*Quest Journals*
*Journal of Research in Applied Mathematics*
*Volume 12 ~ Issue 7 (July 2026) pp: 70-86*
*ISSN (Online): 2394-0743 ISSN (Print): 2394-0735*
*www.questjournals.org*

**Research Paper**

# Generalized Difference Operators with Rough Ideal Statistical Convergence in Picture Fuzzy Normed Spaces

T. Kannan[1], P. Jenifer[2] and M. Jeyaraman[3]

[1]*Associate Professor, Department of Mathematics, Sree Sevugan Annamalai College, Devakottai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.*
[2]*Research Scholar, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.*
[3]*Associate Professor and Head, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai,*
*Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.*

***Abstract.*** *This paper investigates the concept of rough ideal statistical convergence within the framework of picture fuzzy normed spaces, utilizing the generalized difference operator.*
*Furthermore, the generalized difference operator is employed to define and explore a new mode of convergence. Using generalized difference sequences, we have examined the algebraic and topological characteristics of rough ideal statistical limit points.*

***Keywords:*** *Picture fuzzy normed space, Difference Sequence, Rough Ideal statistical convergence*
*AMS Mathematics Subject Classification: 40A35, 46S50, 46S40.*

*Received 11 July., 2026; Revised 26 July, 2026; Accepted 28 July, 2026* 

## 1. Introduction

In 1965, Zadeh[25] introduced fuzzy set theory for the first time. He put forth this theory to deal with the idea of partial truth, in which truth values fall somewhere between being entirely true and being entirely untrue. In situations when traditional binary logic was unable to effectively handle ambiguous or imprecise information, this method proved especially helpful. In 1986, Atanassov[2] introduced the theory of intuitionistic fuzzy sets (IFS). By including both a degree of membership and a degree of non-membership, this theory expands upon the IFS theory. Cuong and Kreinovich[5] invented the picture fuzzy set (PFS), a FS and IFS extension that considers the components' positive, negative, and neutral membership degrees. Cuong[3] (2014) suggested PFSs measurement of distance and looked at a few PFSs properties. Cuong and Hai[4] (2015) defined the main operations for fuzzy inference processes in picture fuzzy systems and investigated the effects of the basic fuzzy logic operators—negations, conjunctions, and disjunctions on PFSs. In order to establish picture fuzzy normed spaces, Pinaki Majumdar[12] and others expanded the concept of fuzzy normed spaces to picture fuzzy sets. Jenifer et al.[10] expanded the concept of picture fuzzy noremd spaces with Fibobacci Ideal convergence. The idea of difference sequence spaces was first put forth by Kizmaz[14] as follows: $Z(\Delta) = \{\mathfrak{f} = (\mathfrak{f}_{\mathfrak{p}}) : (\Delta\mathfrak{f}_{\mathfrak{p}}) \in Z\}$ for $Z = \ell_\infty, c, c_0$.

Kostryko et al.[15], [16] introduced the concept of ideal convergence (I-convergence) in 2000 by using ideals to generalize statistical convergence. In accordance with the concept of density with positive natural numbers, Fast[8] and Steinhaus[24] independently established statistical convergence in 1951. Phu[20], [21] defines rough convergence as a natural development of ordinary convergence in normed spaces of constrained dimensions. In this concept Phu, introduces a "roughness degree" (often denoted as r) that allows for a certain degree of error or "roughness" in the convergence process. Das, Savas, and Ghosal[6] were the first to define the idea of rough ideal statistical($\mathfrak{RI}-\mathfrak{st}$) convergence of a series. This generalization was first presented in 2011 by means of ideals in normed spaces. According to a study on rough ideal statistical($\mathfrak{RI}-\mathfrak{st}$) convergence, Das et al. provided a definition of this term in normed spaces. Rough ideal statistical($\mathfrak{RI}-\mathfrak{st}$) convergence was defined in normed spaces in 2011, according to a different work on the subject. $\mathfrak{RI}-\mathfrak{st}$ convergence utilizing generalized difference operator in IFNS was presented by Manpeer Kaur et al. Rough Ideal statistical convergence in neutrosophic normed spaces and Wijsman Ideal statistical convergence in neutrosophic metric spaces have been extensively studied in the literature [9],[11].

The notions of $\mathfrak{R}-\mathfrak{st}$ convergence and $\mathfrak{RI}-\mathfrak{st}$ convergence for generalized difference sequences in the context of picture fuzzy normed spaces have been introduced in this paper.

For generalized difference sequences, we investigated the topological and algebraic characteristics of rough ideal statistical limit points.

## 2. Preliminaries

The fundamental definitions of PFNS, $\mathfrak{st}$, and $\mathfrak{R}-\mathfrak{st}$ convergence in PFNS are reviewed in this section.

**Definition 2.1.** [10] Picture Fuzzy Normed Space (PFNS) is the name given to the seven tuple $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ if a vector space $\breve{\mathfrak{X}} \neq \emptyset$ and $*$ is a continuous t-norm, $\Diamond$ and $\circledast$ are continuous t-conorms, and $\breve{\mu}, \breve{\nu}$, and $\breve{\omega}$ are FSs on $\breve{\mathfrak{X}} \times (0, \infty)$ that meet the following conditions:

For each $\mathfrak{f}, \mathfrak{y} \in \breve{\mathfrak{X}}$ and $\mathfrak{t}, \mathfrak{n} > 0$:

(i) $\breve{\mu}(\mathfrak{f}, \mathfrak{t}) + \breve{\nu}(\mathfrak{f}, \mathfrak{t}) + \breve{\omega}(\mathfrak{f}, \mathfrak{t}) \leqslant 1$,

(ii) $\breve{\mu}(\mathfrak{f}, \mathfrak{t}) > 0$,

(iii) $\breve{\mu}(\mathfrak{f}, \mathfrak{t}) = 1$ if and only if $\mathfrak{f} = 0$,

(iv) $\breve{\mu}(\mathfrak{c}\mathfrak{f}, \mathfrak{t}) = \breve{\mu}\left(\mathfrak{f}, \frac{\mathfrak{t}}{|\mathfrak{c}|}\right)$ if $\mathfrak{c} \neq 0$,

(v) $\breve{\mu}(\mathfrak{f}, \mathfrak{t}) * \breve{\mu}(\mathfrak{y}, \mathfrak{n}) \leqslant \breve{\mu}(\mathfrak{f} + \mathfrak{y}, \mathfrak{t} + \mathfrak{n})$,

(vi) $\breve{\mu}(\mathfrak{f}, .) : (0, +\infty) \to [0, 1]$ is continuous,

(vii) $\lim_{\mathfrak{t} \to +\infty} \breve{\mu}(\mathfrak{f}, \mathfrak{t}) = 1$ and $\lim_{\mathfrak{t} \to 0} \breve{\mu}(\mathfrak{f}, \mathfrak{t}) = 0$,

(viii) $\breve{\nu}(\mathfrak{f}, \mathfrak{t}) < 1$,

(ix) $\breve{\nu}(\mathfrak{f}, \mathfrak{t}) = 0$ if and only if $\varkappa = 0$,

(x) $\breve{\nu}(\mathfrak{c}\mathfrak{f}, \mathfrak{t}) = \breve{\nu}\left(\mathfrak{f}, \frac{\mathfrak{t}}{|\mathfrak{c}|}\right)$ if $\mathfrak{c} \neq 0$,

(xi) $\breve{\nu}(\mathfrak{f}, \mathfrak{t}) \Diamond \breve{\nu}(\mathfrak{y}, \mathfrak{n}) \geqslant \breve{\nu}(\mathfrak{f} + \mathfrak{y}, \mathfrak{t} + \mathfrak{n})$,

(xii) $\breve{\nu}(\mathfrak{f}, .) : (0, +\infty) \to [0, 1]$ is continuous,

(xiii) $\lim_{\mathfrak{t} \to +\infty} \breve{\nu}(\mathfrak{f}, \mathfrak{t}) = 0$ and $\lim_{\mathfrak{t} \to 0} \breve{\nu}(\mathfrak{f}, \mathfrak{t}) = 1$,

(xiv) $\breve{\omega}(\mathfrak{f}, \mathfrak{t}) < 1$,

(xv) $\breve{\omega}(\mathfrak{f}, \mathfrak{t}) = 0$ if and only if $\mathfrak{f} = 0$,

(xvi) $\breve{\omega}(\mathfrak{c}\mathfrak{f}, \mathfrak{t}) = \breve{\omega}\left(\mathfrak{f}, \frac{\mathfrak{t}}{|\mathfrak{c}|}\right)$ if $\mathfrak{c} \neq 0$,

(xvii) $\breve{\omega}(\mathfrak{f}, \mathfrak{t}) \circledast \breve{\omega}(\mathfrak{y}, \mathfrak{n}) \geqslant \breve{\omega}(\mathfrak{f} + \mathfrak{y}, \mathfrak{t} + \mathfrak{n})$,

(xviii) $\breve{\omega}(\mathfrak{f}, .) : (0, +\infty) \to [0, 1]$ is continuous,

(xix) $\lim_{\mathfrak{t} \to +\infty} \breve{\omega}(\mathfrak{f}, \mathfrak{t}) = 0$ and $\lim_{\mathfrak{t} \to 0} \breve{\omega}(\mathfrak{f}, \mathfrak{t}) = 1$.

