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Research Paper

A Generalized Homotopy Analysis Method for Solving Nonlinear Differential Equations with Applications to Shock Absorber Modeling

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Abstract
This paper introduces a novel hybrid analytical technique, the Generalized Homotopy Analysis Method (GHAM), which combines the Homotopy Analysis Method (HAM) with the Laplace-type integral transform (G-transform). The GHAM is designed to overcome the inherent limitations of traditional analytical methods, such as the difficulty in controlling convergence regions and the complex differentiations and integrations required for higher-order fractional differential equations. The proposed method ensures a convergent series solution through the optimal selection of the auxiliary parameter c_0*. The validity and efficiency of GHAM are demonstrated through several numerical examples, including a fourth-order nonlinear differential equation, the Navier-Stokes equation, and nonlinear partial differential equations with proportional delays. The results show excellent agreement with solutions obtained from established methods like the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), and Variational Iteration Method (VIM). Furthermore, the practical utility of the method is highlighted by its application to the mathematical modeling of shock absorbers with both progressive and degressive nonlinear damping and spring characteristics. The solutions obtained via GHAM are shown to closely match numerical solutions obtained using the Runge-Kutta method, confirming the method's accuracy and computational efficiency for engineering problems.*

Keywords: *Generalized Homotopy Analysis Method (GHAM); G-Transform; Nonlinear Differential Equations; Convergence-Control Parameter; Navier-Stokes Equation; Shock Absorber Modeling; Nonlinear Spring; Nonlinear Damper.*

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I. Introduction

Last few decades, a number of analytical methods have been introduced to solve nonlinear differential equations in different fields. The list of analytical methods includes the Homotopy Perturbation Method (HPM), Non-Perturbation Method as well as the Homotopy Analysis Method (HAM) which provides the solutions in the form of a convergent series. Though these methods are efficient in solving nonlinear differential equations, it possesses inherent limitations. The use of HPM is subject to the existence of physical parameters involved in the differential equations. The convergence region yield by the solutions is also hard to be controlled. Liao (1992) introduced the HAM as a general analytical methodology to produce the solutions in a series form pertaining to a variety of linear as well as nonlinear equations. Liao (1997) proposed a strong solution methodology using HAM for analyzing nonlinear heat transfer problems that arise in the study of plate heating with a microwave. It is interesting to mention that, the HAM controls the convergence region and the rate of approximation of the series can also be handled effectively. In contrast to other analytical techniques, the HAM offers the advantage of ensuring the convergence of the series solution pertaining to nonlinear problems. This is achieved by appropriately determining the convergence-control parameter c_0. The following are the merits of the proposed Generalised Homotopy Analysis Method:
- GHAM avoids multiple differentiation and integration that arise in HAM while solving higher order fractional differential equations.

- The proposed hybrid technique GHAM successfully overcomes the limitation of Laplace typed integral transform and can be effectively used to solve the complicated non-linear fractional differential equations.
- The GHAM can be applied to solve various types of engineering problems and to obtain the solutions by the suitable selection of auxiliary parameter.

II. Homotopy Analysis Method

The concept and principle of 'homotopy' was initiated by Sen (1983). Later, the HAM was suggested by Liao (2004) to obtain an exact solution of a given differential equation. The HAM establishes a continuous mapping that involves an initial approximation based on the best guess, where the rules of solution expression and coefficient ergodicity are incorporated. As a result, the procedure for tackling physical problems can be simplified. Given the nonlinear differential equation

$$N[u(t)] = 0, t > 0 \tag{1}$$

where ′N′ is the nonlinear operator, the unknown function with an independent variable t is denoted as u(t). The zeroth order deformation must be utilized to yield the higher order deformation. At the same time, the solution vector

$u_{\sim n} = u_0(t), u_1(t), ..., u_n(t)$ can be obtained as follows. The n-th order deformation equation is

$$L[u_n(t) - \chi_n u_{n-1}(t)] = hH(t)R_n(\vec{u_n}, t), \tag{2}$$

where

$$\chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & otherwise \end{cases} \tag{3}$$

and

$$R_n(u_n\vec{-}1, t) = \frac{1}{(n-1)!}\frac{\partial^{m-1}(N[\phi(t,s)])}{\partial_{m-1}}|_{s=0}. \tag{4}$$

Given a nonlinear operator N, Equation (4) to represent the term $R_n(u_n - \vec{1}, t)$. This leads us to obtain the following solution,

$$u(t) = \sum_{k=0}^{n} u_k(t) \tag{5}$$

The next subsection presents the fundamental idea pertaining to the proposed HAM model for undertaking nonlinear partial differential equations.

Laplace Typed Integral Transform(G-transform) (2.1) [] The G-transform converts any time domain function u(t) into the associated frequency domain function G(ω) is defined as

$$G[u(t)] = \omega^{\alpha}\int_a^b e^{-t/\omega}u(t)dt. \tag{6}$$

where α is a suitable integer and ω is the frequency variable to be selected. The n-th derivative property of G-transform imply

$$G(u^n) = \frac{1}{\omega^n}G(u) - \frac{1}{\omega^{n-1}}u(0)\omega^{\alpha} - \frac{1}{\omega^{n-2}}u'(0)\omega^{\alpha} - u^{(n-1)}(0)\omega^{\alpha} \tag{7}$$

Generalized Homotopy Analysis method (GHAM) (2.1) [] A systematic procedure for solving the nonlinear partial differential equation

$D^{\alpha}v(x, t) + Rv(x, t) + Nv(x, t) = g(x, t)$. In the following, ordinary differential equation, linear and nonlinear partial differential equations are solved using GHAM. The results show an excellent agreement with those from some of the existing methods.

