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Research Paper

Efficient Numerical Methods with High Convergence for Nonlinear Systems

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Abstract
Nonlinear systems of equations are prevalent in applied mathematics, science, economics, and engineering. Since analytical solutions to these systems are often unavailable, numerical methods become crucial for obtaining approximate solutions. This paper examines some important numerical methods used to solve nonlinear systems of equations, including the Newton-Raphson method, the modified Newton method, the fixed-point iteration method, and the Proudhon quasi-Newton method. The paper discusses the mathematical formulas, convergence properties, advantages, and limitations of these methods. Numerical examples are provided to illustrate their application and performance. A comparative analysis shows that while Newton methods generally exhibit rapid convergence, non-derivative methods offer computational advantages in large-scale applications.

Keywords: *Nonlinear systems, Newton-Raphson method, fixed-point iteration method, numerical analysis, iterative methods.*

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I. Introduction

Many mathematical models in physics, chemistry, engineering, computer science, and economics can lead to systems of nonlinear equations. So, we can write a general nonlinear system like this:

$$F(x)=0 \qquad (1)$$

where

$$F(x)=\begin{bmatrix} f_1(x_1, x_2, & \cdots & x_n) \\ f_2(x_1, x_2, \vdots & \ddots & x_n) \\ f_n(x_1, x_2, & \cdots & x_n) \end{bmatrix}$$

represents a vector of functions (or a system of nonlinear equations).

What each part means

- $x_1, x_2, \ldots, x_n$ are the unknown variables.
- $f_1, f_2, \ldots, f_n$ are function s of those variables.
- The square brackets indicate that these functions are grouped into a single column vector.

We usually write this compactly as

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ . \\ . \\ . \\ f_n(x) \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{bmatrix}$$

Example:

Suppose we have two nonlinear equations:

$$f_1(x, y) = x^2 + y^2 - 4 \qquad (2)$$

$$f_2(x, y) = x - y - 1 \qquad (3)$$

Then the system can be written as

$$F(x, y) = \begin{bmatrix} x2 + y2 - 4 \\ x - y - 1 \end{bmatrix}$$

Solving the system means finding values of x and y such that

$$F(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words,

$$\begin{cases} x2 + y2 - 4 = 0 \\ x - y - 1 = 0 \end{cases}$$

Writing the equations as a vector allows numerical methods such as the Newton–Raphson method to treat all equations simultaneously. Instead of working with each equation separately, we work with the vector function F(x) and its Jacobian matrix. So the expression is simply a compact way of saying: "Here are nnn equations involving nnn unknowns, collected into one vector-valued function."Because exact solutions are rarely obtainable, numerical methods are employed to approximate the roots of these systems. The choice of method depends on factors such as convergence speed, computational cost, and the availability of derivative information.

This paper reviews several widely used numerical methods and compares their effectiveness for solving systems of nonlinear equations.

(1.1) Background(8)

Let nonlinear system be: F(x)=0, where $F:R^n \to R^n$

The objective is to find a vector x* satisfying

$$F(x^*)=0. \qquad (4)$$

An iterative numerical method generates a sequence

$$x^{(0)},\ x^{(1)},\ x^{(2)},\ \ldots\ldots$$

that converges to the exact solution (x*).

A method is said to converge if $\lim_{k\to\infty} x^k = x^*$. Error=$||x^{(k)} - x^*||$

(1.2) Convergence Rate(3)

If: $|| e_{k+1} || \leq C|| e_k ||^p$

then:

p=1 : Linear convergence

p=2 : Quadratic convergence

p > 2 : Superlinear convergence

The efficiency of a numerical method is often measured by its order of convergence and computational complexity.

2. Fixed-Point Iteration Method(1)

The nonlinear system is rewritten as $x = G(x)$ (5)

Iteration proceeds as $x_{k+1}=G(x_k)$ (6)

(2.1) Convergence Condition

The method converges if $\rho(G'(x^*)) < 1$

Continue until the desired tolerance is reached.

