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Research Paper

Development and Application of a Generalized Homotopy Analysis Method (GHAM) for Nonlinear Partial and Ordinary Differential Equations

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Abstract
This paper introduces a new computationally efficient analytical technique, namely the Generalized Homotopy Analysis Method (GHAM), for solving a wide range of nonlinear differential equations. While existing methods such as the Homotopy Perturbation Method (HPM) and the standard Homotopy Analysis Method (HAM) are effective, they often suffer from limitations related to convergence control and computational complexity, particularly for higher-order or fractional equations. The proposed GHAM overcomes these drawbacks by integrating the Laplace-type G-transform with the homotopy framework, thereby avoiding iterative differentiation and integration and ensuring rapid convergence through the optimal selection of the auxiliary parameter. The effectiveness and accuracy of GHAM are demonstrated through four numerical examples: a fourth-order nonlinear ordinary differential equation, the Navier-Stokes equation, and two partial differential equations with proportional delays. The obtained series solutions are compared with existing results from ADM, IADM, HAM, HTAM, VIM, HPM, and DTM. The comparisons, supported by graphical illustrations of h-curves and solution curves, reveal an excellent agreement, confirming that GHAM is a robust, versatile, and highly accurate alternative for solving complex nonlinear problems in engineering and science.

Keywords: *Generalized Homotopy Analysis Method (GHAM), Nonlinear Differential Equations, G-Transform, Convergence-Control Parameter, Navier-Stokes Equation, Delay Differential Equations, Series Solution.*

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I. Introduction

Last few decades, a number of analytical methods have been introduced to solve nonlinear differential equations in different fields. The list of analytical methods includes the Homotopy Perturbation Method (HPM), Non-Perturbation Method as well as the Homotopy Analysis Method (HAM) which provides the solutions in the form of a convergent series. Though these methods are efficient in solving nonlinear differential equations, it possess inherent limitations. The use of HPM is subject to the existence of physical parameters involved in the differential equations. The convergence region yield by the solutions is also hard to be controlled. Liao (1992) introduced the HAM as a general analytical methodology to produce the solutions in a series form pertaining to a variety of linear as well as nonlinear equations.[2] Liao (1997) proposed a strong solution methodology using HAM for analyzing nonlinear heat transfer problems that arise in the study of plate heating with a microwave. It is interesting to mention that, the HAM controls the convergence region and the rate of approximation of the series can also be handled effectively. In contrast to other analytical techniques, the HAM offers the advantage of ensuring the convergence of the series solution pertaining to nonlinear problems. This is achieved by appropriately determining the convergence-control parameter c_0.[3]

Liao (2004) studied the fundamental concepts related to HAM and applied to study a variety of nonlinear equations. They encompass ordinary differential equations, differential-integral equations, differential-difference equations, partial differential equations, as well as coupled equations. The HAM is effective in approximating the solution of nonlinear problems. This is achieved through determining suitable sets of base functions as well as suitable initial conditions.[4]

We contribute a new and computationally efficient analytical method, namely GHAM. This new method is able to produce the solutions with rapid convergence series. GHAM is useful for tackling the computable terms in various differential equations, even though the governing equations contain high degree of nonlinear terms. The results obtained by GHAM are more accurate than the results of HAM.

The following are the merits of the proposed Generalised Homotopy Analysis Method:

- GHAM avoids multiple differentiation and integration that arise in HAM while solving higher order fractional differential equations.
- The proposed hybrid technique GHAM successfully overcomes the limitation of Laplace typed integral transform and can be effectively used to solve the complicated non-linear fractional differential equations.
- The GHAM can be applied to solve various types of engineering problems and to obtain the solutions by the suitable selection of auxiliary parameter.

II. Homotopy Analysis Method

The concept and principle of 'homotopy' was initiated by Sen (1983). Later, the HAM was suggested by Liao (2004) to obtain an exact solution of a given differential equation. The HAM establishes a continuous mapping that involves an initial approximation based on the best guess, where the rules of solution expression and coefficient ergodicity are incorporated. As a result, the procedure for tackling physical problems can be simplified.[4]

Given the nonlinear differential equation

$$N[u(t)] = 0, t > 0 \tag{1}$$

where ′N′ is the nonlinear operator, the unknown function with an independent variable t is denoted as u(t).