4. Numerical Examples

We illustrate the usefulness and computational efficiency of GHAM through the discussion of standard numerical examples.

Example (4.1) [] A fourth-order nonlinear differential equation is expressed as follows:

$$\frac{d^4u(t)}{dt^4} = 24(u(t))^5 + 40(u(t))^3 + 16u(t) \tag{8}$$

The initial conditions are

$$u \quad = \frac{1}{3}.$$

The G-transform is applie

$$G\left[\frac{d^4u(t)}{dt^4}\right] = G\left[24(u(t))^5 + 40(u(t))^3 + 16u(t)\right]. \tag{9}$$

Using the properties of G-Transform, the following equation is obtained

$$G[u(t)] = \omega^{\alpha+2} + \omega^{\alpha+4} + \omega^4 G\left[24(u(t))^5 + 40(u(t))^3 + 16u(t)\right]. \tag{10}$$

$$u_n(x,t) = \chi_n u_{n-1}(x,t) + G^{-1}[hH(x,t)R_n(\vec{u}_{n-1}(x,t))] \tag{11}$$

where

$$R_n(\vec{u}_{n-1}) = G[u_{n-1}(t)] - (1-\chi_n)\omega^{\alpha+1}u(0) - (1-\chi_n)\omega^{\alpha+2}u'(0) - (1-\chi_n)\omega^{\alpha+3}u''(0) - (1-\chi_n)\omega^{\alpha+4}u'''(0) + \omega^4 G[24(u_{n-1}(t))^5 + 40(u_{n-1}(t))^3 + 16u_{n-1}(t)].$$

The above equation is solved with respect to n = 1, 2, 3, …., i.e.,

$$u_0(t) = t + \frac{t^3}{3}$$

$$u_1(t) = \frac{2}{15}ht^5 + \frac{17}{315}ht^7 + \frac{4}{189}t^9 + \frac{2}{297}ht^{11} + \frac{19}{11583}ht^{13} + \frac{2}{7371}t^{15} + \frac{1}{38556}ht^{17} + \frac{1}{941868}t^{19}$$

Similarly, we can estimate u_2, u_3, u_4, …. The series solution is expressed as

$$u(x,t) = \sum_{n=0}^{N} u_n(x,t). \tag{12}$$

Figure 1.1 illustrate the h-curves. It can be observed that, for h ∈ [−1,1], the method provides more accurate results. The solution curve u(t) of equation (8) is depicted in Figure (4.2). Figure (4.3) shows the comparison among GHAM, ADM and IADM. It shows an excellent agreement with the other existing methods.

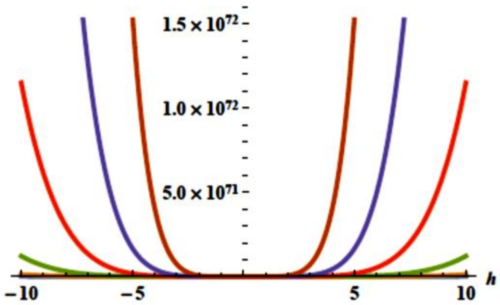

Figure (4.1). The h-curve of Example (4.1).

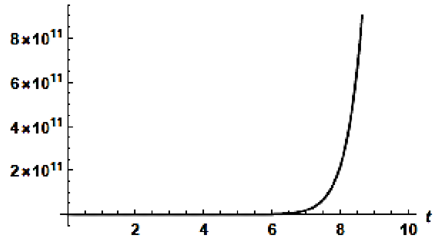

Figure (4.2). The solution curve of Example (4.1).

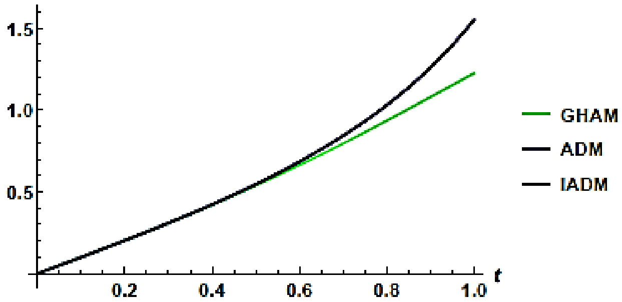

Figure (4.3). Comparison among GHAM, ADM and IADM

It is worth to mention that, as compared with ADM does not require iterative differentiation and integration for solving higher-order differential equations.

Example (4.2) [] The Navier-Stoke's equation is considered as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} \tag{13}$$

The initial condition is u(x,0) = f(x).