**Definition 2.2.** [10] Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS. A sequence $\mathfrak{f} = \{\mathfrak{f}_{\mathfrak{p}}\}$ in $\mathfrak{f}$ is said to be $\mathfrak{st}$ - convergent to $\mathfrak{L} \in \breve{\mathfrak{X}}$ with respect to norm $(\breve{\mu}, \breve{\nu}, \breve{\omega})$ if for every $\epsilon > 0$ and $\mathfrak{t} \in (0, 1)$
$\lim_{n \to \infty} \frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}\}, \breve{\omega}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}\}| = 0$ or
$\delta(\{\mathfrak{p} \leqslant n : \breve{\mu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}\}, \breve{\omega}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}\}) = 0$.
It is denoted by $\mathfrak{f}_{\mathfrak{p}} \xrightarrow{\mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})}} \mathfrak{L}$ or $\mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})} - \lim_{\mathfrak{p} \to \infty} \mathfrak{f}_{\mathfrak{p}} = \mathfrak{L}$.

**Definition 2.3.** Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS. A sequence $\mathfrak{f} = \{\mathfrak{f}_{\mathfrak{p}}\}$ in $\breve{\mathfrak{X}}$ is said to be $\mathfrak{R} - \mathfrak{st}$ - convergent to $\mathfrak{L} \in \breve{\mathfrak{X}}$ with respect to norm $(\breve{\mu}, \breve{\nu}, \breve{\omega})$ for some non-negative number $\mathfrak{r}$ if for every $\epsilon > 0$ and $\mathfrak{t} \in (0, 1)$
$\lim_{n \to \infty} \frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \mathfrak{r} + \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}\}| = 0$ or
$\delta(\{\mathfrak{p} \leqslant n : \breve{\mu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \mathfrak{r} + \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}\}) = 0$.
It is denoted by $\mathfrak{f}_{\mathfrak{p}} \xrightarrow{\mathfrak{R} - \mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})}} \mathfrak{L}$ or $\mathfrak{R} - \mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})} - \lim_{\mathfrak{p} \to \infty} \mathfrak{f}_{\mathfrak{p}} = \mathfrak{L}$. Let $\mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})} - LIM^{\mathfrak{r}}_{\mathfrak{f}_{\mathfrak{p}}}$ indicates the the set of all $\mathfrak{R} - \mathfrak{st}$ limit points of the sequence $\mathfrak{f} = \{\mathfrak{f}_{\mathfrak{p}}\}$.

**Definition 2.4.** ([10]). Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS. A sequence $\mathfrak{f} = \{\mathfrak{f}_m\}$ in $\breve{\mathfrak{X}}$ is said to be $\mathrm{I} - \mathfrak{st}$ convergent to $\rho \in \breve{\mathfrak{X}}$ with respect to norm $(\breve{\mu}, \breve{\nu}, \breve{\omega})$ if for every $\epsilon > 0$ and $\mathfrak{t} \in (0, 1)$:
$\left\{n : \frac{1}{n} |\{m \leqslant n : \breve{\mu}(\mathfrak{f}_m - \rho; \epsilon) \leqslant 1 - \mathfrak{t} \text{ or } \breve{\nu}(\mathfrak{f}_m - \rho; \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\mathfrak{f}_m - \rho; \epsilon) \geqslant \mathfrak{t}\}| > \delta\right\} \in \mathrm{I}$.
It is denoted by $\mathfrak{f}_m \xrightarrow{\mathrm{I} - \mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})}} \rho$.

**Definition 2.5.** Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS. Consider $\Delta^m \mathfrak{f} = (\Delta^m \mathfrak{f}_{\mathfrak{p}})$ be a generalized difference sequence in $\breve{\mathfrak{X}}$ where $m \in N$, is said to be $\Delta^m - \mathfrak{st}$ - convergent to $\mathfrak{L} \in \breve{\mathfrak{X}}$ with respect to norm $(\breve{\mu}, \breve{\nu}, \breve{\omega})$ if for every $\epsilon > 0$ and $\mathfrak{t} \in (0, 1)$:
$\lim_{n \to \infty} \frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}\}| = 0$ or
$\delta(\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}\}) = 0$.
It is denoted by $\Delta^m \mathfrak{f}_{\mathfrak{p}} \xrightarrow{\mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})}} \mathfrak{L}$ or $\mathfrak{st}_{(\breve{\mu}, \breve{\nu}, \breve{\omega})} - \lim_{\mathfrak{p} \to \infty} \Delta^m \mathfrak{f}_{\mathfrak{p}} = \mathfrak{L}$.

## 3. Main Results

In this section, we introduce the notions of $\mathfrak{R}-\mathfrak{st}$ Convergence and $\mathfrak{R}\mathrm{I}-\mathfrak{st}$ Convergence for generalized difference sequences in PFNS. For a generalized difference sequence, we then examined some results for $\mathfrak{R}\mathrm{I}-\mathfrak{st}$ limit points. The article assumes that I is an admissible ideal and that $(\Delta^m \mathfrak{f}_\mathfrak{p}) = \Delta^{m-1}\mathfrak{f}_\mathfrak{p} - \Delta^{m-1}\mathfrak{f}_{\mathfrak{p}+1}$, where $m \in N$ is the generalized difference sequence.

**Definition 3.1.** Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS, and $\Delta^m \mathfrak{f} = (\Delta^m \mathfrak{f}_\mathfrak{p})$ be a generalized difference sequence in $\breve{\mathfrak{X}}$, where $m \in N$ is a $\mathfrak{R}-\Delta^m-\mathfrak{st}$ - convergent to $\mathfrak{L} \in \breve{\mathfrak{X}}$ with respect to norm $(\breve{\mu}, \breve{\nu}, \breve{\omega})$ for some non-negative number. For each $\epsilon > 0$ and $\mathfrak{t} \in (0,1)$, $\mathfrak{r}$ if:
$\lim_{n\to\infty} \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \leqslant 1-\mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}\}| = 0$
or
$\delta(\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \leqslant 1-\mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}\}) = 0.$
The symbol for it is $\Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\mathfrak{R}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}$ or $\mathfrak{R} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - \lim_{\mathfrak{p}\to\infty} \Delta^m \mathfrak{f}_\mathfrak{p} = \mathfrak{L}$.
Consider $\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}}$ indicates the collection of all $\mathfrak{R}-\mathfrak{st}$ limit points of $(\Delta^m \mathfrak{f}_\mathfrak{p})$.

**Remark 3.2.** The $\mathfrak{R}-\Delta^m\mathfrak{st}$ convergent is equal to the $\Delta^m\mathfrak{st}$ convergent for the generalized difference sequence $(\Delta^m \mathfrak{f}_\mathfrak{p})$ in a PFNS if $\mathfrak{r} = 0$.

$(\Delta^m \mathfrak{f}_\mathfrak{p})$'s $\mathfrak{R}-\mathfrak{st}$-limit might not be unique in PFNS. Thus, take into consideration the collection of $\mathfrak{R}-\mathfrak{st}$ limits of $(\Delta^m \mathfrak{f}_\mathfrak{p})$ as $\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \left[\mathfrak{L} : \Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\mathfrak{R}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}\right]$. Take note that $(\Delta^m \mathfrak{f}_\mathfrak{p})$ is $\mathfrak{R}$-convergent if $LIM^{\mathfrak{r}(\breve{\mu},\breve{\nu},\breve{\omega})}_{\Delta^m \mathfrak{f}_\mathfrak{p}} \neq \breve{\varrho}$, where $LIM^{\mathfrak{r}(\breve{\mu},\breve{\nu},\breve{\omega})}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \left[\mathfrak{L}^* \in \breve{\mathfrak{X}} : \Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\mathfrak{R}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}^*\right]$.
Regarding unbounded sequences, $LIM^{\mathfrak{r}(\breve{\mu},\breve{\nu},\breve{\omega})}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \breve{\varrho}$, while $\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \breve{\varrho}$, meaning the sequence may be $\mathfrak{R}-\mathfrak{st}$ convergent.

