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Research Paper

A Study of Fixed Point Theorems for Self-Mappings in Complex-Valued Neutrosophic Metric Spaces

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Abstract: *This paper presents complex-valued neutrosophic metric spaces as an extension of complex-valued fuzzy and intuitionistic fuzzy metric spaces. It explores the existence and uniqueness of common fixed points for self-mappings under diverse contractive conditions within this extended framework. Furthermore, detailed discussions on fixed point theorems for self-mappings are provided. To support the theoretical results and ensure clarity, illustrative examples are included.*

Keywords: *t-norm; t-conorm; Complex valued neutrosophic metric space, Self -mappings, Unique fixed point*
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I. Introduction

Fixed-point theory plays a vital role in mathematical analysis, with diverse and significant applications. The renowned Banach contraction principle, first introduced in [4], serves as a foundational tool for addressing existence problems in nonlinear analysis. Over time, this principle has been extended and refined into various fixed-point theorems using different approaches. Fuzzy set theory has gained substantial recognition in recent years, finding applications across various disciplines and forming the foundation for numerous advanced mathematical models. First proposed by Zadeh [25] in 1965, it provides an effective means of handling uncertainty and ambiguity. By addressing vagueness in complex real-world scenarios, the theory extends beyond the constraints of classical binary logic, enabling the examination of problems characterized by imprecise or undefined boundaries Building on this framework, Kramosil and Michalek [15] introduced fuzzy metric and statistical metric spaces, which extended classical metric spaces to handle uncertainty and variability in a more nuanced way and their work laid important groundwork that has since enabled the development of intuitionistic and neutrosophic fuzzy metric spaces, allowing for even broader applications across mathematics and applied sciences. Expanding on the early development of fuzzy metric spaces, George and Veeramani [6] contributed pivotal results on fixed points, which clarified important aspects of continuity and convergence in these spaces. Grabiec [7] expanded upon the fuzzy metric space concept initially formulated by Kramosil and Michalek by introducing G-Cauchy sequences, a notion that diverges significantly from the classical Cauchy sequence framework set forth by George and Veeramani. Moreover, it has been demonstrated that traditional fuzzy metric spaces lack G-completeness, and compactness within a fuzzy metric space does not necessarily imply G-completeness. In 1986, Atanassov [2] expanded fuzzy set theory by introducing the concept of intuitionistic fuzzy sets. His work delved into the properties of operations and relationships between these sets, as well as their interplay with modal and topological operators, significantly advancing the theoretical groundwork of fuzzy sets. In 2004, Park [17] further expanded the concept by defining intuitionistic fuzzy metric spaces. In 2011, Azam et al. [3] introduced complex-valued metric spaces into fixed-point theory, using ordered complex numbers instead of positive real numbers to obtain fixed-point results for maps under rational inequality conditions. Shukla et al.

[21] later extended this idea to fuzzy metric spaces, defining complex-valued fuzzy metric spaces and deriving fixed-point results for mappings satisfying specific contractive conditions. This field continues to attract significant research, as seen in the work of [5, 26] and the comprehensive studies by Humaira et al. [8]-[10], which produced various results with real-world applications. Smarandache [23] introduced neutrosophic theory in 1998, which extends fuzzy set theory by considering the independent handling of truth, indeterminacy, and falsity. Kiri¸sci and S¸im¸sek [16] contributed to the advancement of this area by proposing neutrosophic metric spaces, offering a strong framework to tackle uncertainty, vagueness, and indeterminacy. Sowndrarajan et al. [24] highlighted the practical applications of neutrosophic metric spaces by deriving fixed point results for contraction theorems within these spaces. This paper introduces an extension of fuzzy metric spaces by developing the concept of complex-valued neutrosophic metric spaces. This framework broadens the scope of both complex-valued fuzzy metric spaces [21] and intuitionistic fuzzy metric spaces [11]. Within this setting, several fixed-point results for self-mappings satisfying contractive conditions are derived.

II. Preliminaries

This study will require the following definitions and results. We use the following symbols to denote certain sets

(a). $\mathbb{N}$: Set of natural numbers,

(b). $\mathbb{N}_0$: Set of non-negative integers,

(C) $\mathbb{C}$: Set of complex numbers.

The following defintions are used

(i) $\zeta = \{(\mathfrak{p}, \mathfrak{q}) : 0 \leq \mathfrak{p} < \infty, 0 \leq \mathfrak{q} < \infty\} \subset \mathbb{C}$

(ii) The points $(0,0)$ and $(1,1)$ within $\mathbb{C}$ as θ and ℓ , repectively,

(iii) The Closed unit complex interval as $\Upsilon = \{(\mathfrak{p}, \mathfrak{q}) : 0 \leq \mathfrak{p} \leq 1, 0 \leq \mathfrak{q} \leq 1\}$,

(iv) The Open unit complex interval $\Upsilon_0 = \{(\mathfrak{p}, \mathfrak{q}) : 0 < \mathfrak{p} < 1, 0 < \mathfrak{q} < 1\}$. Moreover, ζ_0 is designated as $\{(\mathfrak{p}, \mathfrak{q}) : 0 < \mathfrak{p} < \infty, 0 < \mathfrak{q} < \infty\}$.

A partial order $\preceq$ is established on $\mathbb{C}$, where $\tau_1 \preceq \tau_2 \iff \tau_2 - \tau_1 \in \zeta$, where $\tau_1, \tau_2 \in \mathfrak{C}$. We use $\tau_1 \prec \tau_2$ to signify that $Re(\tau_2) > Re(\tau_1)$ and $Im(\tau_2) > Im(\tau_1)$. This implies that $\tau_1 \prec \tau_2$

hilds if and only if $\tau_2 - \tau_1 \in \zeta_0$.

Now, let $\{\tau_n\}$ be a sequence in $\mathbb{C}$. The sequence is monotonic with respect to $\preceq$ if $\tau_{n+1} \preceq \tau_n$ or $\tau_n \preceq \tau_{n+1}$ for each n belonging to $\mathbb{N}$.

The element $\inf \Omega \preceq \eta$ is known as the infimum of Ω in any subst Ω of $\mathbb{C}$. Similarly, the supremum or least upper bound of Ω by $\sup \Omega$.

Definition 2.1. ([21]). Suppose that for every $n \in \mathbb{N}$, $\tau_n \in \zeta$. These attributes are true:

(1)There is a limit $\tau \in \zeta$ such that $\tau_n \to \tau$ as n approaches infinity, if $\{\tau_n\}$ forms a monotonic sequence under $\preceq$ and there exists bounds $\mu, \nu \in \zeta$ with $\mu \preceq \tau_n \preceq \nu$ for each $n \in \mathbb{N}$.

(2) A lattice structure is induced within $\mathbb{C}$ by $\preceq$, even if does not constitude a total order on $\mathbb{C}$.

(3) If every η in subset $\Omega \subset \mathbb{C}$ lies between μ and ν, such that$\mu \preceq \eta \preceq \nu$ for some $\mu, \nu \in \mathbb{C}$, then $\inf \Omega$ and $\sup \Omega$ exist.

Definition 2.2. ([21]). Suppose $\tau_n, \tau_n^{'} \in \zeta_0 \ \forall \ n \in \mathbb{N}$, the statements listed below are true:

(1) If $\tau_n \preceq \tau_n^{'} \preceq \ell \ \forall \ n \in \mathbb{N}$ and $\tau_n \to \ell$ as n tends to ∞, i.e., $\tau_n^{'} = \ell$.

(2) If $\tau_n \preceq \lambda \ \forall \ n \in \mathbb{N}$ and there exists some $\tau \in \zeta$ satisfying $\lim\limits_{n\to\infty} \tau_n = \tau, \tau \preceq \lambda$, this condition holds.

(3) If $\lambda \preceq \tau_n \ \forall \ n \in \mathbb{N}$ and $\exists \tau \in \zeta \ni; \lim\limits_{n\to\infty} \tau_n = \tau, \lambda \preceq \tau$, this condition is satisfied.

Definition 2.3. ([21]). A binary operation $*$, defined as a mapping from Υ^2 to Υ is termed a complex-valued t-norm[CVTN] if it fulfills the following properties:

(1) For every $\tau \in \Upsilon$, $\theta * \tau = \theta, \ell * \tau = \tau$;

(2) $*$ is associative and commutative;

(3) For each $\tau_1, \tau_2, \tau_3, \tau_4$ belonging to Υ, if $\tau_3 * \tau_4 \succeq \tau_2 * \tau_1$ then $\tau_3 succeq \tau_1, \tau_4 \succeq \tau_2$.