Given an equation

$$f(x) = 0$$

rewrite it as

$$x = g(x)$$

and use the iterative formula

$$x_{n+1} = g(x_n)$$

starting from an initial guess x_0.

The iteration converges if, near the root,

$$|g'(x)| < 1.$$

Problem 1: Solve $x^2 - 3 = 0$ (7)

Solution:

Convert to fixed-point form

$$x = (x + 3/x)/ 2$$
$$g(x) = (x + 3/x)/ 2 \quad (8)$$

Choose initial guess

$$x_0 = 1.5$$

Iterate

$$x_1 = (1.5+3/1.5)/2 = 1.5 + 22 = 1.75$$
$$x_2 = (1.75+3/1.75)/2 = 1.732143$$
$$x_3 = (1.732143+3/1.732143)/2=1.732051$$
$$x_4 = 1.732051$$

so

$$x \approx 1.732051$$

Actual value:

$$\sqrt{3} \approx 1.732051$$

Problem 2: Solve $x_3 + x - 1 = 0$ (9)

Solution:

Value of x in terms of x^3

$$x = 1-x_3$$

So

$$g(x) = 1-x_3 \quad (10)$$

Initial guess

$$x_0 = 0.5$$

Iterations

$$x_1 = 1-(0.5)^3 = 0.875$$
$$x_2 = 1-(0.875)^3 = 0.3301$$
$$x_3 = 1-(0.3301)^3 = 0.9640$$
$$x_4 = 1-(0.9640)^3 = 0.1041$$

The values oscillate and do not converge.

Check convergence

$$g'(x) = -3x^2$$

Near the root ($x \approx 0.68$):

$$|g'(x)| \approx 1.39 > 1$$

Therefore this form does not converge.

rearrangement

$$x = (1-x)^{1/3} \quad (11)$$

let $x_0 = 0.5$

n	Xn
0	0.5000
1	0.7937
2	0.5902
3	0.7424
4	0.6368
5	0.7130
6	0.6603
7	0.6980
8	0.6715

Continuing:

$$X \approx 0.6823$$

therefor

$$x \approx 0.6823$$

Problem 3: Solve $e^{-x} - x = 0$ (12)

Rearrange

$$X = e^{-x} \quad (13)$$

And so on

$$g(x) = e^{-x} \quad (14)$$

Initial guess

$$x_0 = 0$$

Iterations

$$\begin{aligned} x_1 &= e^{0} = 1 \\ x_2 &= e^{-1} = 0.367879 \\ x_3 &= e^{-0.367879} = 0.692201 \\ x_4 &= e^{-0.692201} = 0.500474 \\ x_5 &= e^{-0.500474} = 0.606244 \\ x_6 &= e^{-0.606244} = 0.545396 \\ x_7 &= e^{-0.545396} = 0.579612 \\ x_8 &= e^{-0.579612} = 0.560115 \\ x_9 &= e^{-0.560115} = 0.571143 \\ x_{10} &= e^{-0.571143} = 0.564879 \end{aligned}$$

hence:

$$X \approx 0.567143$$

Problem 4: Solve $\cos(x) - x = 0$ (15)

Solution:

(2.2) Rearrange

$$X = \cos(x) \quad (16)$$

Therefore

$$g(x) = \cos(x) \quad (17)$$

Initial guess

$$x_0 = 1$$

(2.3) Iterations

$$\begin{aligned} x_1 &= \cos(1) = 0.540302 \\ x_2 &= \cos(0.540302) = 0.857553 \\ x_3 &= \cos(0.857553) = 0.654290 \\ x_4 &= \cos(0.654290) = 0.793480 \\ x_5 &= \cos(0.793480) = 0.701369 \\ x_6 &= \cos(0.701369) = 0.763960 \\ x_7 &= \cos(0.763960) = 0.722102 \\ x_8 &= \cos(0.722102) = 0.750418 \end{aligned}$$

After several more iterations:

$$x \approx 0.739085$$

Hence

$$x \approx 0.739085$$

(2.4) Summary Table(5)

Equation	Fixed-Point Form g(x)	Root
$x^2 - 3 = 0$	$(x+3/x)/2$	1.732051
$x^3 + x - 1 = 0$	$(1-x)^{1/3}$	0.6823
$e^{-x} - x = 0$	e^{-x}	0.567143
$\mathrm{Cos}(x) - x = 0$	$\mathrm{Cos}(x)$	0.739085

These examples illustrate both successful convergence and failure of convergence, which is an important aspect of the Fixed-Point Iteration Method.