Now, G-transform method is applied to both sides of equation (13),

$$G\left[\frac{\partial u}{\partial t}\right] = G\left[\frac{\partial^2 u}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u}{\partial x}\right] \tag{14}$$

$$\frac{1}{\omega}G[u] - u(0)\omega^{\alpha} = G\left[\frac{\partial^2 u}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u}{\partial x}\right]$$

$$G[u] = \omega^{\alpha+1}f(x) + \omega\left[G\left[\frac{\partial^2 u}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u}{\partial x}\right]\right].$$

Apply GHAM,

$$u(x,t) = \chi_n u_{n-1}(x,t) + G^{-1}\left[hR_n(\vec{u_{n-1}},x,t)\right] \tag{15}$$

where,

$$R_n(\vec{u_{n-1}},x,t) = G[u(x,t)] - (1-\chi_n)\omega^{\alpha+1}u(x,0) + \omega\left[G\left[\frac{\partial^2 u_{n-1}}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u_{n-1}}{\partial x}\right]\right].$$

By solving the above equation for n = 1, 2, 3, ... provides

$$u_0(x,t) = f(x)$$

$$u_1(x,t) = \left[f"(x) + \frac{1}{x}f'(x)\right]\frac{ht}{1!}$$

$$u_2(x,t) = \left[f^{IV}(x) + \frac{2x^2 f'''(x) - xf"(x) + f'(x)}{x^3}\right]\frac{h^2t^2}{2!}$$

Similarly, we estimate u_3, u_4, ..., and this leads to the series solution

$$u(x,t) = \sum_{n=0}^{N} u_n(x,t).$$

Subject to the initial condition f(x) = x, the following solution is obtained.

(16)

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times ... \times (2n-3)^2}{x^{2n-1}} \frac{h^n t^n}{n!}.$$

Figure (4.4) and (4.5) represent the h-curve (h ∈ [−1,1]) and the solution curve of the Navier-Stoke equation (16) respectively. Figure 6 shows an excellent agreement with HAM, HTAM and VIM.

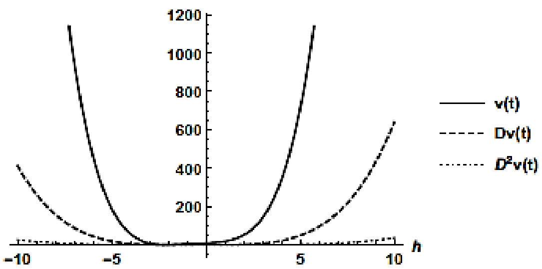

Figure (4.4). The h-curve of Example (4.2).

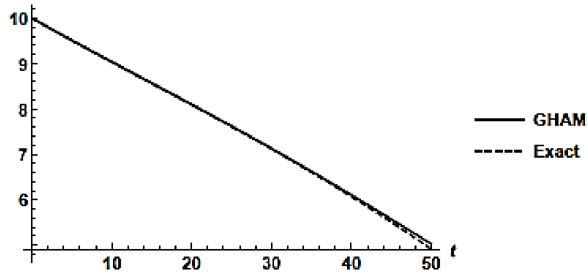

Figure (4.5). The solution curve of Example (4.2).

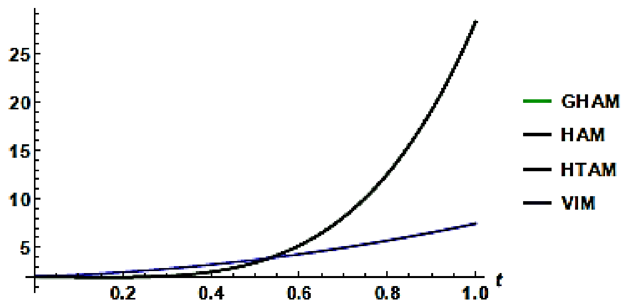

Figure (4.6). Comparison among GHAM, HAM, HTAM and VIM

Example (4.3) [] The generalized Burgers equation along with a proportional delay is given by

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u\left(\frac{x}{2}, \frac{t}{2}\right) \frac{\partial u(x, \frac{t}{2})}{\partial x} + \frac{1}{2}u(x,t) \tag{17}$$

The initial condition is, u(x,0) = 0.
Solving equation (17) for n = 1, 2, 3, ... leads to the following solution

$$u(x,t) = x(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + ...) = xe^{ht}. \tag{18}$$

Figure (4.7) and (4.8) represent the h-curve (h ∈ [−1,1]) and the solution curve of the proportional delay equation (18) respectively. Figure (4.9) shows an excellent agreement with HPM and DTM.

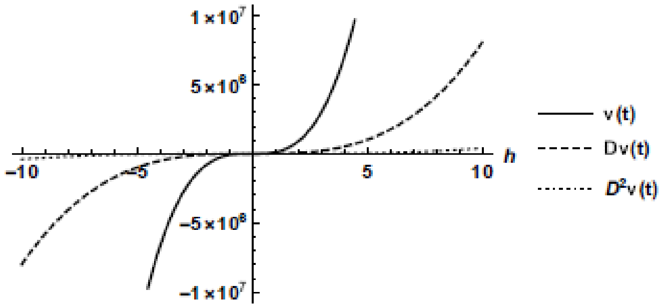

Figure (4.7). The h-curve of Example (4.3).

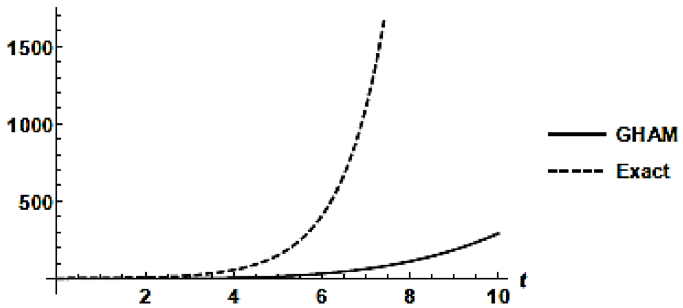

Figure (4.8). The solution curve of Example (4.3).