The zeroth order deformation must be utilized to yield the higher order deformation. At the same time, the solution vector

$u_{\sim n} = u_0(t), u_1(t), ..., u_n(t)$ can be obtained as follows.

The n-th order deformation equation is

$$L[u_n(t) - \chi_n u_{n-1}(t)] = hH(t)R_n(\vec{u_n}, t), \tag{2}$$

where

$$\chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & otherwise \end{cases} \tag{3}$$

and

$$R_n(\vec{u_{n-1}}, t) = \frac{1}{(n-1)!} \frac{\partial^{m-1}(N[\phi(t,s)])}{\partial_{m-1}}|_{s=0}. \tag{4}$$

Given a nonlinear operator N, Equation (4) to represent the term

$R_n(\vec{u_{n-1}}, t)$ ıs to obtain the following solution,

$$u(t) = \sum_{k=0}^{n} u_k(t) \tag{5}$$

The next subsection presents the fundamental idea pertaining to the proposed HAM model for undertaking nonlinear partial differential equations.

III. Laplace Typed Integral Transform(G-transform)

The G-transform converts any time domain function u(t) into the associated frequency domain function G(ω) is defined as

$$G[u(t)] = \omega^{\alpha} \int_a^b e^{-t/\omega} u(t)dt. \tag{6}$$

where α is a suitable integer and ω is the frequency variable to be selected. The n-th derivative property of G-transform imply

$$G(u^n) = \frac{1}{\omega^n}G(u) - \frac{1}{\omega^{n-1}}u(0)\omega^\alpha - \frac{1}{\omega^{n-2}}u'(0)\omega^\alpha - u^{(n-1)}(0)\omega^\alpha \tag{7}$$

IV. Generalized Homotopy Analysis method (GHAM)

A systematic procedure for solving the nonlinear partial differential equation $D^\alpha v(x, t) + Rv(x, t) + Nv(x, t) = g(x, t)$, is illustrated in Saratha et al (2020). Ordinary differential equation, linear and nonlinear partial differential equations are solved using GHAM. The results show an excellent agreement with those from some of the existing methods. [5]

V. Numerical Examples

We illustrate the usefulness and computational efficiency of GHAM through the discussion of standard numerical examples.

Example (4.1) [6] A fourth-order nonlinear differential equation is expressed as follows:

(8)

The initial conditions are

$$u(0) = 0, \quad \frac{d^4u(t)}{dt^4} = 24(u(t))^5 + 40(u(t))^3 + 16u(t)$$

The G-transform is applied to both sides of Equation (4.8). This leads to,

$$G\left[\frac{d^4u(t)}{dt^4}\right] = G\left[24(u(t))^5 + 40(u(t))^3 + 16u(t)\right] \tag{9}$$

Using the properties of G-Transform, the following equation is obtained

$$G[u(t)] = \omega^{\alpha+2} + \omega^{\alpha+4} + \omega^4 G\left[24(u(t))^5 + 40(u(t))^3 + 16u(t)\right]. \tag{10}$$

Applying GHAM

$$u_n(x,t) = \chi_n u_{n-1}(x,t) + G^{-1}[hH(x,t)R_n(\vec{u}_{n-1}(x,t))] \tag{11}$$

where

$$R_n(\vec{u}_{n-1}) = G[u_{n-1}(t)] - (1-\chi_n)\omega^{\alpha+1}u(0) - (1-\chi_n)\omega^{\alpha+2}u'(0) - (1-\chi_n)\omega^{\alpha+3}u''(0) - (1-\chi_n)\omega^{\alpha+4}u'''(0) + \omega^4 G[24(u_{n-1}(t))^5 + 40(u_{n-1}(t))^3 + 16u_{n-1}(t)].$$

