Quest Journals
Journal of Research in Applied Mathematics
Volume 6 ~ Issue 5 (2020) pp: 01-04
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

On Degree of Approximation of the Functions by Product of Summability Means of Lipschitz Class

Sanjay Mukherjee

Research Scholor, MATS University, Raipur(C.G.), India.
sanjaymukherjeeruma@gmail.com

A J Khan
MATS University, Raipur(C.G.), India

ABSTRACT- *In this paper the author has obtained the degree of approximation in the* $Lip\alpha$ *by* $(C,1)(S,\alpha_n)$ *means*

Mathematical Subject Classification : *MSC: 41A 30 , 42A24, 42B05, 42B08*
Keywords : *Lipschiz class, Degree of approximation , Fourier series, Product of summability*

Received 28November, 2020; Accepted 14 December, 2020 © The author(s) 2020.
Published with open access at www.questjournals.org

I. INTRODUCTION

Meyer-Konig introduces so called S_α method of summability which is one of the family of transformation including the Euler, Borel and Taylor (circle method) methods. Later Jakimovski introduced $[F,d_n]$ transformation which methods the Euler method (E,q) Karmata method (K^λ) and Lotosky method as particular cases.

For the first time Meir and Sharma introduced generalization of the S_α method and called it $[S,\alpha_n]$ method. They obtained sufficient condition for the regularity of this method. They also examined the behaviour of its Lebesgue constant.

Let $\{a_j\}$ be a given sequence of real complex numbers. We shall say that $\{a_j\}$ f is the $[S,a_n]$ transformations of $\{S_j\}$; i.e. the sequence of partial sums of the series $\sum a_n$ if

$$\{\sigma_n\}=\sum_{k=0}^{\infty} C_{nk} S_k; (n=0,1,2,3,..........)$$

Converges, where (C_{nk}) is given by the identity

$$\prod \frac{1-a}{1-a_j\theta}=\sum_{k=0}^{\infty} C_{nk}\theta^k$$

The sequence $\{S_j\}$ is said to be $[S,a_n]$ summable to σ if

$$\lim_{n\to\infty} \sigma=\sigma$$

Let f(x) ε L (0, 2π) and be periodic with period 2π outside this range. Let the Fourier series associated with the function be

$$\frac{a_0}{2}+\sum_{n=0}^{\infty}(a_n \cos nx+b_n \sin nx)=\sum_{n=0}^{\infty}A_n(x)$$

and as usual we denote

$$\phi(t)=\phi_x(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}$$

Also

$$V_n=1+2\sum_{j=0}^{n}\frac{a_j}{1-a_j}$$

$$T=2\sum_{j=0}^{n}\frac{a_j}{(1-a_j)^2}$$

$m=\lfloor T_n \rfloor$, the integral part of T_n

and

$$a_n=\frac{2\pi}{m}$$

Meir and Sharma [5] while studying constant established that when V_n and T_n are bounded the $[S, a_n]$ method sums only convergent Fourier series and so here after we assume $T_n \to \infty$ and $V_n \to \infty$ with n.

A function $f \in Lip\alpha$ if

$$|f(x+t)-f(x)|=O(|t|^{\alpha}) \text{ for } 0<\alpha\leq 1$$

and

$f(x) \in Lip(\alpha,p)$, for $0\leq x\leq 2\pi$, if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p dx\right)^{1/p}=O(|t|^{\alpha}), \text{ for } 0<\alpha\leq 1,\ p\geq 1$$

A function $f \in Lip(\phi(t), p)$ class for $p\geq 1$ if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p dx\right)^{1/p}=O(\phi(t)),$$

Where $\phi(t)$ is positive increasing function and $f \in Lip(\phi(t), p)$ if

$$\left(\int_0^{2\pi}\left|[f(x+t)-f(x)]\sin^{\beta}x\right|^p dx\right)^{1/p}=O(\phi(t)), \ (\beta\geq 0)$$

We observe that

$Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\phi(t), p)$, for $0<\alpha\leq 1$, $p\geq 1$

To prove the theorem we need following auxiliary result:

Lemma 1: The following estimates hold:

If

$$K_n(t)=e^{it}\sum_{j=0}^{n}\frac{1-a_j}{1-a_j e^{it}}$$

$$|K_n(t)|=O\left(\frac{1}{t\sqrt{T_n}}\right)$$

and

$$K_n(t)=\exp[V_n it-T_n t^2]+O(T_n t^3) \text{ for t to be very small.}$$

**Corresponding Author: A J Khan*

These are due to Meir and Sharma

Lemma 2: If h(x, t) is a function of two variables defined for $0 \le t \le 2\pi$, then

$$\left\| \int h(x,t)dt \right\| p \le \left\| \int h(x,t) \right\| pdt \; ; \; (p>1)$$

This is due to Hardy, Littlewood and Poly[3].