Example 2.4. ([21]). Here are some examples of complex valued t-norm

(1) $\tau_1 * \tau_2 = (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2)$;

(2) $\tau_1 * \tau_2 = (\min\{\mathfrak{p}_1, \mathfrak{p}_2\}, \min\{\mathfrak{q}_1, \mathfrak{q}_2\})$;

(3) $\tau_1 * \tau_2 = (\max\{\mathfrak{p}_1 + \mathfrak{p}_2 - 1, 0\}, \max\{\mathfrak{q}_1 + \mathfrak{q}_2 - 1, 0\})$.

Definition 2.5. ([11]) A binary operation $\Diamond$, defined as a mapping from Υ^2 to Υ is named to be a complex-valued t-conorm [CVNTCN] if it adheres to the following criteria:

(1) $\tau \Diamond \theta = \tau, \tau \Diamond \ell = \ell \; \forall \tau \in \Upsilon$;

(2) $\Diamond$ is associative and commutative;

(3) $\tau_3 \Diamond \tau_4 \succeq \tau_1 \Diamond \tau_2$ given that $\tau_3 \succeq \tau_1, \tau_4 \succeq \tau_2$ for each $\tau_1, \tau_2, \tau_3, \tau_4$ belonging to Υ.

Example 2.6. Here are examples of complex valued t-conorm:

(1) $\tau_1 \Diamond \tau_2 = (\mathfrak{p}_1 + \mathfrak{p}_2, \mathfrak{q}_1 + \mathfrak{q}_2) - (\mathfrak{p}_1\mathfrak{p}_2, \mathfrak{q}_1\mathfrak{q}_2)$;

(2) $\tau_1 \Diamond \tau_2 = (\max\{\mathfrak{p}_1, \mathfrak{p}_2\}, \max\{\mathfrak{q}_1, \mathfrak{q}_2\})$;

(3) $\tau_1 \Diamond \tau_2 = (\min\{\mathfrak{p}_1 + \mathfrak{p}_2, 1\}, \min\{\mathfrak{q}_1 + \mathfrak{q}_2, 1\})$.

Definition 2.7. ([11]) Consider Ξ as a non-void set, with $*$ and $\Diamond$ denoting continuous CVTN and CVTCN, respectively. Let $\mathcal{A}, \mathcal{B}$ be complex fuzzy sets defined on $\Xi^2 \times \zeta_0$, provided that the conditions listed below are satisfied:

(1) $\mathcal{A}(\mathfrak{o}, \pi, \tau) + \mathcal{B}(\mathfrak{o}, \pi, \tau) \preceq \ell$;

(2) $\mathcal{A}(\mathfrak{o}, \pi, \tau) \succ \theta$;

(3) $\mathcal{A}(\mathfrak{o}, \pi, \tau) = \ell$ for each $\tau \in \zeta_0 \iff \mathfrak{o} = \pi$;

(4) $\mathcal{A}(\mathfrak{o}, \pi, \tau) = \mathcal{A}(\pi, \mathfrak{o}, \tau)$;

(5) $\mathcal{A}(\mathfrak{o}, \varsigma, \tau + \tau^{'}) \succeq \mathcal{A}(\mathfrak{o}, \pi, \tau) * \mathcal{A}(\pi, \varsigma, \tau^{'})$;

(6) The mapping $\mathcal{A}(\mathfrak{o}, \pi, \cdot)$ from ζ_0 to Υ is continuous;

(7) $\mathcal{B}(\mathfrak{o}, \pi, \tau) \prec \ell$;

(8) $\mathcal{B}(\mathfrak{o}, \pi, \tau) = \theta$ for every $\tau \in \zeta_0 \iff \mathfrak{o} = \pi$;

(9) $\mathcal{B}(\mathfrak{o}, \pi, \tau) = \mathcal{B}(\pi, \mathfrak{o}, \tau^{'})$;

(10) $\mathcal{B}(\mathfrak{o}, \varsigma, \tau + \tau^{'}) \preceq \mathcal{B}(\mathfrak{o}, \pi, \tau) \Diamond \mathcal{B}(\pi, \varsigma, \tau^{'})$;

(11) The mapping $\mathcal{B}(\mathfrak{o}, \pi, \cdot)$ from ζ_0 to Υ is continuous;

Here, $\mathfrak{o}, \pi, \varsigma \in \Xi$ and $\tau, \tau^{'} \in \zeta_0$.

Under these conditions, the 5-tuple $(\Xi, \mathcal{A}, \mathcal{B}, *, \Diamond)$ is named to be a complex-valued intuitionistic fuzzy metric space.

3. Main Results

This section primarily focuses on presenting and analyzing the properties of complex-valued neutrosophic metric spaces.

Definition 3.1. Consider Ξ is a non-void set, with $*$ and $\Diamond$ representing continuousCVTN and CVTCN, respectively. Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are complex valued neutrosophic sets defined on $\Xi^2 \times \zeta_0$ whereby the following conditions hold:

(1) $\mathcal{A}(\mathfrak{o}, \pi, \tau) + \mathcal{B}(\mathfrak{o}, \pi, \tau) + \mathcal{C}(\mathfrak{o}, \pi, \tau) \preceq 3\ell$;

(2) $\mathcal{A}(\mathfrak{o}, \pi, \tau) \succ \theta$;

(3) $\mathcal{A}(\mathfrak{o}, \pi, \tau) = \ell$ for each $\tau \in \zeta_0 \iff \mathfrak{o} = \pi$;

(4) $\mathcal{A}(\mathfrak{o}, \pi, \tau) = \mathcal{A}(\pi, \mathfrak{o}, \tau)$;

(5) $\mathcal{A}(\mathfrak{o}, \varsigma, \tau + \tau^{'}) \succeq \mathcal{A}(\mathfrak{o}, \pi, \tau) * \mathcal{A}(\pi, \varsigma, \tau^{'})$;

(6) The mapping $\mathcal{A}(\mathfrak{o}, \pi, \cdot)$ from ζ_0 to Υ is continuous;

(7) $\mathcal{B}(\mathfrak{o}, \pi, \tau) \prec \ell$;

(8) $\mathcal{B}(\mathfrak{o}, \pi, \tau) = \theta$ for every $\tau \in \zeta_0 \iff \mathfrak{o} = \pi$;

(9) $\mathcal{B}(\mathfrak{o}, \pi, \tau) = \mathcal{B}(\pi, \mathfrak{o}, \tau^{'})$;

(10) $\mathcal{B}(\mathfrak{o}, \varsigma, \tau + \tau^{'}) \preceq \mathcal{B}(\mathfrak{o}, \pi, \tau) \Diamond \mathcal{B}(\pi, \varsigma, \tau^{'})$;

(11)The mapping $\mathcal{B}(\mathfrak{o}, \pi, \cdot)$ from ζ_0 to Υ is continuous;

(12) $\mathcal{C}(\mathfrak{o}, \pi, \tau) \prec \ell$;

(13) $\mathcal{C}(\mathfrak{o}, \pi, \tau) = \theta$ for every $\tau \in \zeta_0 \iff \mathfrak{o} = \pi$;

(14) $\mathcal{C}(\mathfrak{o}, \pi, \tau) = \mathcal{B}(\pi, \mathfrak{o}, \tau^{'})$;

(15) $\mathcal{C}(\mathfrak{o}, \varsigma, \tau + \tau^{'}) \preceq \mathcal{C}(\mathfrak{o}, \pi, \tau) \Diamond \mathcal{C}(\pi, \varsigma, \tau^{'})$;

(16) The mapping $\mathcal{C}(\mathfrak{o}, \pi, \cdot)$ from ζ_0 to Υ is continuous;

for each $\mathfrak{o}, \pi, \varsigma \in \Xi$ and $\tau, \tau^{'} \in \zeta_0$.

The structure $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ is named to be complex-valued neutrosophic metric space [CVNMS].

Remark 3.2. A CVNMS can be developed in a complex-valued fuzzy metric space $(\Xi, \mathcal{A}, *)$, by defining it as $(\Xi, \mathcal{A}, \ell - \mathcal{A}, \ell - \mathcal{A}, *, \Diamond)$, where the CVTCN $\Diamond$ and CVTN $*$ are connected. For example, the relation $\tau_1 \Diamond \tau_2 = \ell - ((\ell - \tau_1) * (\ell - \tau_2))$ holds for every $\tau_1, \tau_2 \in \Upsilon$.

Example 3.3. Let (Ξ, d) be a metric space, $d(\mathfrak{o}, \pi) = |\mathfrak{o} - \pi|$. For $\tau_i = (\mathfrak{p}_i, \mathfrak{q}_i) \in \Upsilon$ where $i = 1, 2$, define $*$ and $\Diamond$ by

$\tau_1 * \tau_2 = (\min\{\mathfrak{p}_1, \mathfrak{p}_2\}, \min\{\mathfrak{q}_1, \mathfrak{q}_2\})$ and $\tau_1 \Diamond \tau_2 = (\max\{\mathfrak{p}_1, \mathfrak{p}_2\}, \max\{\mathfrak{q}_1, \mathfrak{q}_2\})$.