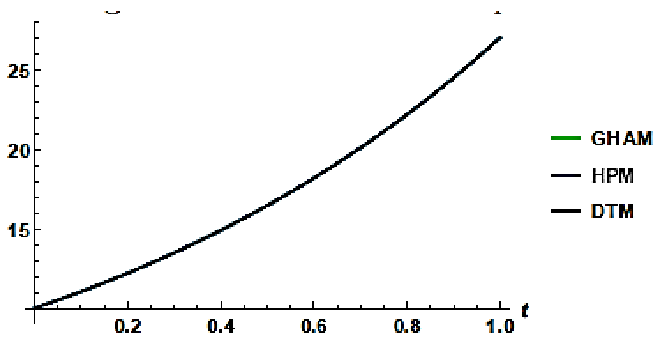

Figure (4.9). Comparison among GHAM, HPM and DTM

Example (4.4) [] An initial value PDE with a proportional delay is constructed as follows

(19)

$$\frac{\partial u}{\partial t} = u(x, \frac{t}{2})\frac{\partial^2 u(x, \frac{t}{2})}{\partial x^2} - u(x,t)$$

The corresponding initial condition is assumed as, u(x,0) = x^2.

Solving equation (12) for n = 1, 2, 3, ... leads to the solution as shown below.

$$u(x,t) = x^2(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + ...) = x^2e^{ht}. \tag{20}$$

Figure (4.10) and (3.11) depict the h-curve (h ∈ [−1,1]) and the solution curve of (4.20), respectively. Figure (4.12) shows an excellent agreement with HPM and DTM.

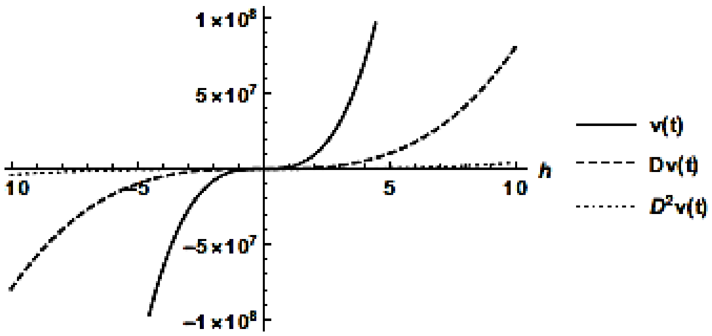

Figure (4.10). The h-curve of Example (4.4).

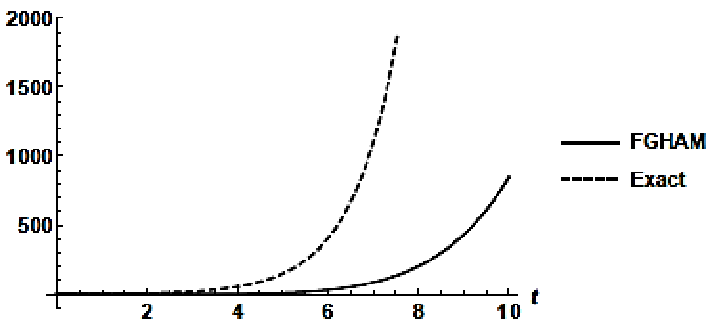

Figure (4.11) The solution curve of Example (4.4).

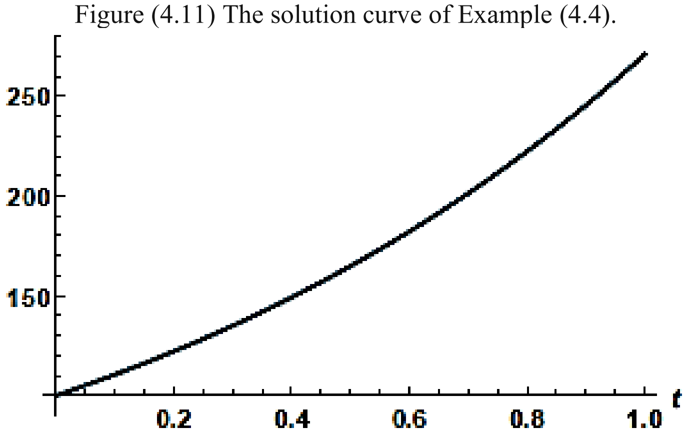

Figure (4.12) Comparison among HPM and DTM

From the above numerical examples, it can be observed that the solution of GHAM excellently agrees with the solutions obtained by the existing method. In addition, GHAM is also computationally efficient.

5. Nonlinear Spring Behavior

A spring element exerts a reaction force in response to a displacement, either compression or extension, of the element. The linear spring relation, f = kx, become less accurate with increasing deflection (for either compression or extension). In such case it is often replaced by:

$$f = k_1x + k_2x^3, \tag{21}$$

where $k_1 > 0$ and has dimension of N/m and x is the spring displacement from its free length. The spring elements is said to be hardening or hard if $k_2 > 0$ and has dimension of N/m^3 and softening or soft if $k_2 < 0$. These cases are shown in Fig. (5.1).

The spring stiffness k is the slope of force-deflection curve and is constant for the linear spring element. A nonlinear spring does not have a single stiffness value since its slope is variable.