**Example 3.3.** The generalized difference sequence$(\Delta^m \mathfrak{f}_\mathfrak{p})$ can be $\mathfrak{R}-\mathfrak{st}$ - convergent but not $\mathfrak{R}$ - convergent, as demonstrated by this example.
Consider $(\breve{\mathfrak{X}}, \|\cdot\|)$ be a normed space. Define the following for every $\mathfrak{q} > 0$ and the generalized difference sequence $\Delta^m \mathfrak{f} = (\Delta^m \mathfrak{f}_\mathfrak{p}) \in \breve{\mathfrak{X}}$,

$$\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}, \mathfrak{q}) = \frac{\mathfrak{q}}{\mathfrak{q} + \|\Delta^m \mathfrak{f}_\mathfrak{p}\|}, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}, q) = \frac{\Delta^m \mathfrak{f}_\mathfrak{p}}{\mathfrak{q} + \|\Delta^m \mathfrak{f}_\mathfrak{p}\|} \text{ and } \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}, q) = \frac{\Delta^m \mathfrak{f}_\mathfrak{p}}{\mathfrak{q}}.$$

Then $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ is a PFNS.
Examine a sequence $(\Delta^m \mathfrak{f}_\mathfrak{p})_{m\in N}$ such that $\Delta^m \mathfrak{f}_\mathfrak{p} = \begin{cases} (-1)^\mathfrak{p}, & \mathfrak{p} \neq n^2, \\ \mathfrak{p}, & \mathfrak{p} = n^2. \end{cases}$

Consequently, $\Delta^m \mathfrak{f}_\mathfrak{p} = (-1,2,3,1,5,6,7,8,-1,\ldots)$, and obviously for each $\epsilon > 0$ and $\mathfrak{t} \in (0,1)$,
$\lim\limits_{n\to\infty} \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \leqslant 1-\mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}\}| = 0.$
Furthermore, $\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \begin{cases} \breve{\varrho}, & \mathfrak{r} < 1, \\ [1-\mathfrak{r}, \mathfrak{r}-1], & \text{otherwise.} \end{cases}$
Additionally, $LIM^{\mathfrak{r}(\breve{\mu},\breve{\nu},\breve{\omega})}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \breve{\varrho}$, for every $\mathfrak{r} \geqslant 0$, because $(\Delta^m \mathfrak{f}_\mathfrak{p})$ is an unbounded sequence.

**Definition 3.4.** Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS. The generalized difference sequence $\Delta^m \mathfrak{f} = (\Delta^m \mathfrak{f}_\mathfrak{p})$, is said to be $\mathrm{I} - \Delta^m$-$\mathfrak{stc}$ to $\mathfrak{L} \in \breve{\mathfrak{X}}$ with respect to norm $(\breve{\mu}, \breve{\nu}, \breve{\omega})$ if for every $\epsilon > 0$ and $\mathfrak{t} \in (0,1)$:

$$\left\{ \begin{array}{c} n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \epsilon) \leqslant 1 - \mathfrak{t} \text{ or} \\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \epsilon) \geqslant \mathfrak{t}\}| \geqslant \delta \end{array} \right\} \in \mathrm{I}.$$

The symbol for it is $\Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\mathrm{I}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}$. Let $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM_{\Delta^m \mathfrak{f}_\mathfrak{p}}$ indicates the collection of all $\mathrm{I} - \mathfrak{st}$ limits of the generalized difference sequence $(\Delta^m \mathfrak{f}_\mathfrak{p})$.

The analysis of $\mathfrak{R}\mathrm{I} - \mathfrak{st}$ convergent for $(\Delta^m \mathfrak{f}_\mathfrak{p})$ in PFNS is the next focus of our attention.

**Definition 3.5.** Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS, $\Delta^m \mathfrak{f} = (\Delta^m \mathfrak{f}_\mathfrak{p})$, is said to be $\mathfrak{R}\mathrm{I} - \Delta^m$-$\mathfrak{st}$ convergent to $\mathfrak{L} \in \breve{\mathfrak{X}}$ with respect to norm $(\breve{\mu}, \breve{\nu}, \breve{\omega})$ for some non-negative number $\mathfrak{r}$ if for every $\epsilon > 0$ and $\mathfrak{t} \in (0,1)$

$$\left\{ \begin{array}{c} n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \leqslant 1 - \mathfrak{t}\text{or} \\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}\}| \geqslant \delta \end{array} \right\} \in \mathrm{I}.$$

It is indicated by $\Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\mathfrak{R}\mathrm{I}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}$. Let $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}}$ indicates the the collection of all $\mathfrak{R}\mathrm{I} - \mathfrak{st}$ limits of $(\Delta^m \mathfrak{f}_\mathfrak{p})$.

**Example 3.6.** Let $(\breve{\mathfrak{X}}, \|\cdot\|)$ be a normed space. For every $\mathfrak{q} > 0$ and for $\Delta^m \mathfrak{f} = (\Delta^m \mathfrak{f}_\mathfrak{p}) \in \breve{\mathfrak{X}}$, define $\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}, \mathfrak{q}) = \frac{\mathfrak{q}}{\mathfrak{q}+\|\Delta^m \mathfrak{f}_\mathfrak{p}\|}$, $\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}, \mathfrak{q}) = \frac{\Delta^m \mathfrak{f}_\mathfrak{p}}{\mathfrak{q}+\|\Delta^m \mathfrak{f}_\mathfrak{p}\|}$, $\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}, \mathfrak{q}) = \frac{\Delta^m \mathfrak{f}_\mathfrak{p}}{\mathfrak{q}}$. Then $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ is a PFNS. If we consider $\mathrm{I}_d = \{\mathfrak{A} \in N \text{ with } \delta(\mathfrak{A}) = 0\}$ where $\delta(\mathfrak{A})$ indicates the natural density of set $\mathfrak{A}$, then, define $(\Delta^m \mathfrak{f}_\mathfrak{p}) m \in N$ such that

$$\Delta^m \mathfrak{f}_\mathfrak{p} = \begin{cases} \mathfrak{p}, & \mathfrak{p} \in \mathfrak{A}, \\ (-1)^\mathfrak{p}, & \mathfrak{p} \notin \mathfrak{A}. \end{cases}$$

Then $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \begin{cases} \breve{\varrho}, & \mathfrak{r} < 1, \\ [1-\mathfrak{r}, \mathfrak{r}-1], & \text{otherwise}. \end{cases}$

Therefore $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} \neq \breve{\varrho}$.

**Definition 3.7.** A sequence $(\Delta^m \mathfrak{f}_\mathfrak{p})$ in PFNS $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \diamondsuit, \circledast)$ is $\mathrm{I} - \Delta^m - \mathfrak{st}$ bounded if for $\epsilon > 0, \mathfrak{t} \in (0,1)$, $\exists\ \mathfrak{J} > 0$ such that

$$\left\{ n \in N : \frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}\}| \geqslant \delta, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}\}| \geqslant \delta \right\} \in \mathrm{I}.$$

For the generalized difference sequence, we got to the following conclusions. $(\Delta^m \mathfrak{f}_\mathfrak{p})$ in PFNS based on the definitions given above.

**Theorem 3.8.** Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \diamondsuit, \circledast)$ be a PFNS with $(\breve{\mu}, \breve{\nu}, \breve{\omega})$. A sequence $(\Delta^m \mathfrak{f}_\mathfrak{p})$ in $\mathfrak{f}$ is $\mathrm{I} - \Delta^m - \mathfrak{st}$-bounded iff $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} \neq \breve{\varrho}$ for some $\mathfrak{r} > 0$.

*Proof.* **(Necessary Component)** Let $(\Delta^m \mathfrak{f}_\mathfrak{p})$ be a sequence which is $\mathrm{I} - \Delta^m - \mathfrak{st}$-bounded in a PFNS $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \diamondsuit, \circledast)$. Then, for every $\epsilon > 0, \mathfrak{t} \in (0,1)$ and some $\mathfrak{r} > 0$, $\exists\ \mathfrak{J} > 0$ such that

$\left\{ n \in N : \frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \leqslant 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}\}| \geqslant \delta \right\} \in \mathrm{I}.$

Due to the admissibility of I, $\mathfrak{D} = \mathfrak{E} \backslash \mathfrak{F}$ is a non-empty set, where

$\mathfrak{F} = \left\{ n \in N : \frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \leqslant 1 - \mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}\}| \geqslant \delta \right\}.$

Choose $\mathfrak{p} \in \mathfrak{D}$, then

$$\begin{aligned} &\frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \leqslant 1 - \mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) \geqslant \mathfrak{t}\}| < \delta \\ \Rightarrow &\frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) > 1 - \mathfrak{t} \,\text{and}\, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) < \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) < \mathfrak{t}\}| \geqslant 1 - \delta. \end{aligned} \tag{3.8.1}$$

Let $\kappa = \{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) > 1 - \mathfrak{t} \,\text{and}\, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) < \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J}) < \mathfrak{t}\}$. Also,

$\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) \geqslant \min\{\breve{\mu}(0, \mathfrak{r}), \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}, \mathfrak{J})\} = \min\{1, \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J})\} > 1 - \mathfrak{t},$

$\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) \leqslant \max\{\breve{\mu}(0, \mathfrak{r}), \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}, \mathfrak{J})\} = \max\{0, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J})\} < \mathfrak{t},$

$\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) \leqslant \max\{\breve{\mu}(0, \mathfrak{r}), \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}, \mathfrak{J})\} = \max\{0, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{J})\} < \mathfrak{t}.$

Therefore, $\kappa \subset \{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) > 1 - \mathfrak{t} \text{ and } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) < \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) < \mathfrak{t}\}$.