Now, we proceed to define the complex neutrosophic sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as:

$$\mathcal{A}(\mathfrak{o}, \pi, \tau) = e^{-\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}}\ell, \mathcal{B}(\mathfrak{o}, \pi, \tau) = (1 - e^{-\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}})\ell$$

and

$$\mathcal{C}(\mathfrak{o}, \pi, \tau) = (1 - e^{\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}})\ell$$

for all $\mathfrak{o}, \pi \in \Xi$ and $\tau = (\mathfrak{p}, \mathfrak{q}) \in \zeta_0$. As a result, the structure $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ forms a CVNMS.

Lemma 3.4. Consider $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ as a CVNMS. the mapping $\mathcal{A}(\mathfrak{o}, \pi, \cdot)$ is monotonic increasing, $\mathcal{B}(\mathfrak{o}, \pi, \cdot)$ and $\mathcal{C}(\mathfrak{o}, \pi, \cdot)$ are monotonic decreasing. Specifically for any $\tau, \tau' \in \zeta_0$ with $\tau \prec \tau'$, it follows that $\mathcal{A}(\mathfrak{o}, \pi, \tau) \preceq \mathcal{A}(\mathfrak{o}, \pi, \tau')$, $\mathcal{B}(\mathfrak{o}, \pi, \tau) \succeq \mathcal{B}(\mathfrak{o}, \pi, \tau')$ and $\mathcal{C}(\mathfrak{o}, \pi, \tau) \succeq \mathcal{C}(\mathfrak{o}, \pi, \tau')$ for every $\mathfrak{o}, \pi \in \Xi$.

Proof. If $\tau, \tau' \in \zeta_0$ where $\tau \prec \tau'$,then it follows that $\tau' - \tau \in \zeta_0$. By applying condition (5) from Definition (3.1), it follows that

$$\begin{aligned}
\mathcal{A}(\mathfrak{o}, \pi, \tau') &= \mathcal{A}(\mathfrak{o}, \pi, \tau' - \tau + \tau) \\
&\succeq \mathcal{A}(\mathfrak{o}, \mathfrak{o}, \tau' - \tau) * \mathcal{A}(\mathfrak{o}, \pi, \tau) \\
&= \ell * \mathcal{A}(\mathfrak{o}, \pi, \tau) \\
&= \mathcal{A}(\mathfrak{o}, \pi, \tau).
\end{aligned}$$

Hence, $\mathcal{A}(\mathfrak{o}, \pi, \tau') \succeq \mathcal{A}(\mathfrak{o}, \pi, \tau)$.However, using condition (10) from Definition (3.1), we have

$$\begin{aligned}
\mathcal{B}(\mathfrak{o}, \pi, \tau') &= \mathcal{B}(\mathfrak{o}, \pi, \tau' - \tau + \tau) \\
&\preceq \mathcal{B}(\mathfrak{o}, \mathfrak{o}, \tau' - \tau) \Diamond \mathcal{B}(\mathfrak{o}, \pi, \tau) \\
&= \theta \Diamond \mathcal{B}(\mathfrak{o}, \pi, \tau) \\
&= \mathcal{B}(\mathfrak{o}, \pi, \tau).
\end{aligned}$$

Hence, $\mathcal{B}(\mathfrak{o}, \pi, \tau') \succeq \mathcal{B}(\mathfrak{o}, \pi, \tau)$.

In addition, using condition (15) from Definition (3.1), we have

$$\begin{aligned}\mathcal{C}(\mathfrak{o}, \pi, \tau') &= \mathcal{C}(\mathfrak{o}, \pi, \tau' - \tau + \tau)\\ &\preceq \mathcal{C}(\mathfrak{o}, \mathfrak{o}, \tau' - \tau) \lozenge \mathcal{C}(\mathfrak{o}, \pi, \tau)\\ &= \theta \lozenge \mathcal{C}(\mathfrak{o}, \pi, \tau)\\ &= \mathcal{C}(\mathfrak{o}, \pi, \tau).\end{aligned}$$

Hence, $\mathcal{C}(\mathfrak{o}, \pi, \tau') \succeq \mathcal{C}(\mathfrak{o}, \pi, \tau)$. □

Definition 3.5. Consider $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \lozenge)$ as a CVNMS. We define the convergence of a sequence $\{\mathfrak{o}_n\}$ in Ξ converges to $\mathfrak{o} \in \Xi$ by the following condition; for each $\rho \in \Upsilon_0$, there exisrs $n_0 \in \mathbb{N}$ such that $\forall\ n > n_0$, the subsequent true $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) \succ \ell - \rho$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$.

Definition 3.6. Consider $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \lozenge)$ as a CVNMS. A sequence $\{\mathfrak{o}_n\}$ in Ξ is termed a Cauchy sequence provided that

$\lim_{n\to\infty} \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \ell$, $\lim_{n\to\infty} \sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$ and $\lim_{n\to\infty} \sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$ for all $\tau \in \zeta_0$. When every Cauchy sequence in Ξ converges, a CVNMS $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \lozenge)$ is considered to be complete.

The following examples aid to clarify the ideas presented in Definitions(3.5) and (3.6).

Example 3.7. In the CVNMS $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \lozenge)$ defined as in Example (3.3), analyze the convergence of the sequence.

Let $\Xi = \mathbb{R}$ and define metric d as $d(\mathfrak{o}, \pi) = |\mathfrak{o} - \pi|$ for all $\mathfrak{o}, \pi \in \Xi$. Consider the sequence $\{\mathfrak{o}_n\} = \{\frac{n+1}{n}\}$ and $\mathfrak{o} = 1$. We now verify that $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) \succ \ell - \rho$ for each $\rho = (\rho_1, \rho_2) \in \Upsilon_0$ and $\tau \in \zeta_0$. For the real part,

$$\begin{aligned}Re(\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) - \ell + \rho) &= e^{-\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}} - 1 + \rho_1\\ &= e^{-\frac{(\frac{n+1}{n}-1)}{\mathfrak{p}+\mathfrak{q}}} - 1 + \rho_1\\ &= e^{-\frac{(1/n))}{\mathfrak{p}+\mathfrak{q}}} - 1 + \rho_1.\end{aligned}$$

As $n \to \infty$, we have $Re(\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) - \ell + \rho) \to \rho_1$. Thus, for each $\rho \in \Upsilon_0$ and $\tau \in \zeta_0$, $\exists \mathfrak{N}_1 \in \mathbb{N}$ in which $Re(\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) - \ell + \rho) > 0$ true $\forall\ n > \mathfrak{N}_1$. The same reasoning for the imaginary part yeilds: $Im(\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) - \ell + \rho) \to \rho_2$ as $n \to \infty$.

Therefore there exists $\mathfrak{N}_2 \in \mathbb{N}$ such that $Im(\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) - \ell + \tau) > 0$ for all $n > \mathfrak{N}_2$. Taking $n_0 = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$, we conclude that$\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) \succ \ell - \rho$ for all $n > n_0$. Now we verify that $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$ for each $\rho = (\rho_1, \rho_2) \in \Upsilon_0$ and $\tau \in \zeta_0$. For the real part,

$$\begin{aligned}Re(\rho - \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau)) &= \rho_1 - 1 + e^{-\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}}\\ &= \rho_1 - 1 + e^{-\frac{\frac{n+1}{n}-1}{\mathfrak{p}+\mathfrak{q}}}\\ &= \rho_1 - 1 + e^{-\frac{\frac{1}{n}}{\mathfrak{p}+\mathfrak{q}}}.\end{aligned}$$

As $n \to \infty$, we have $Re(\rho - \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau)) \to \rho_1$. Thus, for each $\rho \in \Upsilon_0$ and $\tau \in \zeta_0$, there exists $\mathfrak{N}_1 \in \mathbb{N}$ in which $Re(\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) - \ell + \rho) > 0 \; \forall \; n > \mathfrak{N}_1$. The same reasoning for the imaginary part yields: $Im(\rho - \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau)) \to \rho_2$ as $n \to \infty$. Therefore, there exists an $\mathfrak{N}_2 \in \mathbb{N}$ such that $Im(\rho - \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau)) > 0$ for all $n > \mathfrak{N}_2$. By setting $n_0 = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$, we conclude that $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$ for all $n > n_0$.