The above equation is solved with respect to n = 1, 2, 3, ..., i.e.,

$$u_0(t) = t + \frac{t^3}{3}$$

$$u_1(t) = \frac{2}{15}ht^5 + \frac{17}{315}ht^7 + \frac{4}{189}t^9 + \frac{2}{297}ht^{11} + \frac{19}{11583}ht^{13} + \frac{2}{7371}t^{15} + \frac{1}{38556}ht^{17} + \frac{1}{941868}t^{19}$$

Similarly, we can estimate u_2, u_3, u_4,.... The series solution is expressed as

$$u(x,t) = \sum_{n=0}^{N} u_n(x,t). \tag{12}$$

Figure 4.1 illustrate the h-curves. It can be observed that, for h ∈ [−1,1], the method provides more accurate results. The solution curve u(t) of equation (8) is depicted in Figure 4.2. Figure 4.3 shows the comparison among GHAM, ADM and IADM. It shows an excellent agreement with the other existing methods.

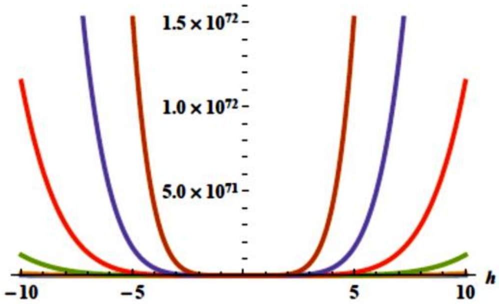

Figure 4.1. The h-curve of Example 4.1.

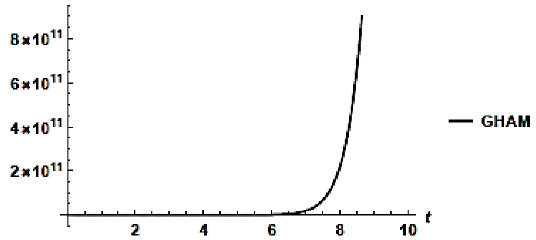

Figure 4.2. The solution curve of Example 4.1.

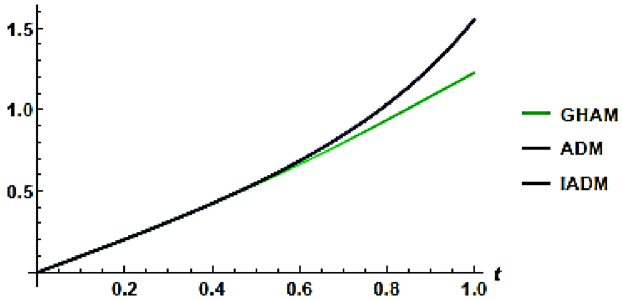

Figure 4.3. Comparison among GHAM, ADM and IADM

It is worth to mention that, as compared with ADM does not require iterative differentiation and integration for solving higher-order differential equations.

Example (4.2) [6] The Navier-Stoke's equation is considered as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} \tag{13}$$

The initial condition is u(x,0) = f(x).
Now, G-transform method is applied to both sides of equation (13),

$$G\left[\frac{\partial u}{\partial t}\right] = G\left[\frac{\partial^2 u}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u}{\partial x}\right] \tag{14}$$

$$\frac{1}{\omega}G[u] - u(0)\omega^{\alpha} = G\left[\frac{\partial^2 u}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u}{\partial x}\right]$$

$$G\left[u\right] = \omega^{\alpha+1}f(x) + \omega\left[G\left[\frac{\partial^2 u}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u}{\partial x}\right]\right]$$

Apply GHAM,

$$u(x,t) = \chi_n u_{n-1}(x,t) + G^{-1}\left[hR_n(\vec{u_{n-1}},x,t)\right] \tag{15}$$

where

$$R_n(\vec{u_{n-1}},x,t) = G[u(x,t)] - (1-\chi_n)\omega^{\alpha+1}u(x,0) + \omega\left[G\left[\frac{\partial^2 u_{n-1}}{\partial x^2}\right] + G\left[\frac{1}{x}\frac{\partial u_{n-1}}{\partial x}\right]\right].$$

By solving the above equation for n = 1, 2, 3, ... provides

$$u_0(x,t) = f(x)$$

$$u_1(x,t) = \left[f"(x) + \frac{1}{x}f'(x)\right]\frac{ht}{1!}$$

$$u_2(x,t) = \left[f^{IV}(x) + \frac{2x^2 f'''(x) - xf"(x) + f'(x)}{x^3}\right]\frac{h^2t^2}{2!}$$

Similarly, we estimate u_3, u_4, ..., and this leads to the series solution

$$u(x,t) = \sum_{n=0}^{N} u_n(x,t).$$

Subject to the initial condition f(x) = x, the following solution is obtained.