Using (3.8.1), we get

$1 - \delta \leqslant \frac{|\kappa|}{n} \leqslant \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) > 1 - \mathfrak{t} \text{ and } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) < \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) < \mathfrak{t}\}|$.

Thus,

$\frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r}+\mathfrak{J}) \leqslant 1-\mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r}+\mathfrak{J}) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p}; \mathfrak{r}+\mathfrak{J}) \geqslant \mathfrak{t}\}| < 1-(1-\delta) < \delta$.

Following that, for $n \in N$,

$\frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) \leqslant 1 - \mathfrak{t} \text{ or } \breve{\nu}(\mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) \geqslant \mathfrak{t}, \breve{\omega}(\mathfrak{f}_\mathfrak{p}; \mathfrak{r} + \mathfrak{J}) \geqslant \mathfrak{t}\}| \geqslant \delta \subset \mathfrak{F} \in \mathrm{I}$.

Hence $0 \in \mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}}$. Therefore, $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}} \neq \breve{\varrho}$.

**(Sufficient component)**

Let $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} \neq \breve{\varrho}$, for some $\mathfrak{r} > 0$.

Then, $\exists$ some $\mathfrak{L} \in \breve{\mathfrak{X}}$ such that $\mathfrak{L} \in \mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}}$.

For every $\epsilon > 0$ and $\mathfrak{t} \in (0, 1)$, we have

$$\left\{ \begin{array}{c} n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \leqslant 1 - \mathfrak{t} \text{ or} \\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}\}| \geqslant \delta \end{array} \right\} \in \mathrm{I}.$$

As a result, all $\Delta^m \mathfrak{f}_\mathfrak{p}$'s have been contained in a ball with center $\mathfrak{L}$ in PFNS.

Therefore, $(\Delta^m \mathfrak{f}_\mathfrak{p})$ is $\mathrm{I} - \Delta^m$-$\mathfrak{st}$ bounded in a PFNS. Following that, we will establish that the algebraic characterization is equally true for $\mathfrak{R}\mathrm{I} - \mathfrak{st}$ convergent for generalized difference sequence $\Delta^m \mathfrak{f}_\mathfrak{p}$ in PFNS. □

**Example 3.9.** Let $(\breve{\mathfrak{X}}, \|\cdot\|)$ be a normed space. For every $q > 0$ and for $\Delta^m f = (\Delta^m f_p) \in \breve{\mathfrak{X}}$, define

$$\hat{\mu}(\Delta^m f_p, q) = \frac{q}{q + \|\Delta^m f_p\|}, \hat{\nu}(\Delta^m f_p, q) = \frac{\Delta^m f_p}{q + \|\Delta^m f_p\|}, \hat{\omega}(\Delta^m f_p, q) = \frac{\Delta^m f_p}{q}.$$

Then $(\breve{\mathfrak{X}}, \hat{\mu}, \hat{\nu}, \hat{\omega}, \star, \diamondsuit, \circledast)$ is a PFNS.

Define the difference of sequence $\{f_p\}$: $\Delta^m f_p = f_{p+1} - f_p$.

Let $f_p = \begin{cases} \frac{1}{p}, & \text{if } p \text{ is a perfect square} \\ 0, & \text{otherwise} \end{cases}$

If neither $p$ or $p+1$ is a space, $\Delta^m f_p = 0$. If $p = n^2$, then $\Delta^m f_{n^2} = f_{n^2+1} - f_{n^2} = 0 - \frac{1}{n^2} = -\frac{1}{n^2}$.

If $p + 1 = n^2$, then $\Delta^m f_p = \frac{1}{n^2} - 0 = -\frac{1}{n^2}$.

All such non-zero differences tend to 0 as $n \to \infty$.

Let $I$ be an ideal on N containing all density-zero sets. Given roughness radius $r > 0$, we say $\Delta^m f_p$ rough ideal statistical converges to 0 if

$$\{p : \mu(\Delta^m f_p - 0, q) \leqslant 1 - t \text{ or } \nu(\Delta^m f_p - 0, q) \geqslant t, \omega(\Delta^m f_p - 0, q) \geqslant t\}.$$

For large $q$, say $q = 1$, we compute:

when $\Delta^m f_p = \pm\frac{1}{n^2}$:

$$\mu = \frac{1}{\frac{1}{n^2}} \approx 1 - \frac{1}{n^2}, \nu = \frac{\frac{1}{n^2}}{1 + \frac{1}{n^2}} \approx \frac{1}{n^2}, \omega = \frac{\frac{1}{n^2}}{1} \approx \frac{1}{n^2}.$$

Therefore,

$$\{p : \mu(\Delta^m f_p - 0, q) \leqslant 1 - t \text{ or } \nu(\Delta^m f_p - 0, q) \geqslant t, \omega(\Delta^m f_p - 0, q) \geqslant t\}.$$

Thus all violations form a density zero set, hence are in the ideal $I$. Therefore, $\Delta^m f_p$ is rough-ideal statistically convergent to 0.

**Theorem 3.10.** In a PFNS $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$, let $(\Delta^m \mathfrak{f}_\mathfrak{p})$ and $(\Delta^m \mathfrak{f}_\mathfrak{q})$ be two generalized difference sequences.The following outcomes are true if $\mathfrak{r}$ is a non-negative number and I is an admissible ideal and :

(i) if $\Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\Re I - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}$ and $\alpha \in R$, then $\alpha\Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\Re I - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \alpha\mathfrak{L}$,

(ii) if $\Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\Re I - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}_1$ and $\Delta^m \mathfrak{f}_\mathfrak{q} \xrightarrow{\Re I - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}_2$, then $(\Delta^m \mathfrak{f}_\mathfrak{p} + \Delta^m \mathfrak{f}_\mathfrak{q}) \xrightarrow{\Re I - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} (\mathfrak{L}_1 + \mathfrak{L}_2)$.

*Proof.* (i) We don't have anything to prove if $\alpha = 0$. Let's assume that $\alpha \neq 0$. As $\Delta^m \mathfrak{f}_\mathfrak{p} \xrightarrow{\Re I - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \mathfrak{L}$, with $\mathfrak{t} > 0$ and $\mathfrak{r} \geqslant 0$ as stated,

$$\mathfrak{F} = \left\{ \begin{array}{c} n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \leqslant 1 - \mathfrak{t} \text{ or} \\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}\}| \geqslant \delta \end{array} \right\} \in \mathrm{I}.$$

There is a non-empty set $\mathfrak{D} = \mathfrak{C} \backslash \mathfrak{F}$ as I is admissible. Next, select $m \in \mathfrak{D}$,

$$\frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \leqslant 1 - \mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}\}| < \delta$$

$$\Rightarrow \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) > 1 - \mathfrak{t} \,\text{or}\, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t}\}| \geqslant 1 - \delta$$

$$\Rightarrow \frac{1}{n}|\kappa| \geqslant 1 - \delta \tag{3.10.1}$$

where $\kappa = \{\mathfrak{p} \in N : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) > 1-\mathfrak{t},\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) < \mathfrak{t}\,\text{and}\,\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}; \mathfrak{r}+\epsilon) < \mathfrak{t}\}$.

It is enough for showing that for every $\mathfrak{t} > 0$ and $\mathfrak{r} \geqslant 0$:

$$\kappa \subset \left\{ \begin{array}{c} m \in N : \breve{\mu}(\alpha\Delta^m \mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) > 1 - \mathfrak{t} \\ \breve{\nu}(\alpha\Delta^m \mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t}\,\text{and}\,\breve{\omega}(\alpha\Delta^m \mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t} \end{array} \right\}.$$

Let $\mathfrak{k} \in \kappa$, then $\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{k} - \mathfrak{L}; \mathfrak{r} + \epsilon) > 1 - \mathfrak{t}\,\text{and}\,\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{k} - \mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{k} - \mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t}$.