3. Advantages of the Fixed-Point Iteration Method(3)

1. Simple to implement

   The method uses the straightforward formula

   $$X_{n+1} = g(x_n)$$

   making it easy to program and calculate.

2. No derivative required

   Unlike the Newton-Raphson Method, it does not require computation of derivatives.

3. Low computational cost per iteration

   Each iteration usually involves only evaluating the function g(x).

4. Useful for nonlinear equations

   It can solve many nonlinear equations that are difficult to handle analytically.

5. Flexible formulation
   The equation $f(x) = 0$ can often be rearranged into different forms x=g(x)x=g(x)x=g(x), allowing the selection of a form that converges more rapidly.

(3.1) Disadvantages of the Fixed-Point Iteration Method(2)

1. Convergence is not guaranteed

   The method converges only if suitable conditions are met, typically:
   $|g'(x)| < 1$ near the fixed point.

2. Sensitive to the choice of g(x).

Different rearrangements of the same equation can lead to convergence, slow convergence, or complete divergence.

3. Slow convergence

   The convergence rate is usually linear, making it slower than methods such as the Newton-Raphson Method, which has quadratic convergence near the root.

4. Requires a good initial guess

   A poor starting value may cause divergence or convergence to an unintended root.

5. May oscillate or diverge

   Iterates can alternate between values or grow without approaching a solution if the convergence conditions are not satisfied.

(3.2) Summary

Advantages	Disadvantages
Easy to understand and implement	Convergence not always guaranteed
No derivative needed	Sensitive to the choice of

Advantages	Disadvantages
	g(x)g(x)g(x)
Low computational effort per step	Usually converges slowly
Suitable for many nonlinear equations	Requires a good initial guess
Flexible equation rearrangement	May oscillate or diverge

In practice: The Fixed-Point Iteration Method is valued for its simplicity, but its success depends heavily on choosing an appropriate iteration function g(x)g(x)g(x) and a reasonable starting value.

(3.3) Limitations of the Fixed-Point Iteration Method(9)

The Fixed-Point Iteration Method is simple and useful, but it has several important limitations:

1. Convergence Is Not Guaranteed

The method converges only when the iteration function g(x)g(x)g(x) satisfies certain conditions, typically

$$|g'(x)| < 1$$

near the fixed point. If this condition is not met, the iterations may diverge.

2. Sensitive to the Choice of Rearrangement. A given equation $f(x) = 0$ can often be rewritten in many forms, $x = g(x)$. Some forms converge, while others diverge or oscillate. Finding a suitable $g(x)$ can be difficult.

3. Slow Rate of Convergence

The method generally exhibits linear convergence, which is slower than methods such as the Newton-Raphson Method. As a result, many iterations may be needed to achieve high accuracy.

4. Dependence on Initial Guess

The success of the method often depends on the starting value x_0. A poor initial guess may cause:
- Divergence,
- Very slow convergence,
- Convergence to an undesired solution.

5. Oscillatory Behavior

If $g'(x)$ is negative and close to -1, the iterates may oscillate around the root and converge very slowly, or fail to converge altogether.

6. Multiple Roots Can Cause Difficulties

For equations with several roots, the method may converge to different roots depending on the initial guess, making the result less predictable.

7. No General Procedure for Choosing g(x)

There is no universal rule for transforming $f(x) = 0$ into a suitable fixed-point form $x = g(x)$. Choosing an effective iteration function often requires experience and trial-and-error.