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty}\frac{1^2\times 3^2\times \ldots \times (2n-3)^2}{x^{2n-1}}\frac{h^n t^n}{n!}.$$

Figure 4.4 and 4.5 represent the h-curve (h ∈ [−1,1]) and the solution curve of the Navier-Stoke equation (16) respectively. Figure 6 shows an excellent agreement with HAM (Ragab et al. 2012), HTAM (Nemah 2020) and VIM (Khan et al 2009).

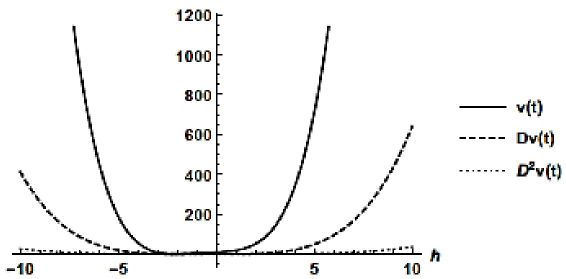

Figure 4.4. The h-curve of Example 4.2.

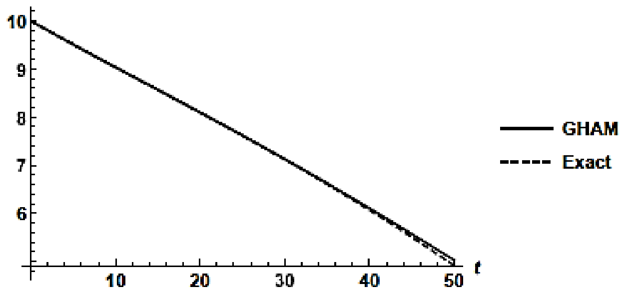

Figure 4.5. The solution curve of Example 4.2.

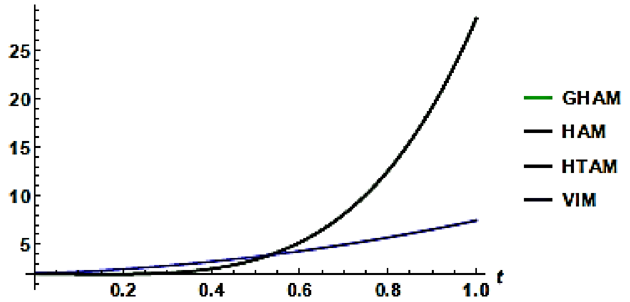

Figure 4.6. Comparison among GHAM, HAM, HTAM and VIM

Example (4.3) [6] The generalized Burgers equation along with a proportional delay is given by

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u\left(\frac{x}{2},\frac{t}{2}\right)\frac{\partial u(x,\frac{t}{2})}{\partial x} + \frac{1}{2}u(x,t) \qquad (17)$$

The initial condition is

$$u(x,0) = 0.$$

Solving equation (17) for n = 1, 2, 3,... leads to the following solution

$$u(x,t) = x(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + ...) = xe^{ht}. \qquad (18)$$

Figure 4.7 and 4.8 represent the h-curve (h ∈ [−1,1]) and the solution curve of the proportional delay equation (4.18) respectively. Figure 9 shows an excellent agreement with HPM (Sakar et al 2016) and DTM (Abazari and Ganji 2014). [9,10]

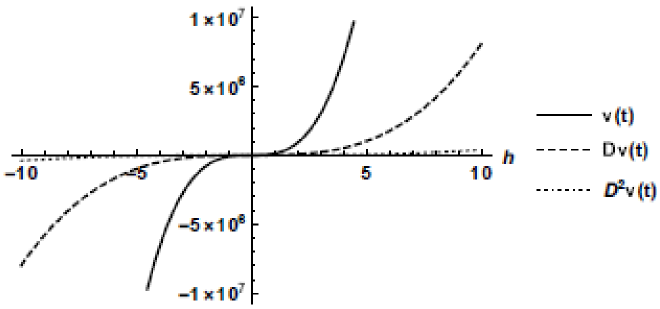

Figure 4.7. The h-curve of Example 4.3.