So,

$$\begin{aligned}
\breve{\mu}(\alpha\mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) &= \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \frac{\mathfrak{r}+\epsilon}{|\alpha|}\right) \\
&\geqslant \min\left\{\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon), \breve{\mu}\left(0, \frac{\mathfrak{r}+\epsilon}{|\alpha|} - (\mathfrak{r}+\epsilon)\right)\right\} \\
&\geqslant \min\{\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon), 1\} \\
&= \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon) > 1 - \mathfrak{t}\,\text{and} \\
\breve{\nu}(\alpha\mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) &= \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \frac{\mathfrak{r}+\epsilon}{|\alpha|}\right) \\
&\leqslant \max\left\{\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon), \breve{\nu}\left(0, \frac{\mathfrak{r}+\epsilon}{|\alpha|} - (\mathfrak{r}+\epsilon)\right)\right\} \\
&\leqslant \max\{\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon), 0\} \\
&= \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon) < \mathfrak{t} \\
\breve{\omega}(\alpha\mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) &= \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \frac{\mathfrak{r}+\epsilon}{|\alpha|}\right) \\
&\leqslant \max\left\{\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon), \breve{\omega}\left(0, \frac{\mathfrak{r}+\epsilon}{|\alpha|} - (\mathfrak{r}+\epsilon)\right)\right\} \\
&\leqslant \max\{\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon), 0\} \\
&= \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{L}, \mathfrak{r} + \epsilon) < \mathfrak{t}
\end{aligned}$$

Hence $\kappa \subset \left\{ \begin{array}{c} m \in N : \breve{\mu}(\alpha\Delta^m \mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) > 1 - \mathfrak{t}, \\ \breve{\nu}(\alpha\Delta^m \mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t}\,\text{and}\,\breve{\omega}(\alpha\Delta^m \mathfrak{f}_\mathfrak{p} - \alpha\mathfrak{L}; \mathfrak{r} + \epsilon) < \mathfrak{t} \end{array} \right\}.$

Using (3.10.1), we have

$$1-\delta \leqslant \frac{|\kappa|}{n} \leqslant \frac{1}{n}\left|\left\{\begin{array}{c}\mathfrak{p} \leqslant n : \breve{\mu}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) > 1-\mathfrak{t}, \\ \breve{\nu}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) < \mathfrak{t} \text{ and } \breve{\omega}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) < \mathfrak{t}\end{array}\right\}\right|.$$

Therefore,

$$\frac{1}{n}\left|\left\{\begin{array}{c}\mathfrak{p} \leqslant n : \breve{\mu}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) \leqslant 1-\mathfrak{t}, \\ \breve{\nu}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t} \text{ and } \breve{\omega}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}\end{array}\right\}\right| < 1-(1-\delta) < \delta.$$

Then $\left\{\begin{array}{c} n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) \leqslant 1-\mathfrak{t} \text{ or} \\ \breve{\nu}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} - \alpha\mathfrak{L}; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}\}| \geqslant \delta \end{array}\right\} \subset \mathfrak{F} \in \mathrm{I},$

which shows that $\alpha\Delta^m \mathfrak{f}_{\mathfrak{p}} \xrightarrow{\mathfrak{RI}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}} \alpha\mathfrak{L}$.

(ii) We can demonstrate the part(ii) in a similar way. Therefore, we are leaving out its proof. □

**Example 3.11.** Let $(\breve{\mathfrak{X}}, \|\cdot\|)$ be a normed space and $\Delta^m x_k$ denote the m-th generalized difference operator. Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS and let the sequences $\{x_k\}$ and $\{y_k\}$ in PFNS be such that

$$\Delta^m x_k \overset{RI-ST}{\to} L_x, \Delta^m y_k \overset{RI-ST}{\to} L_y.$$

Choose, $x_k = \frac{(-1)^k}{k}$ and $y_k = \frac{1}{\sqrt{k}}$. Generalized first-order difference:

$$\Delta x_k = x_{k+1} - x_k, \Delta y_k = y_{k+1} - y_k.$$

As $k \to \infty$, both $\Delta x_k \to 0$ and $\Delta y_k \to 0$. Within PFNS $(\mu, \gamma, \omega)$, we can show that

$$\Delta x_k \overset{RI-ST}{\to} 0, \Delta y_k \overset{RI-ST}{\to} 0.$$

Infact, each sequence is $RI-ST$ convergent to 0. By the algebraic properties:

(i) Sum: $\Delta(x_k + y_k) = \Delta x_k + \Delta y_k$.

Hence by linearity, $\Delta(x_k + y_k) \overset{RI-ST}{\to} L_x + L_y = 0 + 0 = 0$.

(ii) Scalar multiplication: For any scalar $\alpha \in \mathbb{R}$,

$$\Delta(\alpha x_k) = \alpha\Delta x_k; \Delta(\alpha x_k) \overset{RI-ST}{\to} \alpha L_x = 0.$$

These follow the theorems establishing closure under addition and scalar multiplication of $RI-ST$ limit points in such spaces.

**Theorem 3.12.** Let $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ be a PFNS. The set $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}}$ of generalized difference sequence $(\Delta^m \mathfrak{f}_\mathfrak{p})$ is closed in $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$.

*Proof.* Suppose $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} = \breve{\varrho}$, then the outcome is clear as follows:

$\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}}$ can be either a singleton or empty set.

Let $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}} \neq \breve{\varrho}$, for some $\mathfrak{r} > 0$.

Let $\Delta^m \mathfrak{f} = (\Delta^m \mathfrak{f}_\mathfrak{p})$ be a convergent sequence in $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ with respect to $(\breve{\mu}, \breve{\nu}, \breve{\omega})$, which converges to $\mathfrak{f}_0 \in \mathfrak{f}$.

For $\epsilon > 0$ and $\mathfrak{t} \in (0,1)$ then $\exists\ m_0 \in N$ such that $\breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \frac{\epsilon}{2}\right) > 1 - \mathfrak{t}$, $\breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \frac{\epsilon}{2}\right) < \mathfrak{t}$ and $\breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \frac{\epsilon}{2}\right) < \mathfrak{t}$ for all $p \geqslant m_0$.

First, let's take $\Delta^m \mathfrak{f}_{m_1} \in \mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}}$ with $m_1 > m_0$ such that

$$\mathfrak{A} = \left\{ \mathfrak{p} \in N : \frac{1}{n} \left| \left\{ \begin{array}{c} \mathfrak{p} \leqslant n : \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) \leqslant 1 - \mathfrak{t} \,\text{or} \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t}, \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t} \end{array} \right\} \right| \geqslant \delta \right\} \in \mathrm{I}.$$

Since I is admissible so $\mathfrak{F} = \mathfrak{C} \backslash \mathfrak{A}$ is a non-empty set. Consider $n \in \mathfrak{F}$, then

$$\frac{1}{n} \left| \left\{ \begin{array}{c} \mathfrak{p} \leqslant n : \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) \leqslant 1 - \mathfrak{t} \,\text{or} \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t}, \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t} \end{array} \right\} \right| < \delta$$

$$\frac{1}{n} \left| \left\{ \begin{array}{c} \mathfrak{p} \leqslant n : \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) > 1 - \mathfrak{t}, \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) < \mathfrak{t} \,\text{and}\, \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) < \mathfrak{t} \end{array} \right\} \right| \geqslant 1 - \delta.$$

Put $$\mathfrak{B}_n = \left\{ \begin{array}{c} \mathfrak{p} \leqslant n : \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) > 1 - \mathfrak{t}, \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) < \mathfrak{t}, \,\text{and}\, \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right) < \mathfrak{t} \end{array} \right\}.$$

Following this, for $j \in \mathfrak{B}_n, j \geqslant m_0$, we have

$\breve{\mu}(\Delta^m \mathfrak{f}_j - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \geqslant \min\left\{\breve{\mu}\left(\Delta^m \mathfrak{f}_j - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right), \breve{\mu}\left(\Delta^m x_{m_1} - \mathfrak{f}_0; \frac{\epsilon}{2}\right)\right\} > 1 - \mathfrak{t}$

$\breve{\nu}(\Delta^m \mathfrak{f}_j - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \leqslant \max\left\{\breve{\nu}\left(\Delta^m \mathfrak{f}_j - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right), \breve{\nu}\left(\Delta^m \mathfrak{f}_{m_1} - \mathfrak{f}_0; \frac{\epsilon}{2}\right)\right\} < \mathfrak{t}$ and

$\breve{\omega}(\Delta^m \mathfrak{f}_j - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \leqslant \max\left\{\breve{\omega}\left(\Delta^m \mathfrak{f}_j - \Delta^m \mathfrak{f}_{m_1}; \mathfrak{r} + \frac{\epsilon}{2}\right), \breve{\omega}\left(\Delta^m \mathfrak{f}_{m_1} - \mathfrak{f}_0; \frac{\epsilon}{2}\right)\right\} < \mathfrak{t}.$

As a result,

$j \in \{\mathfrak{p} \in N : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) > 1 - \mathfrak{t}, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) < \mathfrak{t} \,\text{and}\, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) < \mathfrak{t}\}.$

Hence

$\mathfrak{B}_n \subset \{\mathfrak{p} \in N : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) > 1 - \mathfrak{t}, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) < \mathfrak{t} \,\text{and}\, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) < \mathfrak{t}\},$

which shows

$$1 - \delta \leqslant \frac{|\mathfrak{B}_n|}{n} \leqslant \frac{1}{n} \left| \left\{ \begin{array}{c} \mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) > 1 - \mathfrak{t}, \\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) < \mathfrak{t} \,\text{and}\, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) < \mathfrak{t} \end{array} \right\} \right|.$$

Hence, $$\frac{1}{n} \left| \left\{ \begin{array}{c} \mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or} \\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\mathfrak{d}^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_\delta; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t} \end{array} \right\} \right| < 1 - (1 - \delta) = \delta.$$

Next, $$\left\{ \begin{array}{c} n \in N : \frac{1}{n} |\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \leqslant 1 - \mathfrak{t} \,\text{or} \\ \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \mathfrak{f}_0; \mathfrak{r} + \epsilon) \geqslant \mathfrak{t}\}| \geqslant \delta \end{array} \right\} \subset \mathfrak{A} \in \mathrm{I}.$$

which implies that $\mathfrak{f}_0 \in \mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_\mathfrak{p}}$ in $(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$.