Lastly, we verify that $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$ for every $\rho = (\rho_1, \rho_2) \in \Upsilon_0$ and $\tau \in \zeta_0$. For the real part,

$$\begin{aligned} Re(\rho - \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau)) &= \rho_1 - 1 + e^{\frac{\frac{n+1}{n} - 1}{\mathfrak{p}+\mathfrak{q}}} \\ &= \rho_1 - 1 + e^{\frac{\frac{1}{n}}{\mathfrak{p}+\mathfrak{q}}}. \end{aligned}$$

As $n \to \infty$, we have $Re(\rho - \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau)) \to \rho_1$. Thus, for each $\rho \in \Upsilon_0$ and $\tau \in \zeta_0$, there exists an $\mathfrak{N}_1 \in \mathbb{N}$ in which $Re(\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) - \ell + \tau) > 0$ for all $n > \mathfrak{N}_1$. The same steps for the imaginary part give: $Im(\rho - \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau)) \to r_2$ as $n \to \infty$. Thus for any $n > \mathfrak{N}_2$, there exists $\mathfrak{N}_2 \in \mathbb{N}$ such that $Im(\rho - \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau)) > 0$. Taking $n_0 = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$, we conclude that $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$ for all $n > n_0$

As all the conditions in Definition (3.5) are fully satisfied, we can deduce that $\{\frac{n+1}{n}\} \to 1$

.

Example 3.8. Following the framework of the previous example, our goal is to establish $\frac{n+1}{n}$ is a Cauchy sequence. For any $n, m \in \mathbb{N}$, $n < m$ and $\forall \; \tau \in \zeta_0$. Define $d(\mathfrak{o}_n, \mathfrak{o}_m) = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right|$

$$\mathcal{A}(\mathfrak{o}, \pi, \tau) = e^{-\frac{d(\mathfrak{o}, \pi)}{\mathfrak{p}+\mathfrak{q}}} \ell = e^{-\frac{\left|\frac{1}{n} - \frac{1}{m}\right|}{\mathfrak{p}+\mathfrak{q}}} \ell$$

.

As $n, m \to \infty$, $\left|\frac{1}{n} - \frac{1}{m}\right| \to \infty$, $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \to \ell$

$$\mathcal{B}(\mathfrak{o}, \pi, \tau) = (1 - e^{-\frac{d(\mathfrak{o}, \pi)}{\mathfrak{p}+\mathfrak{q}}}) \ell = (1 - e^{-\frac{\left|\frac{1}{n} - \frac{1}{m}\right|}{\mathfrak{p}+\mathfrak{q}}}) \ell$$

As $n, m \to \infty, \left|\frac{1}{n} - \frac{1}{m}\right| \to \infty$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \to \theta$ in addition,

$$\mathcal{C}(\mathfrak{o}, \pi, \tau) = (e^{-\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}} - 1)\ell = (e^{-\frac{\left|\frac{1}{n}-\frac{1}{m}\right|}{\mathfrak{p}+\mathfrak{q}}} - 1)\ell$$

.

As $n, m \to \infty, \left|\frac{1}{n} - \frac{1}{m}\right| \to \infty, \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \to \theta$. From the above computations, $\lim_{n\to\infty} \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \ell$, $\lim_{n\to\infty} \sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$ and $\lim_{n\to\infty} \sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$. Since $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau), \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau), \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$ satisfies the required conditions as $n, m \to \infty$. Hence, $\mathfrak{o}_n \in \Xi$ is a Cauchy.

Lemma 3.9. Let $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ be a CVNMS. The sequence $\{\mathfrak{o}_n\}$ in Ξ converges to $\mathfrak{o} \in \Xi$ $\iff \lim_{n\to\infty} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \ell$, $\lim_{n\to\infty} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \theta$ and $\lim_{n\to\infty} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \theta$ are satisfied for each $\tau \in \zeta_0$.

Proof. Assume that $\{\mathfrak{o}_n\}$ in Ξ converges to $\mathfrak{o} \in \Xi$. Then for any $\rho \in \Upsilon_0$, there exist $n_0 \in \mathbb{N}$ such that $\forall\ n > n_0$, $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) \succ \ell - \rho$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$.

Now consider a complex number $\rho \in \Upsilon_0$ with $|\rho| < \epsilon$. By the properties of CVNMS, we have: $|\ell - \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau)| < |\rho| < \epsilon$, $|\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau)| < |\rho| < \epsilon$ and $|\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau)| < |\rho| < \epsilon\ \forall\ n > n_0$.

This implies that as $n \to \infty$, $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) \to \ell$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) \to \theta$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) \to \theta$.

Hence, the conditions of convergence are satisfied for every $e \in \zeta_0$.

Now assume that $\lim_{n\to\infty} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \ell$, $\lim_{n\to\infty} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \theta$ and $\lim_{n\to\infty} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \theta$ for each $\tau \in \zeta_0$. Fix an element τ in ζ_0 and take any $\rho \in \Upsilon_0$.

Since the limits exists as $n \to \infty$, there exists an positive real number $\epsilon > 0 \ni$ every complex number $z \in \mathbb{C}$ with $|z| < \epsilon$ satisfies $z \prec \rho$.

Because $\lim_{n\to\infty} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \ell$, $\exists n_0 \in \mathbb{N} \ni |\ell - \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau)| < \epsilon$. This inequality implies $\ell - \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$, $\ell - \rho \prec \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}, \tau)$.

Similarly, since $\lim_{n\to\infty} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \theta$ and $\lim_{n\to\infty} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) = \theta$, $\exists n_0 \in \mathbb{N} \ni$ for all $n > n_0$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}, c) \prec \rho$, $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}, \tau) \prec \rho$. Thus $\{\mathfrak{o}_n\}$ in Ξ converges to $\mathfrak{o} \in \Xi$.

Lemma 3.10. Given that $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ is a CVNMS. $\{\mathfrak{o}_n\}$ is a Cauchy sequence in Ξ $\iff$ for each $\rho \in \Upsilon_0$ and $\tau \in \zeta_0$, it is possible to find a natural number $n_0 \in \mathbb{N}$ such that $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \succ \ell - \rho$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho\ \forall\ n, m > n_0$.

Proof. Assume the sequence $\{\mathfrak{o}_n\}$ is Cauchy. Fix $\tau \in \zeta_0$. For any $\rho \in \Upsilon_0$, there exists an $n_0 \in \mathbb{N}$ such that $\ell - \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$, $\sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$ and $\sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$, for all $n > n_0$. Consider three distinct cases:

Case1: $m > n > n_0$

$m > n > n_0$. It follows that $\ell - \rho \prec \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \rho) \prec \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$.

Case 2: $m = n > n_0$

Here, $\ell - \rho \prec \ell = \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta \prec \rho$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta \prec \rho$.

Case 3: $n > m > n_0$

In this scenario, $\ell - \rho < \inf_{n>m} \mathcal{A}(\mathfrak{o}_m, \mathfrak{o}_n, \tau) \preceq \mathcal{A}(\mathfrak{o}_m, \mathfrak{o}_n, \tau) = \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$, $\mathcal{B}(\mathfrak{o}_m, \mathfrak{o}_n, \tau) = \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \sup_{n>m} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$ and $\mathcal{C}(\mathfrak{o}_m, \mathfrak{o}_n, \tau) = \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \sup_{n>m} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$.

From all the three cases, we deduce that for $n, m > n_0$, $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \succ \ell - \rho$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$.

Conversely, suppose $\tau \in \zeta_o$ fixed and let $\epsilon > 0$. Assume that for any $\rho \in \Upsilon_0$, there exists $n_0 \in \mathbb{N}$ in which $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \succ \ell - \rho$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \prec \rho$ for any $n, m > n_0$. Then $\ell - 2\rho \prec \ell - \rho \preceq \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$, $\sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \rho \prec 2\rho$ and $\sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \rho \prec 2\rho$ for all $n > n_0$. Choose $\rho \in \Upsilon_0$ such that $|\rho| < \frac{\epsilon}{2}$. Then $|\ell - \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)| \prec 2|\rho| < \epsilon$, $|\sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)| \prec 2|\rho| < \epsilon$ and $|\sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)| \prec 2|\rho| < \epsilon \ \forall\ n > n_0$.

Thus, $\lim\limits_{n\to\infty} \inf\limits_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \ell$, $\lim\limits_{n\to\infty} \sup\limits_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$ and $\lim\limits_{n\to\infty} \sup\limits_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$. Therefore, $\{\mathfrak{o}_n\}$ is a Cauchy sequence. □

Definition 3.11. Let $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \lozenge)$ be a CVNMS. A couple of self-mappings $\mathfrak{g}, \mathfrak{h} : \Xi \to \Xi$ is defined as a neutrosophic Banach contraction, if there exists $0 < \eta < 1 \ni; \forall\ \mathfrak{o}, \pi \in \mathfrak{Z}$ and $\tau \in \mathfrak{P}_0$, the following inequalities are satisfied:

$$\ell - \mathcal{A}(\mathfrak{g}\mathfrak{o}, \mathfrak{h}\pi, \tau) \preceq \eta(\ell - \mathcal{A}(\mathfrak{o}, \pi, \tau)), \mathcal{B}(\mathfrak{g}\mathfrak{o}, \mathfrak{h}\pi, \tau) \preceq \eta\mathcal{B}(\mathfrak{o}, \pi, \tau) \text{ and } \mathcal{C}(\mathfrak{g}\mathfrak{o}, \mathfrak{h}\pi, \tau) \preceq \eta\mathcal{C}(\mathfrak{o}, \pi, \tau) \tag{3.11.1}$$

Theorem 3.12. Consider $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \lozenge)$ is a complete CVNMS, if the self-mappings $\mathfrak{g}, \mathfrak{h} : \Xi \to \Xi$ that satisfy the conditions of a neutrosophic Banach contraction. Then $\mathfrak{g}$ and $\mathfrak{h}$ share a unique, which belongs to Ξ.