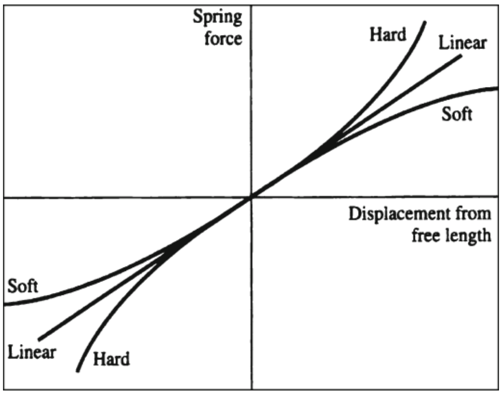

Fig. (5.1) Hard, soft and linear spring function

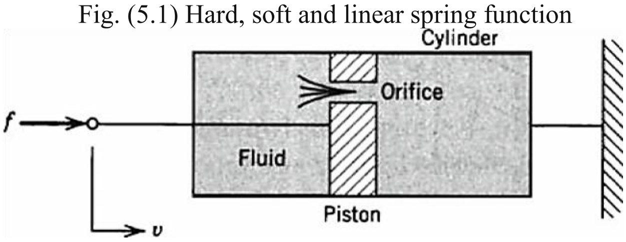

Fig. (5.2) A fluid damper consists of a piston moving inside a cylinder filled with a viscous fluid

A damper element is an element that resists relative velocity across it. A simple way to achieve damping is with a dashpot or damper, which is the basis of shock absorber. It consists of a piston moving inside a cylinder that is sealed and filled with a viscous fluid (Fig. 5.2) the piston has a hole or orifice through which the fluid can flow when the piston moves relative to the cylinder, but the fluid' viscosity resists this motion.

Shock damper (5.1) [] The telescopic Stockalper's used in many vehicles. A cutaway view atypical shock damper is very complex, but the basic principle of its operation is the damper concept illustrated in Fig. (5.3). The damping can be designed to be progressive, which means that the slope of the damping force-versus-velocity curve increasing with velocity, or degressive, where the slope decreases with velocity (Fig. 5.4).

Homotopy perturbation method (5.2) [] The widely applied techniques (i.e., perturbation method) are of great interest to be used in engineering systems. To eliminate the limitation of "small parameter", which is assumed in the perturbation method, a new technique based on the homotopy terminology has been proposed. Accordingly, a nonlinear problem is transformed into an infinite number of simple problems without using the perturbation technique. Effectively, letting the small parameter float and converge to the unity, the problem will be converted into a special perturbation problem with a small embedding parameter. So the method receives the name, homotopy perturbation method (HPM). The effectiveness of the new technique is presented. This approach can take full advantage of the traditional perturbation technique and homotopy analysis method. The new scheme has been applied to linear and nonlinear ordinary and partial differential equations. These equations usually describe a dynamic system incorporating the perturbation value (i.e., HPM). It has been applied to a wide class of differential equations, such as Duffing equation, numerical and algebraic methods and heat transfer. To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, \quad r \in \Omega. \tag{5.2}$$

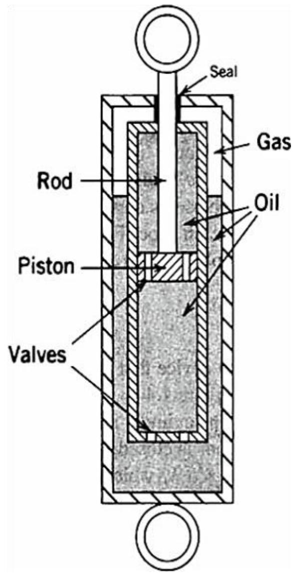

Fig. (5.3) A shock damper in common parlance (or damper in technical use) is a mechanical device designed to smooth out or damp shock impulse, and dissipate kinetic energy

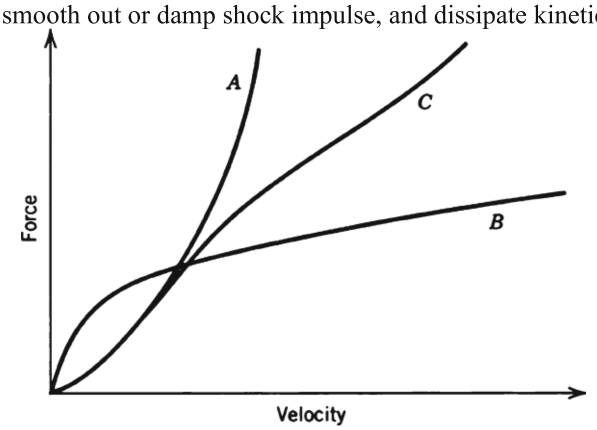

Fig. (5.4) Damping curves

With the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{23}$$

where A is a general differential operator, B a boundary operator, f (r) a known analytical function and is the boundary of the domain. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Equation (2) can therefore be rewritten as follows:

$$L(u) - N(r) - f(r) = 0, \quad r \in \Omega. \tag{24}$$

Homotopy perturbation structure is shown as follows:

$$H(\nu, p) = (1 - p)\,[L(\nu) - L(u_0)] + p\,[A(\nu) - f(r)] = 0, \tag{25}$$

where,

$$\nu(r, p) : \Omega \times [0,1] \to R, \tag{26}$$

In Eq. (25), $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (14) can be written as a power series in p, as following:

$$\nu = \nu_0 + p\nu_1 + p_2\nu_2 + \cdots, \tag{27}$$

and the best approximation for solution is:

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_1 + \ldots, \quad (5.8)$$

Enhanced homotopy perturbation method (5.3) [] Using HPM in some equations may lead to a different or unexpected answer. This problem specifically occurs when a stable nonlinear differential equation has an unstable linear part. This problem has to be treated to match the actual solution. A possible way of treatment is briefly introduced. In this method, the stability property of the nonlinear differential equation will be examined first. If it is provided, the stability of the usual distinguished linear part will be tested. It is hoped that the usual HPM leads to a satisfactory result. If not, there is no guarantee to have a well-behaved result. At this point, an additional zero-term (an extra term with a globally zero effect, i.e., $\eta(x) \in \Omega(x)$ space of linear function with real coefficients) may cope with the problem. At this stage, an extra term will be added to the linear part and subtracted from the rest, at the same time, such that a linear part is stabilized. The rest of the statement establishes the nonlinear part, i.e., N(v). By this alteration and using HPM, a so-called enhanced homotopy perturbation method (EHPM) will produce a stable v_0, and therefore the whole solutions can be achieved.