The convexity of the set $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}}$ is proved in the upcoming result. □

DOI: 10.35629/0743-12077086

**Theorem 3.13.** For some non-negative number $\mathfrak{r}$ the collection $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_{\mathfrak{p}}}$ of generalized difference sequence $\Delta^m \mathfrak{f}_{\mathfrak{p}}$ in PFNS$(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$ is a convex set.

*Proof.* Consider $\breve{\varrho}_1, \breve{\varrho}_2 \in \mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_{\mathfrak{p}}}$.

To prove convexity, we must demonstrate $(1-\lambda)\breve{\varrho}_1 + \lambda\breve{\varrho}_2 \in \mathrm{I} - \mathfrak{st} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}}$ for any real number $\lambda \in (0,1)$.

Since $\breve{\varrho}_1, \breve{\varrho}_2 \in \mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_{\mathfrak{p}}}$, then there is a $\mathfrak{p} \in N$ for all $\epsilon > 0$ and $\mathfrak{t} \in (0,1)$ such that

$$\mathfrak{A}_0 = \left\{ \begin{array}{c} \mathfrak{p} \in N : \breve{\mu}\left(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \breve{\varrho}_1; \frac{\mathfrak{r}+\epsilon}{2(1-\lambda)}\right) \leqslant 1 - \mathfrak{t} \text{ or} \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \breve{\varrho}_1; \frac{\mathfrak{r}+\epsilon}{2(1-\lambda)}\right) \geqslant \mathfrak{t}, \breve{\omega}\left(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \breve{\varrho}_1; \frac{\mathfrak{r}+\epsilon}{2(1-\lambda)}\right) \geqslant \mathfrak{t} \end{array} \right\} \text{ and}$$

$$\mathfrak{A}_1 = \left\{ \begin{array}{c} \mathfrak{p} \in N : \breve{\mu}\left(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \breve{\varrho}_2; \frac{\mathfrak{r}+\epsilon}{2\lambda}\right) \leqslant 1 - \mathfrak{t} \text{ or} \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \breve{\varrho}_2; \frac{\mathfrak{r}+\epsilon}{2\lambda}\right) \geqslant \mathfrak{t}, \breve{\omega}\left(\Delta^m \mathfrak{f}_{\mathfrak{p}} - \breve{\varrho}_2; \frac{\mathfrak{r}+\epsilon}{2\lambda}\right) \geqslant \mathfrak{t} \end{array} \right\}.$$

For $\delta > 0$, we have $\left\{n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_0 \cup \mathfrak{A}_1|\} \geqslant \delta\right\} \in \mathrm{I}$.

Now take $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$.

Let $\mathfrak{A} = \left\{n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_0 \cup \mathfrak{A}_1|\} \geqslant \delta_1\right\} \in \mathrm{I}$.

Now for $n \notin \mathfrak{A}$, $\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_0 \cup \mathfrak{A}_1|\} < 1 - \delta_1$, $\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \notin \mathfrak{A}_0 \cup \mathfrak{A}_1|\} \geqslant 1 - (1-\delta_1) = \delta_1$.

This gives that $\{\mathfrak{p} \leqslant n : m \notin \mathfrak{A}_0 \cup \mathfrak{A}_1\} \neq \breve{\varrho}$.

Consider $m_0 \in (\mathfrak{A}_0 \cup \mathfrak{A}_1)^c = \mathfrak{A}_0^c \cap \mathfrak{A}_1^c$. Then

$$\begin{aligned} &\breve{\mu}(\Delta^m \mathfrak{f}_{m_0} - [(1-\lambda)\breve{\varrho}_1 + \lambda\breve{\varrho}_2]; \mathfrak{r} + \epsilon) \\ &= \breve{\mu}[(1-\lambda)(\Delta^m \mathfrak{f}_{m_0} - \breve{\varrho}_1) + \lambda(\Delta^m \mathfrak{f}_{m_0} - \breve{\varrho}_2); \mathfrak{r} + \epsilon] \end{aligned}$$

$$\geqslant \min\left\{\breve{\mu}\left((1-\lambda)(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1);\frac{\mathfrak{r}+\epsilon}{2}\right),\breve{\mu}\left(\lambda(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2);\frac{\mathfrak{r}+\epsilon}{2}\right)\right\}$$

$$=\min\left\{\breve{\mu}\left(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1;\frac{\mathfrak{r}+\epsilon}{2(1-\lambda)}\right),\breve{\mu}\left(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2;\frac{\mathfrak{r}+\epsilon}{2\lambda}\right)\right\}$$

$$>1-\mathfrak{t}\text{ and}$$

$$\breve{\nu}(\Delta^m \mathfrak{f}_{m_0}-[(1-\lambda)\breve{\varrho}_1+\lambda\breve{\varrho}_2];\mathfrak{r}+\epsilon)$$

$$=\breve{\nu}[(1-\lambda)(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1)+\lambda(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2);\mathfrak{r}+\epsilon]$$

$$\leqslant \max\left\{\breve{\nu}\left((1-\breve{\omega})(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1);\frac{\mathfrak{r}+\epsilon}{2}\right),\breve{\nu}\left(\lambda(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2);\frac{\mathfrak{r}+\epsilon}{2}\right)\right\}$$

$$=\max\left\{\breve{\nu}\left(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1;\frac{\mathfrak{r}+\epsilon}{2(1-\lambda)}\right),\breve{\nu}\left(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2;\frac{\mathfrak{r}+\epsilon}{2\lambda}\right)\right\}$$

$$<\mathfrak{t}$$

$$\breve{\omega}(\Delta^m \mathfrak{f}_{m_0}-[(1-\lambda)\breve{\varrho}_1+\lambda\breve{\varrho}_2];\mathfrak{r}+\epsilon)$$

$$=\breve{\omega}[(1-\lambda)(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1)+\lambda(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2);\mathfrak{r}+\epsilon]$$

$$\leqslant \max\left\{\breve{\omega}\left((1-\breve{\omega})(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1);\frac{\mathfrak{r}+\epsilon}{2}\right),\breve{\omega}\left(\lambda(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2);\frac{\mathfrak{r}+\epsilon}{2}\right)\right\}$$

$$=\max\left\{\breve{\omega}\left(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_1;\frac{\mathfrak{r}+\epsilon}{2(1-\lambda)}\right),\breve{\omega}\left(\Delta^m \mathfrak{f}_{m_0}-\breve{\varrho}_2;\frac{\mathfrak{r}+\epsilon}{2\lambda}\right)\right\}$$

$$<\mathfrak{t}$$

This implies that $\mathfrak{A}_0^c \cap \mathfrak{A}_1^c \subset \mathfrak{B}^c$, where

$$\mathfrak{B}=\left\{\begin{array}{c}\mathfrak{p}\in N:\breve{\mu}(\Delta^m \mathfrak{f}_{m_0}-[(1-\lambda)\breve{\varrho}_1+\lambda\breve{\varrho}_2];\mathfrak{r}+\epsilon)\leqslant 1-\mathfrak{t}\text{ or}\\ \breve{\nu}(\Delta^m \mathfrak{f}_{m_0}-[(1-\lambda)\breve{\varrho}_1+\lambda\breve{\varrho}_2];\mathfrak{r}+\epsilon)\geqslant \mathfrak{t},\breve{\omega}(\Delta^m \mathfrak{f}_{m_0}-[(1-\lambda)\breve{\varrho}_1+\lambda\breve{\varrho}_2];\mathfrak{r}+\epsilon)\geqslant \mathfrak{t}\end{array}\right\}.$$

So for $n\notin\mathfrak{A}$,

$\delta_1 \leqslant \frac{1}{n}|\{\mathfrak{p}\leqslant n:\mathfrak{p}\notin \mathfrak{A}_0\cup\mathfrak{A}_1\}| \leqslant \frac{1}{n}|\{\mathfrak{p}\leqslant n:\mathfrak{p}\notin\mathfrak{B}\}|$ or $\frac{1}{n}|\{\mathfrak{p}\leqslant n:\mathfrak{p}\in\mathfrak{B}\}|<1-\delta_1<\delta$.

Hence $\mathfrak{A}^c\subset\left\{n\in N:\frac{1}{n}|\mathfrak{p}\leqslant n:\mathfrak{p}\in\mathfrak{B}|<\delta\right\}$. Since $\mathfrak{A}^c\in\mathfrak{F}(\mathrm{I})$, then

$\left\{n\in N:\frac{1}{n}|\mathfrak{p}\leqslant n:\mathfrak{p}\in\mathfrak{B}|<\delta\right\}\in\mathfrak{F}(\mathrm{I})$, which gives $\left\{n:\frac{1}{n}|\mathfrak{p}\leqslant n:\mathfrak{p}\in\mathfrak{B}|\geqslant\delta\right\}\in\mathrm{I}$.