Proof. Let $\mathfrak{o}_0 \in \Xi$ be any chosen point. Construct a sequence $\{\mathfrak{o}_n\}$ in Ξ by $\mathfrak{o}_{2n+1} = \mathfrak{g}\mathfrak{o}_{2n}, \mathfrak{o}_{2n+2} = \mathfrak{h}\mathfrak{o}_{2n+1}$ for any $n \in \mathbb{N}_0$. If $\exists n_0 \in \mathbb{N}$ such that $\mathfrak{o}_{n_0} = \mathfrak{o}_{n_0+1}$ then $\mathfrak{o}_{n_0}$ serves as a common invariant point of $\mathfrak{g}$ and $\mathfrak{h}$. If $\mathfrak{o}_{2n} = \mathfrak{o}_{2n+1}$ for some $n \in \mathbb{N}_0$ it directly follows that $\mathfrak{o}_{2n}$ is an invariant point of $\mathfrak{g}$. Using the contraction condition (3.11.1), we further have

$$\begin{aligned}
\ell - \mathcal{A}(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n+2}, \tau) &= \ell - \mathcal{A}(\mathfrak{g}\mathfrak{o}_{2n}, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau) \\
&\preceq \eta(\ell - \mathcal{A}(\mathfrak{o}_{2n}, \mathfrak{o}_{2n+1}, \tau)) \\
&= \eta(\ell - \ell) \\
&= \theta
\end{aligned}$$

$$\ell - \mathcal{B}(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n+2}, \tau) = \mathcal{B}(\mathfrak{g}\mathfrak{o}_{2n}, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau)$$

$$\begin{aligned}&\preceq \eta\mathcal{B}(\mathfrak{o}_{2n},\mathfrak{o}_{2n+1},\tau)\\&=\eta(\theta)\\&=\theta.\end{aligned}$$

and

$$\begin{aligned}\ell-\mathcal{C}(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n+2},\tau)&=\mathcal{C}(\mathfrak{g}\mathfrak{o}_{2n},\mathfrak{h}\mathfrak{o}_{2n+1},\tau)\\&\preceq \eta\mathcal{C}(\mathfrak{o}_{2n},\mathfrak{o}_{2n+1},\tau)\\&=\eta(\theta)\\&=\theta.\end{aligned}$$

for every $c\in\mathfrak{P}_0$. We find that $\mathcal{A}(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n+2},\tau)=\ell$, $\mathcal{B}(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n+2},\tau)=\theta$ and $\mathcal{C}(\mathfrak{o}_{2n+1},\mathfrak{o}_{2n+2},\tau)=\theta$. Based on conditions (3), (8) and (13) of Definition (3.1), this implies that $\mathfrak{o}_{2n+1}=\mathfrak{o}_{2n+2}=\mathfrak{h}\mathfrak{o}_{2n+1}$, conforms that $\mathfrak{o}_{2n+1}$ is an invariant point of $\mathfrak{h}$. Because $\mathfrak{o}_{2n}=\mathfrak{o}_{2n+1}$, it can be concluded that $\mathfrak{o}_{2n}$ is a common invariant point of $\mathfrak{g}$ and $\mathfrak{h}$. Similarly, if there exists $n\in\mathbb{N}_0$ such that $\mathfrak{o}_{2n+1}=\mathfrak{o}_{2n+2}$, utilizing (3.11.1), it follows that $\mathfrak{o}_{2n+1}$ is a common fixed point of $\mathfrak{g}$ and $\mathfrak{h}$.

Suppose $\mathfrak{o}_n,\mathfrak{o}_{n+1}$ remain distinct for every $n\in\mathbb{N}_0$. To address this, consider two scenarios. In the first scenario, assume n is odd. By substituting $\mathfrak{o}=\mathfrak{o}_{n-1}$ and $\pi=\mathfrak{o}_n$ in (3.11.1), it follows that for all $\tau\in\mathfrak{P}_0$ we get

$$\begin{aligned}\ell-\mathcal{A}(\mathfrak{o}_n,\mathfrak{o}_{n+1},\tau)&=\ell-\mathcal{A}(\mathfrak{g}\mathfrak{o}_{n-1},\mathfrak{h}\mathfrak{o}_n,\tau)\\&\preceq \eta(\ell-\mathcal{A}(\mathfrak{o}_{n-1},\mathfrak{o}_n,\tau))\\&\prec \ell-\mathcal{A}(\mathfrak{o}_{n-1},\mathfrak{o}_n,\tau)\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\mathfrak{o}_n,\mathfrak{o}_{n+1},\tau)&=\mathcal{B}(\mathfrak{g}\mathfrak{o}_{n-1},\mathfrak{h}\mathfrak{o}_n,\tau)\\&\preceq \eta\mathcal{B}(\mathfrak{o}_{n-1},\mathfrak{o}_n,\tau)\\&\prec \mathcal{B}(\mathfrak{o}_{n-1},\mathfrak{o}_n,\tau).\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}(\mathfrak{o}_n,\mathfrak{o}_{n+1},\tau)&=\mathcal{C}(\mathfrak{g}\mathfrak{o}_{n-1},\mathfrak{h}\mathfrak{o}_n,\tau)\\&\preceq \eta\mathcal{C}(\mathfrak{o}_{n-1},\mathfrak{o}_n,\tau)\\&\prec \mathcal{C}(\mathfrak{o}_{n-1},\mathfrak{o}_n,\tau).\end{aligned}$$

This implies that $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \succ \mathcal{A}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau)$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \prec \mathcal{B}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau)$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \prec \mathcal{C}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau)$ $\forall\ \tau \in \mathfrak{P}_0$. For the second scenario, assume n is even.

By Substituting $\mathfrak{o} = \mathfrak{o}_n$ and $\pi = \mathfrak{o}_{n-1}$ in (3.11.1), we get the following:

$$\begin{aligned}\ell - \mathcal{A}(\mathfrak{o}_{n+1}, \mathfrak{o}_n, \tau) &= \ell - \mathcal{A}(\mathfrak{g}\mathfrak{o}_n, \mathfrak{h}\mathfrak{o}_{n-1}, \tau)\\ &\preceq \eta(\ell - \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau))\\ &\prec \ell - \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau)\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\mathfrak{o}_{n+1}, \mathfrak{o}_n, \tau) &= \mathcal{B}(\mathfrak{g}\mathfrak{o}_n, \mathfrak{h}\mathfrak{o}_{n-1}, \tau)\\ &\preceq \eta\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau)\\ &\prec \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau).\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}(\mathfrak{o}_{n+1}, \mathfrak{o}_n, \tau) &= \mathcal{C}(\mathfrak{g}\mathfrak{o}_n, \mathfrak{h}\mathfrak{o}_{n-1}, \tau)\\ &\preceq \eta\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau)\\ &\prec \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau).\end{aligned}$$

It follows that $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \succ \mathcal{A}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau)$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \prec \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau)$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \prec \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau)$ for any $\tau \in \mathfrak{P}_0$.

Hence, we conclude that $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \succ \mathcal{A}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau)$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \prec \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau)$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) \prec \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n-1}, \tau)$ for every $n \in \mathbb{N}_0$ and $\tau \in \mathfrak{P}_0$. Let $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) = \mathfrak{A}_n$, $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) = \mathfrak{B}_n$ and $\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) = \mathfrak{C}_n$ $\forall\ n \in \mathbb{N}_0$.

Because $\ell \succeq \mathfrak{A}_n \succ \mathfrak{A}_{n-1} \succ \theta$, $\theta \preceq \mathfrak{B}_n \prec \mathfrak{B}_{n-1} \prec \ell$ and $\theta \preceq \mathfrak{C}_n \prec \mathfrak{C}_{n-1} \prec \ell$ $\forall\ n \in \mathbb{N}_0$, this implies that the sequences $\{\mathfrak{A}_n\}$, $\{\mathfrak{B}_n\}$and $\{\mathfrak{C}_n\}$ are monotonic in $\mathfrak{P}$.