6. Mathematical modeling of shock absorber

There are many applications where we can derive an expression for the spring force as a function of deflection or the damping force as a function of velocity; in practice forms for the spring relation are

– the linear model: f (x) = k(x)
– the cubic model as shown in Eq. (5.1): $f(x) = k_1(x) + k_2(x)^3$

In which the constant term and the x^2 term cannot appear if the force function is anti-symmetric about the origin; i.e., if f (−x) = − f (x).

The most common model forms for the damping relation are:

– The linear model: f (v) = cv
– The square-law model: $f(v) = cv^2$
– The general progressive model (of which the square law is a special case); $f(v) = cv^n$, n > 1
– The degressive model: $f(v) = cv^n$, n < 1

The difference between the progressive and the degressive model is that the slope of the progressive curve increase with v. The slope of the linear model is constant, and its line between the progressive and the degressive curves (Fig. 5.4).

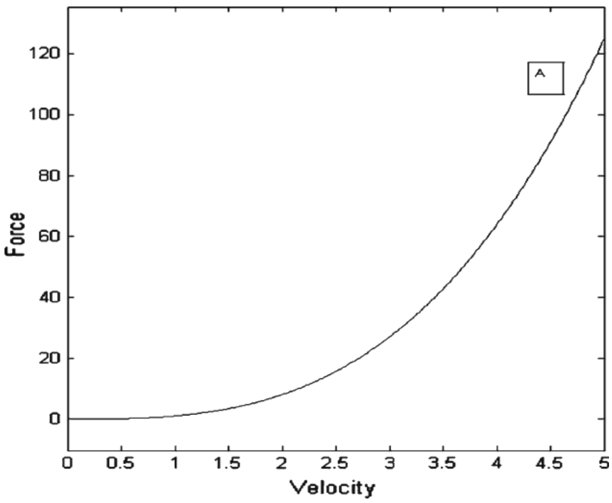

Fig. (6.1) Progressive nonlinear damping force (N)-versus-velocity (m/s)

Case 1 (6.1) [] The shock absorber with a progressive nonlinear damper and linear spring is modeled. The damping force and spring force be described by the following functions, respectively:

$$F_d = v^2 \text{ and } F_s = 100x. \tag{29}$$

where the v is velocity (m/s) and x is displacement (m).

The damping force-versus-velocity curve is shown in Fig. (6.1):

We consider the equation of motion and the initial conditions as:

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 + 100x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1, \tag{30}$$

The linear and the nonlinear parts can be distinguished as:

(31)

$$L(x) = \frac{d^2x}{dt^2} + 100x,$$

and

(32)

$$N(x) = \left(\frac{dx}{dt}\right)^3 .$$

By adding and subtracting $\left(a\,\frac{d^2x}{dt^2} + b\,\frac{dx}{dt}\right)$ in Eq. (30) we have:

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 + 100x + \left(a\frac{d^2x}{dt^2} + b\frac{dx}{dt}\right) - \left(a\frac{d^2x}{dt^2} + b\frac{dx}{dt}\right) = 0,$$
$$x(0) = 0,\ \frac{dx}{dt}(0) = 1$$

(33)

The new L(x) and N(x) are:

$$L(x) = \frac{d^2x}{dt^2} + a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + 100x,$$

(34)

and

$$N(x) = \left(\frac{dx}{dt}\right)^3 - a\frac{d^2x}{dt^2} - b\frac{dx}{dt},$$

(35)

According to Eq. (35), the homotopy function can be written as:

$$H(p,x) = (1-p)[L(v)] + p[L(v) + N(v)].$$

(36)

We choose a = 0.2 and b = 0.5. By using ν as $\nu = \nu_0 + p\nu_1 + p_2\nu_2 + \ldots$ and substituting into Eq. (36) and rearranging based on powers of p-terms, we can obtain:

(37)

and

(38)

$$p^1 : 1.2\frac{d^2v_1}{dt^2} + 0.5\frac{dv_1}{dt} + 100v_1 + \left(\frac{dv_0}{dt}\right)^3 - 0.5\frac{dv_0}{dt} - 0.2\frac{d^2v_0}{dt^2} = 0,$$
$$v_1(0) = 0,\quad \frac{dv_1}{dt}(0) = 0$$

By solving the simple equation in (37), we have the solution as:

$$v_0(t) = 0.11e^{-0.208t}\sin(9.11t). \qquad (39)$$

The approximation for solution is:

(40)

$$x(t) = \lim_{p\to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots,$$

The first order of EHPM solution is chosen. The solution of the nonlinear shock absorber equation (40), when p → 1, will be as follows:

$$x(t) = 0.11e^{-0.208t}\sin(9.11t), \qquad (41)$$

By the drawing figures of numerical solution of Runge–Kutta method and EHPM solution, we see that figure similar to each other (Fig. 6.2):