It follows that $\mathrm{I}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}-LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_{\mathfrak{p}}}$ is convex. □

**Theorem 3.14.** Let generalized difference sequence $(\Delta^m \mathfrak{f}_{\mathfrak{p}})$ in PFNS$(\breve{\mathfrak{X}},\breve{\mu},\breve{\nu},\breve{\omega},*,\diamondsuit,\circledast)$ be $\mathfrak{R}\mathrm{I}-\Delta^m$-$\mathfrak{st}$ convergent to $\rho\in\breve{\mathfrak{X}}$ with respect to the norm $(\breve{\mu},\breve{\nu},\breve{\omega})$ for some $\mathfrak{r}>0$ if there exists a sequence $\Delta^m\mathfrak{y}=(\Delta^m\mathfrak{y}_{\mathfrak{p}})$ in $\breve{\mathfrak{X}}$ such that $\mathrm{I}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}-LIM_{\Delta^m\mathfrak{y}_{\mathfrak{p}}}=\rho$ in $\mathfrak{f}$ and for every $\mathfrak{t}\in(0,1)$

have $\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}+\epsilon) > 1-\mathfrak{t}, \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}+\epsilon) < \mathfrak{t}$ and $\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}+\epsilon) < \mathfrak{t}$, for all $\mathfrak{p} \in N$.

*Proof.* Since $\Delta^m \mathfrak{y} = (\Delta^m \mathfrak{y}_\mathfrak{p})$ be a generalized difference sequence in $\breve{\mathfrak{X}}$, which is $\mathrm{I} - \Delta^m$-$\mathfrak{st}$ convergent to $\rho \in \breve{\mathfrak{X}}$ and

$\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}+\epsilon) > 1-\mathfrak{t}$ and $\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}+\epsilon) < \mathfrak{t}$, $\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}+\epsilon) < \mathfrak{t}$ for all $\mathfrak{p} \in N$ and $\mathfrak{t} \in (0,1)$.

Then by definition, for any $\epsilon, \delta > 0$ and $\mathfrak{t} \in (0,1)$ the set

$$\mathfrak{D} = \left\{ \begin{array}{c} n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \breve{\mu}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon) \leqslant 1-\mathfrak{t} \text{ or} \\ \breve{\nu}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon) \geqslant \mathfrak{t}\}| \geqslant \delta \end{array} \right\} \in \mathrm{I}.$$

Define $\mathfrak{A}_1 = \{\mathfrak{p} \in N : \breve{\mu}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon) \leqslant 1-\mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon) \geqslant \mathfrak{t}\}$,

$\mathfrak{A}_2 = \{\mathfrak{p} \in N : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}) \leqslant 1-\mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}) \geqslant \mathfrak{t}\}$.

For $\delta > 0$, we have $\left\{n \in N : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2|\} \geqslant \delta\right\} \in \mathrm{I}$.

Select $0 < \delta_1 < 1$ so that $0 < 1-\delta_1 < \delta$ and let

$\mathfrak{A} = \left\{n : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2|\} \geqslant \delta_1\right\} \in \mathrm{I}$.

Now for $n \notin \mathfrak{A}$, $\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2|\} < 1-\delta_1$, $\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \notin \mathfrak{A}_1 \cup \mathfrak{A}_2|\} \geqslant 1-(1-\delta_1) = \delta_1$.

That means $\{\mathfrak{p} \leqslant n : \mathfrak{p} \notin \mathfrak{A}_1 \cup \mathfrak{A}_2\} \neq \breve{\varrho}$.

Let $\mathfrak{p} \in (\mathfrak{A}_1 \cup \mathfrak{A}_2)^c = \mathfrak{A}_1^c \cap \mathfrak{A}_2^c$.

Then $\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \rho; \mathfrak{r}+\epsilon) \geqslant \min\{\breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}), \breve{\mu}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon)\} > 1-\mathfrak{t}$ and

$\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \rho; \mathfrak{r}+\epsilon) \leqslant \max\{\breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}), \breve{\nu}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon)\} < \mathfrak{t}$ and

$\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \rho; \mathfrak{r}+\epsilon) \leqslant \max\{\breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \Delta^m \mathfrak{y}_\mathfrak{p}; \mathfrak{r}), \breve{\omega}(\Delta^m \mathfrak{y}_\mathfrak{p} - \rho; \epsilon)\} < \mathfrak{t}$

which gives $\mathfrak{A}_1^c \cap \mathfrak{A}_2^c \subset \mathfrak{B}^c$, where

$\mathfrak{B} = \left\{\mathfrak{p} \in N : \breve{\mu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \rho; \mathfrak{r}+\epsilon) \leqslant 1-\mathfrak{t} \text{ or } \breve{\nu}(\Delta^m \mathfrak{f}_\mathfrak{p} - \rho; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}, \breve{\omega}(\Delta^m \mathfrak{f}_\mathfrak{p} - \rho; \mathfrak{r}+\epsilon) \geqslant \mathfrak{t}\right\}$.

Hence, for $n \notin \mathfrak{A}$, $\delta_1 \leqslant \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \notin \mathfrak{A}_1 \cup \mathfrak{A}_2\}| \leqslant \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \notin \mathfrak{B}\}|$

or $\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{B}\}| < 1-\delta_1 < \delta$.

Thus $\mathfrak{A}^c \subset \left\{n : \frac{1}{n}|\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{B}| < \delta\right\}$.

Since $\mathfrak{A}^c \in \mathfrak{F}(\mathrm{I})$, So, $\left\{n : \frac{1}{n}|\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{B}| < \delta\right\} \in \mathfrak{F}(\mathrm{I})$, which gives

$\left\{n : \frac{1}{n}|\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{B}| \geqslant \delta\right\} \in \mathrm{I}$.

Therefore, $\Delta^m \mathfrak{f}_\mathfrak{p} \overset{\mathfrak{R}\mathrm{I}-\mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})}}{\longrightarrow} \rho$ in PFNS$(\mathfrak{f}, \breve{\mu}, \breve{\nu}, \breve{\omega})$. □

**Theorem 3.15.** Let generalized difference sequence $(\Delta^m \mathfrak{f}_\mathfrak{p})$ be in PFNS$(\breve{\mathfrak{X}}, \breve{\mu}, \breve{\nu}, \breve{\omega}, *, \Diamond, \circledast)$. Then there does not exist two elements $\eta_1, \eta_2 \in \mathrm{I} - \mathfrak{st} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}}$ for $\mathfrak{r} > 0$ and $\mathfrak{t} \in (0,1)$ such that $\breve{\mu}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) \leqslant 1 - \mathfrak{t}$ and $\breve{\nu}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) \geqslant \mathfrak{t}$ and $\breve{\omega}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) \geqslant \mathfrak{t}$ for $\mathfrak{b} > 2$.

*Proof.* Assume that, if possible $\eta_1, \eta_2 \in \mathrm{I} - \mathfrak{st} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}}$ such that

$$\breve{\mu}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) \leqslant 1 - \mathfrak{t}, \breve{\nu}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) \geqslant \mathfrak{t} \text{ and } \breve{\omega}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) \geqslant \mathfrak{t}, \text{ for } \mathfrak{b} > 2. \tag{3.15.1}$$

As $\eta_1, \eta_2 \in \mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}}$ then for every $\epsilon > 0$ and $\mathfrak{t} \in (0,1)$. Define,

$$\mathfrak{A}_1 = \left\{ \begin{array}{c} \mathfrak{p} \in N : \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_1; \mathfrak{r} + \frac{\epsilon}{2}\right) \leqslant 1 - \mathfrak{t} \text{ or} \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_1; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t}, \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_1; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t} \end{array} \right\}$$

$$\mathfrak{A}_2 = \left\{ \begin{array}{c} \mathfrak{p} \in N : \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_2; \mathfrak{r} + \frac{\epsilon}{2}\right) \leqslant 1 - \mathfrak{t} \text{ or} \\ \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_2; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t}, \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_2; \mathfrak{r} + \frac{\epsilon}{2}\right) \geqslant \mathfrak{t} \end{array} \right\}$$

Then $\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2\}| \leqslant \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1\}| + \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_2\}|$.

According to the I-convergence property, we thus have

$$\mathrm{I} - \lim_{n \to \infty} \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2\}|$$
$$\leqslant \mathrm{I} - \lim_{n \to \infty} \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1\}| + \mathrm{I} - \lim_{n \to \infty} \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_2\}| = 0.$$

Thus $\left\{n : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2\}| \geqslant \delta\right\} \in \mathrm{I}$, for all $\delta > 0$.

Now choose $0 < \delta_1 = 1/2 < 1$ such that $0 < 1 - \delta_1 < \delta$.

Let $\kappa = \left\{n : \frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2\}| \geqslant \delta_1\right\} \in \mathrm{I}$.

Now for $n \notin \kappa$,

$\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \in \mathfrak{A}_1 \cup \mathfrak{A}_2\}| < 1 - \delta_1 = 1/2$, $\frac{1}{n}|\{\mathfrak{p} \leqslant n : \mathfrak{p} \notin \mathfrak{A}_1 \cup \mathfrak{A}_2\}| \geqslant 1 - (1 - \delta_1) = 1/2$.