By Definition (2.1), there exists $\alpha, \beta, \gamma \in \mathfrak{P}$ such that

$\lim_{n\to\infty} \mathfrak{A}_n = \alpha$, $\lim_{n\to\infty} \mathfrak{B}_n = \beta$, $\lim_{n\to\infty} \mathfrak{C}_n = \gamma$.

Using (3.11.1), for $n \in \mathbb{N}_0$ and $\tau \in \mathfrak{P}_0$ we derive

$$\begin{aligned}\ell - \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) &\preceq \eta(\ell - \mathcal{A}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau))\\ \ell - \mathfrak{A}_n &\preceq \eta(\ell - \mathfrak{A}_{n-1})\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) &\preceq \eta\mathcal{B}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau)\\ \mathfrak{B}_n &\preceq \eta\mathfrak{B}_{n-1}.\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) &\preceq \eta\mathcal{C}(\mathfrak{o}_{n-1}, \mathfrak{o}_n, \tau)\\ \mathfrak{C}_n &\preceq \eta\mathfrak{C}_{n-1}.\end{aligned}$$

Taking $n \to \infty$ in these inequalities, we find $\ell - \alpha \preceq \eta(\ell - \alpha)$ and $\beta \preceq \eta\beta$.

Since $\eta \in (0,1)$, if $\alpha \prec \ell$ and $\beta \succ \theta$, a contradiction arises. Hence, $\alpha = \ell$ and $\beta = \theta$, leading to the conclusion $\lim_{n\to\infty} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) = \ell$, $\lim_{n\to\infty} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \tau) = \theta$ for every $n \in \mathbb{N}_0$ and $\tau \in \mathfrak{P}_0$.

Now, we will show that $\{\mathfrak{o}_n\}$ is Cauchy sequence . For any $n \in \mathbb{N}_0$ and $\tau \in \mathfrak{P}_0$, define the sets $\mathfrak{A}_n = \{\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) : m > n\} \subseteq \mathfrak{I}$, $\mathfrak{B}_n = \{\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) : m > n\} \subseteq \mathfrak{I}$,
and $\mathfrak{C}_n = \{\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) : m > n\} \subseteq \mathfrak{I}$. Since $\theta \prec \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \ell$, $\theta \preceq \mathcal{B}(\mathfrak{o}_m, \mathfrak{o}_n, \tau) \prec \ell$ and $\theta \preceq \mathcal{C}(\mathfrak{o}_m, \mathfrak{o}_n, \tau) \prec \ell$. According to Definition (2.1), the infimum of complex fuzzy set $\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$ and the supremum of complex neutrosophic sets $\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$,$(\mathfrak{o}_n, \mathfrak{o}_m, \tau)$ exist. For $m > n$, repeatedly using condition (5) of Definition (3.1) yields:

$$\mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \succeq \mathcal{A}\left(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \frac{\tau}{m-n}\right) * \mathcal{A}\left(\mathfrak{o}_{n+1}, \mathfrak{o}_{n+2}, \frac{\tau}{m-n}\right) * \cdots * \mathcal{A}\left(\mathfrak{o}_{m-1}, \mathfrak{o}_m, \frac{c}{m-n}\right).$$

Taking the $n \to \infty$. we obtain $\lim_{n\to\infty} \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \succeq \ell * \ell * \cdots * \ell = \ell$, which implies $\lim_{n\to\infty} \inf_{m>n} \mathcal{A}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \ell$ for every $\tau \in \mathfrak{P}_0$. Similarly, for any $n < m$, by utilizing condition (10) of Definition (3.1) repeatedly gives:

$$\mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \mathcal{B}\left(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \frac{\tau}{m-n}\right) \Diamond \mathcal{B}\left(\mathfrak{o}_{n+1}, \mathfrak{o}_{n+2}, \frac{\tau}{m-n}\right) \Diamond \ldots \Diamond \mathcal{B}\left(\mathfrak{o}_{m-1}, \mathfrak{o}_m, \frac{\tau}{m-n}\right).$$

Taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} \sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \theta \Diamond \theta \Diamond \ldots \Diamond \theta = \theta$, which implies $\lim_{n\to\infty} \sup_{m>n} \mathcal{B}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$ for all $\tau \in \mathfrak{P}_0$. Using condition (15) of Definition (3.1) repeatedly, we get

$$\mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \mathcal{C}\left(\mathfrak{o}_n, \mathfrak{o}_{n+1}, \frac{\tau}{m-n}\right) \Diamond \mathcal{C}\left(\mathfrak{o}_{n+1}, \mathfrak{o}_{n+2}, \frac{\tau}{m-n}\right) \Diamond \ldots \Diamond \mathcal{C}\left(\mathfrak{o}_{m-1}, \mathfrak{o}_m, \frac{\tau}{m-n}\right).$$

Taking the limit $n \to \infty$, we conclude $\lim_{n\to\infty} \sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) \preceq \theta \Diamond \theta \Diamond \ldots \Diamond \theta = \theta$, which implies $\lim_{n\to\infty} \sup_{m>n} \mathcal{C}(\mathfrak{o}_n, \mathfrak{o}_m, \tau) = \theta$ for all $\tau \in \mathfrak{P}_0$ Thus, $\{\mathfrak{o}_n\}$ is a Cauchy sequence.

Since $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ is complete, Lemma (3.9) guarantees the existence of $\varrho \in \Xi$ such that $\lim_{n\to\infty} \mathcal{A}(\mathfrak{o}_n, \varrho, \tau) = \ell$, $\lim_{n\to\infty} \mathcal{B}(\mathfrak{o}_n, \varrho, \tau) = \theta$ and $\lim_{n\to\infty} \mathcal{C}(\mathfrak{o}_n, \varrho, \tau) = \theta$ for all $\tau \in \mathfrak{P}_0$. For any $n \in \mathbb{N}_0$ and $\tau \in \mathfrak{P}_0$, by (3.11.1) we find

$\ell - \mathcal{A}(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau) \preceq \eta(\ell - \mathcal{A}(\varrho, \mathfrak{o}_{2n+1}, \tau)) \prec \ell - \mathcal{A}(\varrho, \mathfrak{o}_{2n+1}, \tau)$

$\mathcal{B}(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau) \preceq \eta\mathcal{B}(\varrho, \mathfrak{o}_{2n+1}, \tau) \prec \mathcal{B}(\varrho, \mathfrak{o}_{2n+1}, \tau)$ and

$\mathcal{C}(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau) \preceq \eta\mathcal{C}(\varrho, \mathfrak{o}_{2n+1}, \tau) \prec \mathcal{C}(\varrho, \mathfrak{o}_{2n+1}, \tau)$.

These imply that

$$\mathcal{A}(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau) \succ \mathcal{A}(\varrho, \mathfrak{o}_{2n+1}, \tau) \tag{3.12.1}$$

$$\mathcal{B}(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau) \prec \mathcal{B}(\varrho, \mathfrak{o}_{2n+1}, \tau) and \tag{3.12.2}$$

$$\mathcal{C}(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \tau) \prec \mathcal{C}(\varrho, \mathfrak{o}_{2n+1}, \tau) \tag{3.12.3}$$

for each $n \in \mathbb{N}_0$ and $\tau \in \mathfrak{P}_0$. From conditions (5), (10) and (15) of Definition (3.1), (3.12.1), (3.12.2) and (3.12.3 for any $n \in \mathbb{N}_0$ and $\tau \in \mathfrak{P}_0$, we conclude that

$$\begin{aligned}
\mathcal{A}(\varrho, \mathfrak{g}\varrho, \tau) &\succeq \mathcal{A}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) * \mathcal{A}\left(\mathfrak{o}_{2n+2}, \mathfrak{g}\varrho, \frac{\tau}{2}\right) \\
&= \mathcal{A}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) * \mathcal{A}\left(\mathfrak{h}\mathfrak{o}_{2n+1}, \mathfrak{g}\varrho, \frac{\tau}{2}\right) \\
&= \mathcal{A}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) * \mathcal{A}\left(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \frac{\tau}{2}\right) \\
&\succeq \mathcal{A}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) * \mathcal{A}\left(\varrho, \mathfrak{o}_{2n+1}, \frac{\tau}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}(\varrho, \mathfrak{g}\varrho, \tau) &\preceq \mathcal{B}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{B}\left(\mathfrak{o}_{2n+2}, \mathfrak{g}\varrho, \frac{\tau}{2}\right) \\
&= \mathcal{B}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{B}\left(\mathfrak{h}\mathfrak{o}_{2n+1}, \mathfrak{g}\varrho, \frac{\tau}{2}\right) \\
&= \mathcal{B}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{B}\left(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \frac{\tau}{2}\right) \\
&\preceq \mathcal{B}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{B}\left(\varrho, \mathfrak{o}_{2n+1}, \frac{\tau}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}(\varrho, \mathfrak{g}\varrho, \tau) &\preceq \mathcal{C}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{C}\left(\mathfrak{o}_{2n+2}, \mathfrak{g}\varrho, \frac{\tau}{2}\right) \\
&= \mathcal{C}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{C}\left(\mathfrak{h}\mathfrak{o}_{2n+1}, \mathfrak{g}\varrho, \frac{\tau}{2}\right) \\
&= \mathcal{C}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{C}\left(\mathfrak{g}\varrho, \mathfrak{h}\mathfrak{o}_{2n+1}, \frac{\tau}{2}\right) \\
&\preceq \mathcal{C}\left(\varrho, \mathfrak{o}_{2n+2}, \frac{\tau}{2}\right) \Diamond \mathcal{C}\left(\varrho, \mathfrak{o}_{2n+1}, \frac{\tau}{2}\right)
\end{aligned}$$