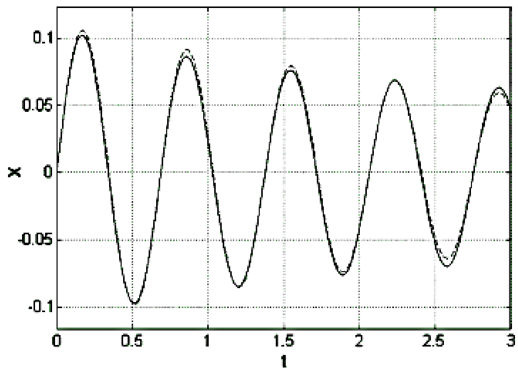

Fig. (6.2) Comparison of the first order of EHPM solution with the numerical solution of Runge–Kutta method. Dashed line the EHPM solution and solid line numerical solution (displacement (m)-versus-time (s))

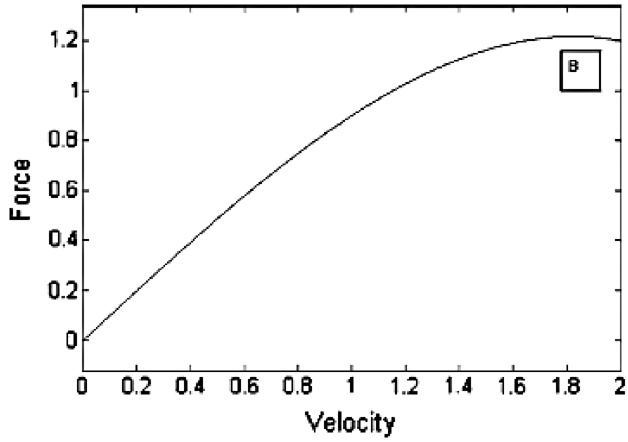

Fig.(6.3) Degressive nonlinear damping force (force (N)-versus-velocity (m/s))

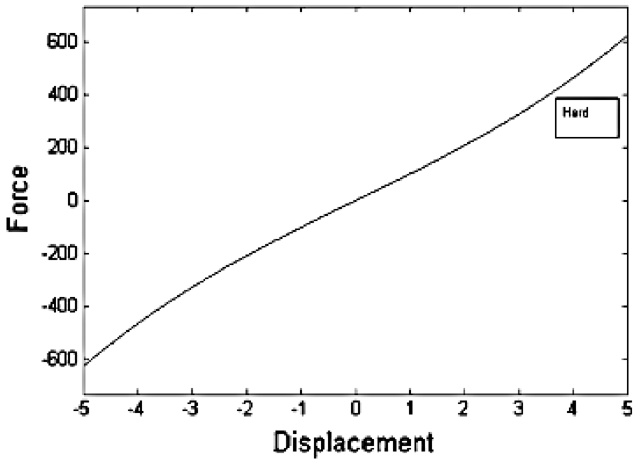

Fig. (6.4) Hard-spring function (force (N)-versus-displacement (m))

Case 2 (6.2) [] The shock absorber with a digressive nonlinear damper and nonlinear spring is modeled. The damping force and spring force be described by the following functions, respectively:

$$F_d = v - 0.1v^3 \quad \text{and} \quad Fs = x^3 + 100x \tag{42}$$

where the v is velocity (m/s) and x is displacement (m).

The damping force-versus-velocity curve is shown in Fig. (6.3):
The spring force-versus-displacement curve is shown in Fig. (6.4):
The spring element is hard because its slope thus its stiffness increases with deflection.

The equation of motion and the initial conditions are assumed as follows:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 0.1\left(\frac{dx}{dt}\right)^3 + x^3 + 100x = 0, \quad x(0) = 0, \frac{dx}{dt}(0) = 1. \tag{43}$$

The linear and the nonlinear parts can be distinguished as:

$$L(x) = \frac{d^2x}{dt^2} + \frac{dx}{dt} + 100x, \tag{5.24}$$

and

$$N(x) = -0.1\left(\frac{dx}{dt}\right)^3 + x^3. \tag{5.25}$$

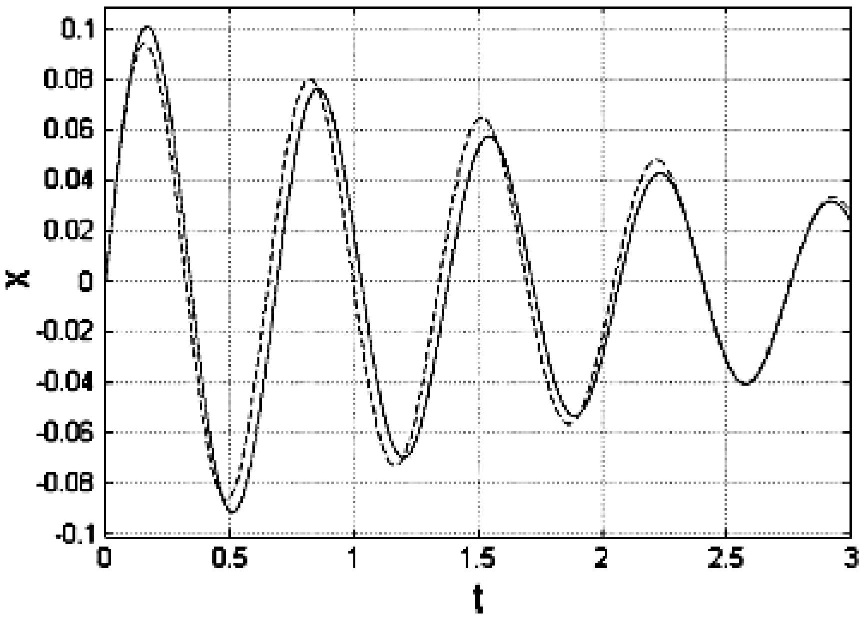

Fig. (6.5) Comparison of the second order of EHPM solutions of equation of motion (m) with the numerical solution of Runge–Kutta method. Dashed line the EHPM solution and solid line numerical solution