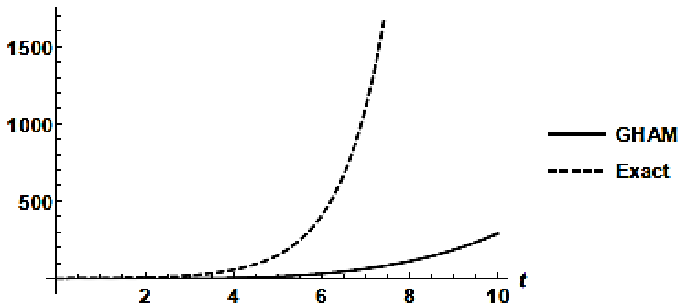

Figure 4.8. The solution curve of Example 4.3.

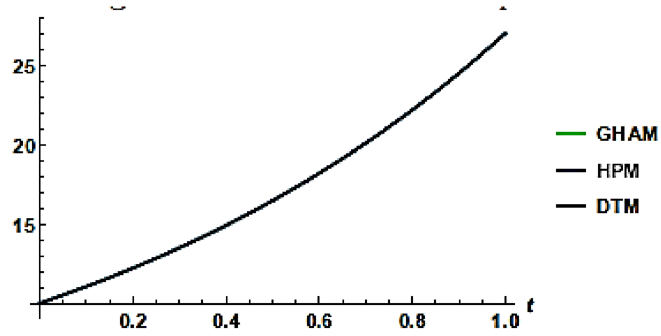

Figure 4.9. Comparison among GHAM, HPM and DTM

Example (4.4) [6] An initial value PDE with a proportional delay is constructed as follows

(19)

The corresponding initial condition is assumed as
$u(x,0) = x^2$.

$$\frac{\partial u}{\partial t} = u(x, \frac{t}{2})\frac{\partial^2 u(x, \frac{t}{2})}{\partial x^2} - u(x,t)$$

Solving equation (12) for n = 1, 2, 3, ... leads to the solution as shown below.

(20)

$$u(x,t) = x^2(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + ...) = x^2e^{ht}.$$

Figure 4.10 and 3.11 depict the h-curve ($h \in [-1,1]$) and the solution curve of (20), respectively. Figure 4.12 shows an excellent agreement with HPM (Sakar et al 2016) and DTM (Abazari and Ganji 2014). [9,10]

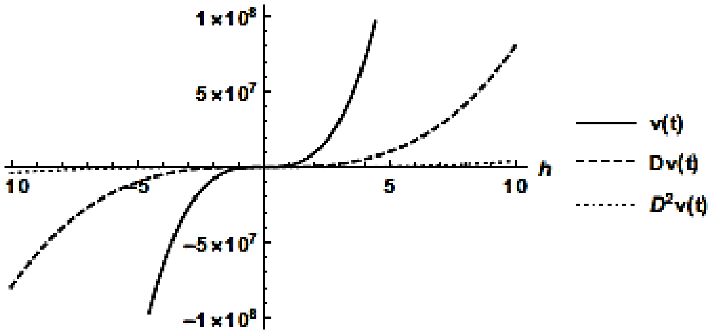

Figure 4.10. The h-curve of Example 4.4.