(3.4) Summary of Limitations(4)

1. Convergence is not always guaranteed.
2. Highly dependent on the choice of $g(x)$.
3. Usually converges slowly.
4. Requires a good initial approximation.
5. May oscillate or diverge.
6. Can converge to different roots for different starting points.
7. No systematic way to find the best fixed-point form.

These limitations make the Fixed-Point Iteration Method most useful for problems where a convergent iteration function is known and high-speed convergence is not essential.

- Slow convergence
- Convergence not always guaranteed

4. Newton-Raphson Method(7)

The Newton–Raphson method (also called Newton's method) is a numerical technique used to find approximate solutions of equations of the form: f(x) = 0

It is one of the most important methods in numerical analysis because it often converges very quickly to a solution when the initial guess is sufficiently close to the root. The method was developed from ideas by Isaac Newton and later refined by Joseph Raphson.

(4.1) Idea

Suppose we want to find a root of a function (f(x)).

Starting from an initial estimate (x_0), we:

1. Draw the tangent line to the curve (y=f(x)) at (x_0).
2. Find where this tangent intersects the x-axis.
3. Use that intersection as a better approximation.

Repeating this process generates increasingly accurate estimates.

(4.2) Derivation of the Formula(9)

The tangent line to $(f(x))$ at (xn) is: $Y = f(xn) + f''(xn)(x - xn)$

At the root of the tangent line, $(y = 0)$: $0 = f(xn) + f''(xn)(x\{n + 1\} - xn)$

Solving for $(x\{n + 1\})$: $x\{n + 1\} = xn - f(xn) / f''(xn)$

This is the Newton–Raphson iteration formula.

(4.3) Algorithm(6)

Given:

- Function $(f(x))$
- Derivative $(f'(x))$
- Initial guess (x_0)

Repeat: $x_{n+1} = x_n - \{f(x_n) / f'(x_n))\}$

Until

$$|x_{n+1} - x_n| < \varepsilon$$

Or

$$|f(x_n)| < \varepsilon$$

Example :

Find the root of the function:

$$f(x) = x^3 + 4x^2 - 10$$

(18)

by Newton- Raphson method , in the interval [1 , 2] , with error $\in = 10^{-4}$, $x_0 = 1$

Solution:

$$f(x) = x^3 + 4x^2 - 10$$

$$f'(x) = 3x^2 + 8x$$

$$f''(x) = 6x + 8$$

$$f(1) = (1)^3 + 4(1)^2 - 10 = -5$$

$$f(2) = (2)^3 + 4(2)^2 - 10 = 14$$

$$f(1) * f(2) = (-5) * (14) < 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$$

$$f(x_n) = x_n^3 + 4x_n^2 - 10$$

$$f'(x_n) = 3x_n^2 + 8x_n$$

$$x_{n+1} = x_n - \frac{x_n^3 + 4x_n^2 - 10}{3x_n^2 + 8x_n}$$

$$\in = 10^{-4} \;;\; [1, 2] \;;\; x_0 = 1$$

$$x_{n+1} = \frac{3x_n^3 + 8x_n^2 - x_n^3 - 4x_n^2 + 10}{3x_n^2 + 8x_n}$$

$$x_{n+1} = \frac{2x_n^3 + 4x_n^2 + 10}{3x_n^2 + 8x_n}$$

n	x_n	x_{n+1}
0	x_1	1.45454
1	x_2	1.36889
2	x_3	1.36523
3	x_4	1.36523

So the root of the funcation is ; $X_4 = 1.36523$

Example :

Find the root of the function

$$f(x) = e^{-x} - 4x \qquad (19)$$

using Newton-Raphson method , with $x_0 = 0$

Solution:

$$f(x) = e^{-x} - 4x$$

$$f'(x) = -e^{-x} - 4$$

$$f''(x) = e^{-x}$$

$$f(0) = -1 , f(1) = 0.63$$

$$f(0) * f(1) < 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n - e^{x_n}}{1 - e^{-x_n}}$$

n	x_n	x_{n+1}
0	x_1	0.5
1	x_2	0.56631
2	x_3	0.567143
3	x_4	0.5671432

$$x_3 = 0.567143$$

Example :

Use Newton – Raphson method to find the root near 2 of the equation

$$x^4 - 11x + 8 = 0 \qquad (20)$$

accurate to five decimal places .