This implies $\{\mathfrak{p} \leqslant n : \mathfrak{p} \notin \mathfrak{A}_1 \cup \mathfrak{A}_2\} \neq \breve{\varrho}$. Then, for $\mathfrak{p} \in \mathfrak{A}_1^c \cap \mathfrak{A}_2^c$, we have

$\breve{\mu}(\eta_1 - \eta_2; 2\mathfrak{r} + \epsilon) \geqslant \min\left\{\breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_2 : \mathfrak{r} + \frac{\epsilon}{2}\right), \breve{\mu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_1; \mathfrak{r} + \frac{\epsilon}{2}\right)\right\} > 1 - \mathfrak{t}$,

$\breve{\nu}(\eta_1 - \eta_2; 2\mathfrak{r} + \epsilon) \leqslant \max\left\{\breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_2 : \mathfrak{r} + \frac{\epsilon}{2}\right), \breve{\nu}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_1; \mathfrak{r} + \frac{\epsilon}{2}\right)\right\} < \mathfrak{t}$ and

$\breve{\omega}(\eta_1 - \eta_2; 2\mathfrak{r} + \epsilon) \leqslant \max\left\{\breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_2 : \mathfrak{r} + \frac{\epsilon}{2}\right), \breve{\omega}\left(\Delta^m \mathfrak{f}_\mathfrak{p} - \eta_1; \mathfrak{r} + \frac{\epsilon}{2}\right)\right\} < \mathfrak{t}$.

Hence,

$$\breve{\mu}(\eta_1 - \eta_2; 2\mathfrak{r} + \epsilon) > 1 - \mathfrak{t}, \breve{\nu}(\eta_1 - \eta_2; 2\mathfrak{r} + \epsilon) < \mathfrak{t} \text{ and } \breve{\omega}(\eta_1 - \eta_2; 2\mathfrak{r} + \epsilon) < \mathfrak{t}. \tag{3.15.2}$$

Then from (3.15.2), we have $\breve{\mu}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) > 1 - \mathfrak{t}$, $\breve{\nu}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) < \mathfrak{t}$ and $\breve{\omega}(\eta_1 - \eta_2; \mathfrak{b}\mathfrak{r}) < \mathfrak{t}$ for $\mathfrak{b} > 2$, which is in opposition to (3.15.1).

Consequently, two elements do not exist such that

$\breve{\mu}(\eta_1 - \eta_2; \mathfrak{br}) \leqslant 1 - \mathfrak{t}, \quad \breve{\nu}(\eta_1 - \eta_2; \mathfrak{br}) \geqslant \mathfrak{t}$ and $\breve{\omega}(\eta_1 - \eta_2; \mathfrak{br}) \geqslant \mathfrak{t}$ for $\mathfrak{b} > 2$. □

## 4. Conclusion

In this study, we introduced the notion of $\mathfrak{R}\mathrm{I} - \mathfrak{st}$ convergence for difference sequences in PFNS, which extends the current theories on sequence convergence in PFNS. We have examined new concepts of $\mathfrak{R}$ - convergence and $\mathfrak{R}\mathrm{I} - \mathfrak{st}$-convergence, with a particular emphasis on how difference sequences behave in PFNS. Furthermore, we have thoroughly examined the features and characteristics of a mathematical concept that represents the $\mathfrak{R}\mathrm{I} - \mathfrak{st}$ limit set of the difference sequence and is denoted as $\mathrm{I} - \mathfrak{st}_{(\breve{\mu},\breve{\nu},\breve{\omega})} - LIM^{\mathfrak{r}}_{\Delta^m \mathfrak{f}_{\mathfrak{p}}}$. In the context of $\mathfrak{R}\mathrm{I} - \mathfrak{st}$- convergence in PFNS, our analysis of these characteristics has provided important light on the characteristics and behavior related to the $\mathfrak{R}\mathrm{I} - \mathfrak{st}$ limit established.

**Acknowledgement**: We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

## References

[1] Antal. R, Chawla. M and Kumar. V, Rough statistical convergence in intuitionistic fuzzy normed spaces, Filomat 35(13) (2021), 4405 - 4416, DOI: 10.2298/FIL2113405A.

[2] Atanassov. K.T, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1) (1986), 87 - 96, DOI: 10.1016/S0165-0114(86)80034-3.

[3] Cuong, Bui., Picture fuzzy sets, Journal of Computer Science and Cybernetics, 30(4),(2014), 409-420.

[4] Cuong, Bui., Hai V.Pham., Some fuzzy logic operators for picture fuzzy sets, Seventh International Conference on Knowledge and Systems Engineering, IEEE.

[5] Cuong, Bui.,Kreinovich, V., Ngan, R.T., A classification of representable t- norm operators for picture fuzzy sets, Departmental Technical Reports(CS).

[6] Das. P, Savas. E and Ghosal. S.K, On generalizations of certain summability methods using ideals, Applied Mathematics Letters 24(9) (2011), 1509 - 1514, DOI: 10.1016/j.aml.2011.03.036.

[7] Debnath. S and Subramanian. N, Rough statistical convergence on triple sequences, Proyecciones (Antofagasta) 36(4) (2017), 685 - 699, DOI: 10.4067/S0716-09172017000400685.

[8] Fast. H, Sur la convergence staistique, Colloquium Mathematicae 2(3-4) (1951), 241 - 244, URL: http://eudml.org/doc/209960.

[9] Jenifer, P., Jeyaraman, M., Jafari, S., and Pigazzini, A., A Study on Rough Ideal Statistical Convergence in Neutrosophic Normed Spaces, Axioms, 2025, 14, 659. https://doi.org/10.3390/axioms14090659

[10] Jenifer, P., and Jeyaraman, M., Picture fuzzy normed spaces with Fibonacci ideal convergence of double sequnces, Journal of Inequalities and Special Functions 16(1),(2025).

[11] Johnsy, J., Jeyaraman, M., Wijsman Ideal Convergence Sequence in Neutrosophic Metric Spaces, Indian Journal of Science and Technology, 17(SP1),(2024),71-78. http s://doi.org/10.17485/IJST/v17sp1.135

[12] Kalyan Sinha, Pinaki Majumdar, Picture Fuzzy Normed Linear Spaces, Bol. Soc. Paran. Mat. (2024), (42), 1-9.

[13] Karakus. S, Demirci. K and Duman. O, Statistical convergence on intuitionistic fuzzy normed spaces, Chaos Salitons & Fractals 35(4) (2008), 763 - 769, DOI: 10.1016/j.chaos.2006.05.046.

[14] Kizmaz. H, On certain sequence spaces, Canadian Mathematical Bulletin 24(2) (1981), 169 - 176, DOI: 10.4153/CMB-1981-027-5.

[15] Kostyrko. P, Macaj. M, Salat. T and Sleziak. M, I-convergence and extremal I-limit points, Mathematica Slovaca 55(4) (2005), 443 - 464, URL: http://dml.cz/dmlcz/132906.

[16] Kostyrko. P, Wilczynski. W and Salat. T, I-convergence, Real Analysis Exchange 26(2) (2000/2001), 669 - 685, URL: https://projecteuclid.org/journals/real-analysis-exchange/volume-26/issue-2/I-Convergence/rae/1214571359.full?tab=ArticleLink.

[17] Malik. P and Maity. M, On rough convergence of double sequence in normed linear spaces, Bulletin of the Allahabad Mathematical Society 28 (2013), 89 - 99, URL: https://www.amsallahabad.org/pdf/bams281.pdf.

[18] Malik. P and Maity. M, On rough statistical convergence of double sequences in normed linear spaces, Afrika Matematika 27 (2016), 141 - 148, DOI: 10.1007/s13370-015-0332-9.

[19] Park. J.H, Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 22(5) (2004), 1039 - 1046, DOI: 10.1016/j.chaos.2004.02.051.

[20] Phu. H.X, Rough convergence in normed linear spaces, Numerical Functional Analysis and Optimization 22(1-2) (2001), 199 - 222, DOI: 10.1081/NFA-100103794.

[21] Phu. H.X, Rough convergence in infinite dimensional normed spaces, Numerical Functional Analysis and Optimization 24(3-4) (2003), 285 - 301, DOI: 10.1081/NFA-120022923.

[22] Saadati. R and Park. J.H, On the intuitionistic fuzzy topological spaces, Chaos, Solitons & Fractals 27(2) (2006), 331 - 344, DOI: 10.1016/j.chaos.2005.03.019.

[23] Savas. E and Gurdal. M, Certain summability methods in intuitionistic fuzzy normed spaces, Journal of Intelligent & Fuzzy Systems 27(4) (2014), 1621 - 1629, DOI: 10.3233/IFS-141128.

[24] Steinhaus. H, Sur la convergence ordinaire et la convergence asymptotique, Colloquium Mathematicum 2 (1951), 73 - 74.

[25] Zadeh. L.A, Fuzzy sets, Information and Control 8(3) (1965), 338 - 353, DOI: 10.1016/S0019-9958(65)90241-X.