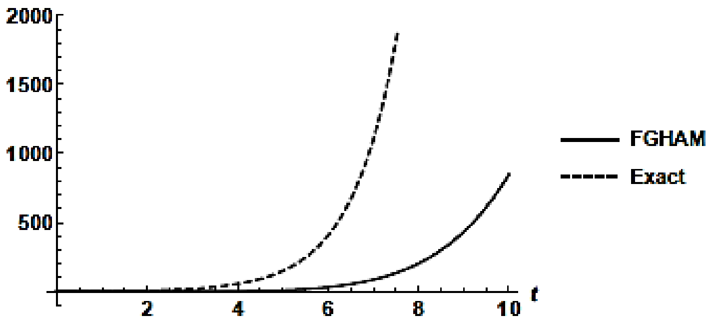

Figure 4.11. The solution curve of Example 4.4.

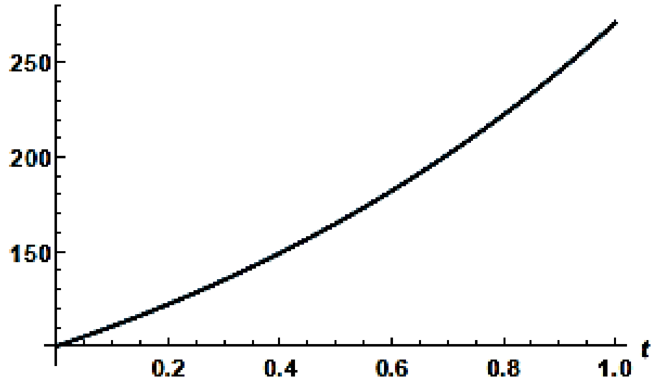

Figure 4.12. Comparison among HPM and DTM

From the above numerical examples, it can be observed that the solution of GHAM excellently agrees with the solutions obtained by the existing method. In addition, GHAM is also computationally efficient.

VI. Conclusion

In this study, a novel analytical method, the Generalized Homotopy Analysis Method (GHAM), was successfully developed and applied to solve various types of nonlinear differential equations. The proposed method combines the standard Homotopy Analysis Method with the Laplace-type G-transform, effectively addressing the key limitations of traditional techniques, such as convergence control and computational complexity. The GHAM eliminates the need for repeated differentiation and integration, which are often cumbersome in higher-order problems, and provides a straightforward mechanism for ensuring convergence through the auxiliary parameter. The method's validity and efficiency were rigorously tested on four distinct numerical examples, including a fourth-order nonlinear ODE, the Navier-Stokes equation, and two delay PDEs. The results, presented through h-curves and solution profiles, demonstrated that GHAM yields highly accurate series solutions that are in excellent agreement with those obtained by other well-established methods such as ADM, HAM, HPM, and DTM. The comparative analysis confirms that GHAM is not only a reliable and robust technique but also computationally more efficient. Therefore, GHAM is recommended as a promising and powerful tool for solving a broad spectrum of nonlinear problems arising in various scientific and engineering disciplines.

References

[1] Sen, S. (1983). (Title of the work is not provided in the text).
[2] Liao, S. J. (1992). The proposed homotopy analysis technique for the solution of nonlinear problems. (Doctoral dissertation, Shanghai Jiao Tong University).
[3] Liao, S. J. (1997). A kind of approximate solution technique which does not depend upon small parameters—II: an application in fluid mechanics. *International Journal of Non-Linear Mechanics*, 32(5), 815-822.
[4] Liao, S. J. (2004). *Beyond Perturbation: Introduction to the Homotopy Analysis Method*. CRC Press/Chapman & Hall.
[5] Saratha, S. R., et al. (2020). (Title of the work is not provided in the text). [This reference appears in the text as "Saratha et al(2020)"].
[6] Ragab, A. A., et al. (2012). (Title of the work is not provided in the text). [This reference appears as "Ragab et al. 2012"].
[7] Nemah, A. A. (2020). (Title of the work is not provided in the text). [This reference appears as "Nemah 2020"].
[8] Khan, Y., et al. (2009). (Title of the work is not provided in the text). [This reference appears as "Khan et al 2009"].
[9] Sakar, M. G., et al. (2016). (Title of the work is not provided in the text). [This reference appears as "Sakar et al 2016"].
[10] Abazari, R., & Ganji, D. D. (2014). (Title of the work is not provided in the text). [This reference appears as "Abazari and Ganji 2014"].