By adding and subtracting $\left(a\frac{d^2x}{dt^2} + b\frac{dx}{dt}\right)$ in Eq. (43) we have

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 0.1\left(\frac{dx}{dt}\right)^3 + x^3 + 100x + \left(a\frac{d^2x}{dt^2} + b\frac{dx}{dt}\right) - \left(a\frac{d^2x}{dt^2} + b\frac{dx}{dt}\right) = 0,$$
$$x(0) = 0, \frac{dx}{dt}(0) = 1. \tag{46}$$

The new L(x) and N(x) are

$$L(x) = \frac{d^2x}{dt^2} + \frac{dx}{dt} + 100x + \left(a\frac{d^2x}{dt^2} + b\frac{dx}{dt}\right), \tag{47}$$

and:

$$N(x) = -0.1\left(\frac{dx}{dt}\right)^3 + x^3 - \left(a\frac{d^2x}{dt^2} + b\frac{dx}{dt}\right) \tag{48}$$

According to Eq. (5.5), the homotopy function can be written as similar as Eq. (5.16):

$$H(p, x) = (1 - p)[L(v)] + p[L(v) + N(v)], \tag{49}$$

We consider a = 0.3 and b = 1.2. By using ν as $\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \ldots$ and substituting ν into Eq. (49) and rearranging based on powers of p-terms, we can obtain:

$$p^0 : 1.3\frac{d^2v_0}{dt^2} + 2.2\frac{dv_0}{dt} + 100v_0 = 0, \quad v_0(0) = 0, \quad \frac{dv_0}{dt}(0) = 1, \tag{50}$$

and

$$p^1 : 1.3\frac{d^2v_1}{dt^2} + 2.2\frac{dv_1}{dt} + 100v_1 - 0.1\left(\frac{dv_0}{dt}\right)^3 - 0.3\frac{d^2v_0}{dt^2} - 1.2\frac{dv_0}{dt} + v_0^3 = 0, \tag{51}$$

$$v_1(0) = 0, \quad \frac{dv_1}{dt}(0) = 0,$$

By solving the simple equations in (50), (51) we have the solution as:

$$\nu_0(t) = 0.114e^{-0.846t}\sin(8.72t), \tag{52}$$

and:

$$\nu_1(t) = -0.0049e^{-0.846t}\sin(8.72t) - 0.0017e^{-0.0846t}\cos(8.72t)$$
$$+6.45 \times 10^{-22}[(1.85 \times 10^{20}t + 2.71 \times 10^{18})e^{-2.54t}e^{1.69t} - 4.51 \times 10^{16}e^{-2.54t}]\cos(8.72t) + 0.0305[-0.0647 + (-0.224 + te^{1.69t})]e^{-2.54t}\sin(8.72t)$$
$$+6.45 \times 10^{-22}[6.39 \times 10^{15}\sin(26.2t) - 4.87 \times 10^{16}\cos(26.2t)]e^{-2.54t} \tag{53}$$

The approximation for solution is:

$$x(t) = \lim_{p\to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots, \tag{54}$$

The second-order EHPM is chosen. The solution of the nonlinear shock absorber equation (52), when p → 1, will be as follows:

$$x(t) = \nu_0 + \nu_1. \tag{55}$$

By the figures of numerical solution of Runge–Kutta method and EHPM solution, we see that the figures are similar to each other.

7. Result

Let a nonlinear differential equation be represented as $N[u(t)] = 0$, where N is a nonlinear operator. The Generalized Homotopy Analysis Method (GHAM) provides a convergent series solution $u(t) = \sum_{n=0}^{\infty} u_n(t)$ if there exists an auxiliary parameter $\hbar \neq 0$ and an auxiliary function $H(t) \neq 0$ such that the sequence of functions $u_n(t)$ derived from the n-th order deformation equation,

$$\mathcal{L}[u_n(t) - \chi_n u_{n-1}(t)] = \hbar H(t) R_n(\vec{u}_{n-1}, t),$$

is convergent for all $t > 0$. The convergence region and rate of the series are effectively controlled by the auxiliary parameter $\hbar$. The GHAM, by incorporating the G-transform, further ensures that the method avoids the complex iterative differentiations and integrations required by standard HAM, making it particularly robust for solving higher-order and fractional nonlinear differential equations. The solution obtained by GHAM is unique and converges to the exact solution within a chosen range of the auxiliary parameter.

8. Conclusion

The Generalized Homotopy Analysis Method (GHAM) has been successfully developed and applied to a range of challenging nonlinear differential equations.

In summary, the GHAM is a powerful, efficient, and versatile analytical tool for solving a wide class of nonlinear ordinary and partial differential equations, with significant potential for application in various scientific and engineering disciplines.

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