As $n \to \infty$ in the above inequalities, we obtain
$\mathcal{A}(\varrho, \mathfrak{g}\varrho, \tau) = \ell$, $\mathcal{B}(\varrho, \mathfrak{g}\varrho, \tau) = \theta$ and $\mathcal{C}(\varrho, \mathfrak{g}\varrho, \tau) = \theta$ for all $\tau \in \mathfrak{P}_0$. Applying conditions (3), (8) and (13) of Definition (3.1), this implies that $\varrho = \mathfrak{g}\varrho$. Following a similar reasoning, one can show that $\mathcal{A}(\varrho, \mathfrak{h}\varrho, \tau) = \ell$, $\mathcal{B}(\varrho, \mathfrak{h}\varrho, \tau) = \theta$ and $\mathcal{C}(\varrho, \mathfrak{h}\varrho, \tau) = \theta$ for all $\tau \in \mathfrak{P}_0$. By the same conditions (3), (8) and (13) of Definition (3.1), it follows that $\varrho = \mathfrak{h}\varrho$. Therefore, $\varrho = \mathfrak{g}\varrho = \mathfrak{h}\varrho$ meaning ϱ is common fixed point of $\mathfrak{g}$ and $\mathfrak{h}$.

To prove the uniqueness, suppose χ is another fixed point of f , distinct fromϱ, i.e., $\chi \neq \varrho$. Therefore there exists $\tau \in \mathfrak{P}_0$ such that $\mathcal{A}(\varrho, \chi, \tau) \neq \ell$, $\mathcal{B}(\varrho, \chi, \tau) \neq \theta$ and $\mathcal{C}(\varrho, \chi, \tau) \neq \theta$. Using (3.11.1),

$$\ell - \mathcal{A}(\varrho, \chi, \tau) = \ell - \mathcal{A}(\mathfrak{g}\varrho, \mathfrak{h}\chi, \tau)$$

$$\preceq \eta(\ell - \mathcal{A}(\varrho, \chi, \tau))$$

$$\prec \ell - \mathcal{A}(\varrho, \chi, \tau)$$

$$\begin{aligned}\mathcal{B}(\varrho, \chi, \tau) &= \mathcal{B}(\mathfrak{g}\varrho, \mathfrak{h}\chi, \tau)\\ &\preceq \eta\mathcal{B}(\varrho, \chi, \eta)\\ &\prec \mathcal{B}(\varrho, \chi, \tau)\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}(\varrho, \chi, \tau) &= \mathcal{C}(\mathfrak{g}\varrho, \mathfrak{h}\chi, \tau)\\ &\preceq \eta\mathcal{C}(\varrho, \chi, \tau)\\ &\prec \mathcal{C}(\varrho, \chi, \tau)\end{aligned}$$

which is a contradiction. Thus $\mathcal{A}(\varrho, \chi, \tau) = \ell$, $\mathcal{B}(\varrho, \chi, \tau) = \theta$ and $\mathcal{C}(\varrho, \chi, \tau) = \theta$ for all $\tau \in \mathfrak{P}_0$. By utilizing (3), (8) and (13) of Definition (3.1), it follows that $\varrho = \chi$. This establish the fixed point is unique. □

Corollary 3.13. Condiser $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ is a complete CVNMS. Let $\mathfrak{g} : \Xi \to \Xi$ be a self-mapping such that, for all $\mathfrak{o}, \pi \in \Xi$ and $\tau \in \mathfrak{P}_0$, the conditions given below are fulfilled: $\ell - \mathcal{A}(\mathfrak{g}\mathfrak{o}, \mathfrak{g}\pi, \tau) \preceq \eta(\ell - \mathcal{A}(\mathfrak{o}, \pi, \tau))$, $\mathcal{B}(\mathfrak{g}\mathfrak{o}, \mathfrak{g}\pi, \tau) \preceq \eta\mathcal{B}(\mathfrak{o}, \pi, \tau)$ and $\mathcal{C}(\mathfrak{g}\mathfrak{o}, \mathfrak{g}\pi, \tau) \preceq \eta\mathcal{C}(\mathfrak{o}, \pi, \tau)$, where $0 < \eta < 1$. Then $\mathfrak{g}$ has a unique invariant point in Ξ.

Proof. The conclusion follows directly by setting $\mathfrak{g} = \mathfrak{h}$ in Theorem (3.12). □

The following example illustrate the concept presented in Corollary (3.13).

Example 3.14. Let (Ξ, d) be a metric space, $\Xi = [0, 1]$ and $d(\mathfrak{o}, \pi) = |\mathfrak{o} - \pi|$. For $\tau_i = (\mathfrak{p}_i, \mathfrak{q}_i) \in \Upsilon$ where $i = 1, 2$, define $*$ and $\Diamond$ by

$\tau_1 * \tau_2 = (\min\{\mathfrak{p}_1, \mathfrak{p}_2\}, \min\{\mathfrak{q}_1, \mathfrak{q}_2\})$ and $\tau_1 \Diamond \tau_2 = (\max\{\mathfrak{p}_1, \mathfrak{p}_2\}, \max\{\mathfrak{q}_1, \mathfrak{q}_2\})$.

Now define the complex neutrosophic sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as:

$$\mathcal{A}(\mathfrak{o},\pi,\tau)=e^{-\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}}\ell,\mathcal{B}(\mathfrak{o},\pi,\tau)=(1-e^{-\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}})\ell$$

and

$$\mathcal{C}(\mathfrak{o},\pi,\tau)=(1-e^{\frac{d(\mathfrak{o},\pi)}{\mathfrak{p}+\mathfrak{q}}})\ell$$

for all $\mathfrak{o},\pi\in\Xi$ and $\tau=(\mathfrak{p},\mathfrak{q})\in\zeta_0$. As a result, the structure $(\Xi,\mathcal{A},\mathcal{B},\mathcal{C},*,\Diamond)$ forms a CVNMS.

Define $\mathfrak{g}:\Xi\to\Xi$ by $\mathfrak{g}=\mathfrak{o}/8$. It follows that

$$\mathcal{A}(\mathfrak{go},\mathfrak{g}\pi,\tau)=e^{-\frac{d(\mathfrak{go},\mathfrak{g}\pi)}{\mathfrak{p}+\mathfrak{q}}}\ell=e^{-\frac{|\frac{\mathfrak{o}}{8}-\frac{\pi}{8}|}{\mathfrak{p}+\mathfrak{q}}}\ell\succeq e^{-\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}}\ell \implies \ell-\mathcal{A}(\mathfrak{go},\mathfrak{g}\pi,\tau)\preceq\ell-e^{-\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}}\ell\preceq\eta(\ell-e^{-\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}}\ell)$$
$$\preceq\eta(\ell-\mathcal{A}(\mathfrak{o},\pi,\tau)).$$

Take $\eta\in(0,1)$, we have $\ell-\mathcal{A}(\mathfrak{go},\mathfrak{g}\pi,\tau)\preceq\eta(\ell-\mathcal{A}(\mathfrak{o},\pi,\eta))$ for every $\mathfrak{o},\pi\in\Xi$ and $\tau=(a,b)\in\mathfrak{P}_0$. Similarly,

$$\mathcal{B}(\mathfrak{go},\mathfrak{g}\pi,\tau)=(1-e^{-\frac{d(\mathfrak{go},\mathfrak{g}\pi)}{\mathfrak{p}+\mathfrak{q}}})\ell=(1-e^{-\frac{|\frac{\mathfrak{o}}{8}-\frac{\pi}{8}|}{\mathfrak{p}+\mathfrak{q}}})\ell\succeq(1-e^{-\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}})\ell$$
$$\implies\mathcal{B}(\mathfrak{go},\mathfrak{g}\pi,\tau)\succeq(1-e^{-\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}})\ell\succeq\eta(1-e^{-\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}}\ell)$$
$$\succeq\eta(\mathcal{B}(\mathfrak{o},\pi,\tau)).$$

and

$$\mathcal{C}(\mathfrak{go},\mathfrak{g}\pi,\tau)=(e^{\frac{d(\mathfrak{go},\mathfrak{g}\pi)}{\mathfrak{p}+\mathfrak{q}}}-1)\ell=(e^{\frac{|\frac{\mathfrak{o}}{8}-\frac{\pi}{8}|}{\mathfrak{p}+\mathfrak{q}}}-1)\ell\succeq(e^{\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}}-1)\ell$$
$$\implies\mathcal{C}(\mathfrak{go},\mathfrak{g}\pi,\tau)\succeq(e^{\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}}-1)\ell\succeq\eta(e^{\frac{|\mathfrak{o}-\pi|}{\mathfrak{p}+\mathfrak{q}}}-1)\ell$$
$$\succeq\eta(\mathcal{C}(\mathfrak{o},\pi,\tau)).$$