Solution:

$$x_0 = 2 ; f(x) = x^4 - 11x + 8$$

$$\rightarrow f'(x) = 4x^3 - 11$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\rightarrow x_n - \frac{x_n^4 - 11x_n + 8}{4x_n^3 - 11}$$

$$x_1 = x_0 - \frac{x_0^4 - 11x_0 + 8}{4x_0^3 - 11} = 2 - \frac{2^4 - 11(2) + 8}{4(2)^3 - 11} = 2 - {}^2/_{21} = 1.90476$$

$$x_2 = 1.90476 - \frac{(1.90476)^4 - 11(1.90476) + 8}{4(1.90476)^3 - 11} = 1.89209$$

$$x_3 = 1.89209 - \frac{(1.89209)^4 - 11(1.89209) + 8}{4(1.89209)^3 - 11} = 1.89188$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.89188 - \frac{(1.89188)^4 - 11(1.89188) + 8}{4(1.89188)^3 - 11} = 1.89188$$

Hence the root of the equation is 1.89188

5. Convergence

Convergence determines how quickly approximations approach the real root. For a simple root (r): f(r)=0

(5.1) Advantages(2)

1. Very Fast Convergence: Quadratic convergence makes it much faster than many alternatives.

2. High Accuracy: Can achieve extremely accurate solutions with only a few iterations.

3. Widely Applicable. Works for:

- Algebraic equations
- Transcendental equations
- Systems of nonlinear equations

4. Computational Efficiency: Often requires fewer iterations than methods such as the bisection method.

5. Foundation of Many Numerical Algorithms: Used in optimization, machine learning, engineering simulations, and scientific computing.

(5.2) Disadvantages(2)

1. Requires Derivatives: Must compute: f '(x), which may be difficult or expensive.

2. Sensitive to Initial Guess: A poor starting value can cause:

- Slow convergence
- Convergence to the wrong root
- Divergence

3. Failure When Derivative Is Zero: If: $f'(x_n)=0$, the formula becomes undefined.

4. Oscillation: The method may bounce between points without approaching a root.

5. No Guaranteed Convergence: Unlike some bracketing methods, Newton–Raphson does not always converge.

(5.3) Summary

The Newton–Raphson method is a powerful iterative technique for solving (f(x)=0). It uses the tangent line of a function to generate increasingly accurate approximations of a root according to: $X_{n+1}=x_n - \frac{f(x_n)}{f'(x_n)}$

Its greatest strength is quadratic convergence, making it extremely fast near a solution. Its main weaknesses are the need for derivatives and sensitivity to the initial guess. Because of its efficiency and broad applicability, it remains one of the most important algorithms in numerical analysis, scientific computing, engineering, and optimization.

6. Result

The Newton–Raphson method is one of the most efficient numerical techniques for finding the roots of nonlinear equations. Its quadratic convergence enables rapid and highly accurate solutions when a suitable initial guess is chosen. However, its performance depends on the availability of the derivative and the selection of an appropriate starting point, as poor initial estimates may lead to slow convergence or divergence. Despite these limitations, the method remains a fundamental and widely used algorithm in numerical analysis, engineering, scientific computing, and optimization due to its speed, accuracy, and versatility.

7. Conclusion

When comparing the fixed-point method and the Newton-Raphson method, the Newton-Raphson method is often preferred when the derivative is available and a reasonable initial value can be chosen, as it converges more quickly and requires fewer iterations. The fixed-point iteration method is simpler to implement and does not require derivatives, but its convergence is slower and depends heavily on the choice of the iteration function g(x). In short, the Newton-Raphson method is more efficient and accurate, while the fixed-point iteration method is simpler but less robust and slower.

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