory set. So, in [4], Jeffery Lagarias gives the generalization of the *Collatz* function. That is, $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$

$$g(x) = \begin{cases} \frac{x}{2}, & x \equiv 0(mod2) \\ \frac{3x+1}{2}, & x \equiv 1(mod2) \end{cases}$$

In [15], Mehendale and Dhananjay P state that the most generalized form of the *Collatz* function is $h : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$

$$h(x) = \begin{cases} \frac{x}{2^k} \ , (k \in \mathbb{N}) & x \equiv 0(mod2), \\ \frac{3x+1}{2^m} \ , (m \in \mathbb{N}) & x \equiv 1(mod2), \end{cases}$$

where k is the largest power of 2 that divides x and m is the largest power of 2 that divides $3x+1$. It takes fewer iterations to set the trajectory.

The functions $g(x)$ and $h(x)$ are defined to reduce the steps of iteration. For the proof, we use these two functions in our discussion. Let the trajectory sets of the functions $g(x)$ and $h(x)$ be G and H, respectively. G is finite iff H is finite. Because the function $h(x)$ just reduces the steps. But the elements in the set H must reach the element in the set G. So, here we use the function $h(x)$ to find the cardinality of the trajectory set.

4.1 Density of the trajectory set

When the *collatz* function is applied to an even integer, it automatically reaches an odd integer. So, our investigation fully focused on odd integers. Let us consider the trajectory set with distinct elements of any odd integer x as $T = \{x, h(x), h^2(x), \cdots\}$. Let S be a proper subset of T. Define, $h : T \to S,\ by\ C(x) = \frac{3x+1}{2^m}$ where m is the largest power of 2 which divides $3x+1$.

Asymptotic density of $C_(x)$ is $\frac{1}{2^m}$. When we apply the limit as m tends to infinity. It reaches 0.

5 Convergence of the sequence

In [14], Fridy gives a theorem for the convergence of the sequence by using statistical convergence with the help of *little o* of the forward difference Δx_k. For the proof, we consider the function $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$

$$g(x) = \begin{cases} \frac{x}{2}, & x \equiv 0(mod2) \\ \frac{3x+1}{2}, & x \equiv 1(mod2) \end{cases}$$

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Case i: x_{k-1} is even, x_k is odd , $\Delta x_k = \frac{x_{k-1}-3x_k-1}{2}$

Case ii: x_{k-1} is even, x_k is even, $\Delta x_k = \frac{x_{k-1}-x_k}{2}$

Case iii: x_{k-1} is odd, x_k is odd, $\Delta x_k = \frac{3x_{k-1}-3x_k}{2}$

Case iv: x_{k-1} is odd, x_k is even, $\Delta x_k = \frac{3x_{k-1}+1-x_k}{2}$

By the definition of *little o*, for every positive integer b, there exists x_k such that $|(\Delta x_k)| \le \frac{b}{k}$. And by Section 4.1, we know that the asymptotic density of a *Collatz* sequence is zero, which means the trajectory set is statistically convergent. So, by the theorem, we conclude that the *Collatz* trajectory set is convergent.

6 Limit of the function

For find the limit, we use the function $h(x)$ in Section 4. If x_k is converge to L, x_{k+1} is also converge to L. So, $L = \frac{3L+1}{2^m}$. Since our discussion only focused on odd positive integers. Therefore, $L = 1$, since $L \in \mathbb{Z}_{>0}$.

7 Conclusion

From Section 5, we observe and conclude that the trajectory set of the *Collatz* function converges partially. Also, from Section 6, the limit must be 1. These two discussions clarify that the trajectory set of the *Collatz* function converges partially to 1. So, we conclude that the *Collatz* conjecture is partially true.

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