We obtain that $\mathcal{B}(\mathfrak{go},\mathfrak{g}\pi,\tau)\preceq\eta\mathcal{B}(\mathfrak{o},\pi,\tau)$ and $\mathcal{C}(\mathfrak{go},\mathfrak{g}\pi,c)\preceq\eta\mathcal{C}(\mathfrak{o},\pi,\tau)$ for every $\mathfrak{o},\pi\in\Xi$ and $\tau=(\mathfrak{p},\mathfrak{q})\in\mathfrak{P}_0$. As a result, all the conditions outlined in Corollary (3.13) are satisfied.

Notably,0 is identified as the unique fixed point of $\mathfrak{g}$.

The graphical representation of these inequalities is provided below.

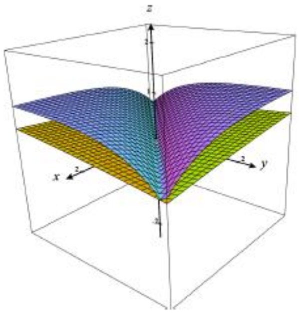

FIGURE 1. Graphical view of inequality $\ell - \mathcal{A}(\mathfrak{g}\mathfrak{o}, \mathfrak{g}\pi, \tau) \preceq \eta(\ell - \mathcal{A}(\mathfrak{o}, \pi, \tau))$ where the plane in blue color plane represents the right-hand side and the plane yellow color plane represents the left-hand side of the inequality.

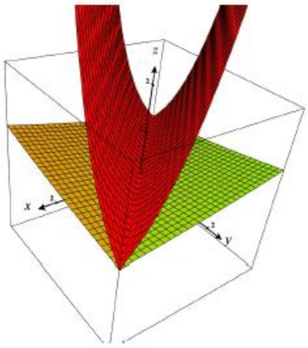

FIGURE 2. Graphical view of the inequality $\mathcal{B}(\mathfrak{g}\mathfrak{o}, \mathfrak{g}\pi, \tau) \preceq \eta\mathcal{B}(\mathfrak{o}, \pi, \tau)$, where the blue plane represents the right-hand side of the inequality and the yellow plane represents the left-hand side of the inequality.

Theorem 3.15. Let $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ be a complete CVNMS. Assume that $\mathfrak{g}, \mathfrak{h} : \Xi \to \Xi$ are commuting self- mappings satisfying the conditions:

$\ell - \mathcal{A}(\mathfrak{g}^n\mathfrak{o}, \mathfrak{h}^n\pi, \tau) \preceq \eta(\ell - \mathcal{A}(\mathfrak{o}, \pi, \tau))$, $\mathcal{B}(\mathfrak{g}^n\mathfrak{o}, \mathfrak{h}^n\pi, \tau) \preceq \eta\mathcal{B}(\mathfrak{o}, \pi, \tau)$ and

$\mathcal{C}(\mathfrak{g}^n\mathfrak{o}, \mathfrak{h}^n\pi, \tau) \preceq \eta\mathcal{C}(\mathfrak{o}, \pi, \tau)$

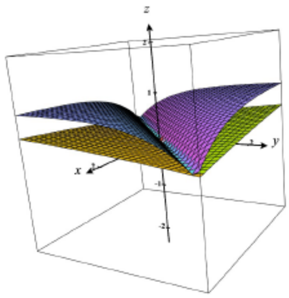

FIGURE 3. Graphical view of inequality $\mathcal{C}(\mathfrak{g}\mathfrak{o}, \mathfrak{g}\pi, \tau) \preceq \eta(\mathcal{C}(\mathfrak{o}, \pi, \tau))$ where the plane in red color represents the right-hand side othe inequality and the plane in yellow color represents the left-hand side of the inequality.

for all $\mathfrak{o}, \pi \in \Xi, \tau \in \mathfrak{P}_0$ and $n \in \mathbb{N}$, where $\eta \in (0, 1)$. Then $\mathfrak{g}$ and $\mathfrak{h}$ have a unique common invariantt in Ξ.

Proof. All the conditions of Theorem (3.12) are fulfilled by both $\mathfrak{g}_n$ and $\mathfrak{h}_n$. Thus, they admit a unique common fixed point ϱ in Ξ, such that, $\mathfrak{g}^n\varrho = \mathfrak{h}^n\varrho = \varrho$. From the relation $\mathfrak{g}^n\mathfrak{g}\varrho = \mathfrak{g}\mathfrak{g}^n\varrho = \mathfrak{g}\varrho$, it follows that $\mathfrak{g}\varrho$ is an invariant point of $\mathfrak{g}^n$. Because $\mathfrak{g}$ and $\mathfrak{h}$ commute, we can write $\mathfrak{h}^n\mathfrak{g}\varrho = \mathfrak{g}\mathfrak{h}^n\varrho = \mathfrak{g}\varrho$, showing that $\mathfrak{g}\varrho$ is a point fixed by $\mathfrak{h}^n$. Thus, $\mathfrak{g}\varrho$ is a common fixed point of $\mathfrak{g}^n$ and $\mathfrak{h}^n$.

Similarly, using $\mathfrak{h}^n\mathfrak{h}\varrho = \mathfrak{h}\mathfrak{h}^n\varrho = \mathfrak{h}\varrho$, it follows that $\mathfrak{h}\varrho$ is an invariant point of $\mathfrak{h}^n$.

Since mappings $\mathfrak{g}$ and $\mathfrak{h}$ commute, we can write $\mathfrak{g}^n\mathfrak{h}\varrho = \mathfrak{h}\mathfrak{g}^n\varrho = \mathfrak{h}\varrho$ showing that $\mathfrak{h}\varrho$ is a point fixed by $\mathfrak{g}^n$. As a result, $\mathfrak{h}\varrho$ serves as common fixed point of $\mathfrak{g}^n$ and $\mathfrak{h}^n$.
Since the common fixed point of $\mathfrak{g}^n$ and $\mathfrak{h}^n$ is unique, it follows that $\varrho = \mathfrak{h}\varrho = \mathfrak{g}\varrho$.

Therefore, ϱ is the unique common fixed point of $\mathfrak{g}$ and $\mathfrak{h}$. Evidently, any common fixed point of $\mathfrak{g}$ and $\mathfrak{h}$ is also a common fixed point of $\mathfrak{g}^n$ and $\mathfrak{h}^n$. Hence, the common fixed point of $\mathfrak{g}$ and $\mathfrak{h}$ is unique. □

Corollary 3.16. Consider $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ is a complete CVNMS. Let $\mathfrak{g} : \Xi \to \Xi$ be a self-mapping that satisfies the following conditions for all $\mathfrak{o}, \pi \in \Xi, \tau \in \mathfrak{P}_0$ and $n \in \mathbb{N}$

$\ell - \mathcal{A}(\mathfrak{g}^n\mathfrak{o}, \mathfrak{g}^n\pi, \tau) \preceq \eta(\ell - \mathcal{A}(\mathfrak{o}, \pi, \tau))$, $\mathcal{B}(\mathfrak{g}^n\mathfrak{o}, \mathfrak{g}^n\pi, \tau) \preceq \eta\mathcal{B}(\mathfrak{o}, \pi, \tau)$ and $\mathcal{C}(\mathfrak{g}^n\mathfrak{o}, \mathfrak{g}^n\pi, \tau) \preceq \eta\mathcal{C}(\mathfrak{o}, \pi, \tau)$, where $\eta \in (0, 1)$. Under these conditions, the mapping $\mathfrak{g}$ has a unique fixed point in Ξ.

Proof. The conclusion is obtained by applying Theorem (3.15) with $\mathfrak{g} = \mathfrak{h}$. □

4. Conclusion

This study has successfully proposed and explored complex-valued neutrosophic metric spaces, providing a novel and comprehensive extension to existing metric space frameworks. By demonstrating fixed-point existence and uniqueness under diverse contractive conditions, our findings address important gaps in the literature and offer significant theoretical advancements.

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