The series $\sum_{k=0}^{\infty} A_k$ is said to be $(C,1)$summable to s if

$$(C,1) = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \; as \; n \to \infty$$

Then $(C,1)$ transform of the $[S,\alpha_n]$ transform defines to $(C,1)(S,\alpha_n)$ transform of s_n of the series $\sum_{k=0}^{\infty} A_k$

Thus if $(CS)_n^{\alpha} = \frac{1}{n+1}\sum_{k=0}^{n} S_n^{\alpha} \to s \; as \; s \to \infty$

Where S_n^{α} denotes (S,α_n), then the series $\sum_{k=0}^{\infty} A_k$ is said to be summable to $(C,1)(S,\alpha_n)$means.

In the present chapter we have extended the above result to obtained the degree of approximation in the Lipschiz class by $(C,1)(S,\alpha_n)$. The theorem is as follows:

Theorem- If $f: R \to R$ is 2πperiodic, Lebesgue integrable on $[-\pi,\pi]$ and belonging to Lipschiz class then the degree of approximation of f by the $(C,1)(S,\alpha_n)$ product means of Fourier series satisfies for n=0,1,2,…

$|(CS)_n^{\alpha} - f(x)| = o(1)$

Proof of theorem : The $[s,a_n]$ transform of partial sums of Fourier series is given by

$$\sigma_n - f(x) = \frac{2}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{t} \operatorname{Im}\left\{ \sum_{k=0}^{n} C_{nk} \sin\left(k+\frac{1}{2}\right)t \right\} dt$$

$$= f(x) + \frac{2}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{t} \operatorname{Im}\left\{ \exp\left(\frac{it}{2}\right) \sum_{k=0}^{n} C_{nk} \exp(ikt) \right\} dt$$

Therefore $(C,1)(S,\alpha_n)$ means of the series are

$$(CS)_n^{\alpha} = \frac{1}{n+1}\sum_{k=0}^{n} S_n^{\alpha} \quad \text{(n= 0,1,2,3,...)}$$

$$= f(x) + \frac{1}{\pi(n+1)}\int_0^{\pi} \frac{\emptyset_x(t)}{sin\frac{t}{2}} Im\left\{ \sum_{k=0}^{\infty} C_{nk}\, sin\left(k+\frac{1}{2}\right)t \right\} dt$$

$$(CS)_n^{\alpha} = f(x) + \frac{1}{n+1}\int_0^{\pi} \frac{\emptyset_x(t)}{sin\frac{t}{2}} \left\{ \sum_{k=0}^{n} 2k+1 \right\} dt$$

$$(CS)_n^{\alpha} - f(x) = \frac{1}{n+1} o(n+1)$$

$$|(CS)_n^{\alpha} - f(x)| = 0(1)$$

This completes the theorem.

II. CONCLUSION

If $f: R \to R$ is 2πperiodic, Lebesgue integrable on $[-\pi,\pi]$ and belonging to Lipschiz class then the degree of approximation of f by the $(C,1)(S,\alpha_n)$ product means of Fourier series satisfies for n=0,1,2,…

$|(CS)_n^{\alpha} - f(x)| = o(1)$

ACKNOWLEDGEMENT

All great works done requires either active or passive support of some person, surrounding and one's close quarters. Thus is it not hard to conclude how active assistance from senior could positively impact the execution of the paper. I am highly thankful to our faculty head Dr. A. J. khan Sir for his active guidance throughout the completion of the paper. I am thankful to my better half Ruma Mukherjee and my son Saptarshi Mukherjee for giving me the encouragement and mental support for my work. I also thanks my father Salil Kumar Mukherjee , my mother in law Bina Pani Majumder for his whole hearted support. I also thank to my late mother Uma Mukherjee and father in law Priyalal Majumder who supports me by their blessings.

Last but not the least, I would also want to extend my appreciation to those who could not be mentioned here but have well played their role in inspire me behind the certain.

REFERENCES:

[1]. Agnew,R.P., E uler transformations,Amer.J.Math,66(1944)
[2]. Alexits,G.,Uber die Annaherung einer metigen durch die ceasaroschen Mittel ihrer fourier-reihe.Math.Anual.100.264-277(1928).
[3]. Bernstein,S.N,Sur-Iordre de la meilleure approximation des functions continues perdes polynomes de degree donne,Memories Acad., Royale Belgiyve,(2),4,1-103(1912).
[4]. Chandra, P.,Degree of approximation of function in Holder metric,Journal of the Indian Math Soc. 53,99-114(1988).
[5]. Holland,A.S.B., Sahney,B.N. and J. Tzimbalario, On degree of approximation of class of functions by means of fourier series,Acta Sci. Math,38,69-72(1976).
[6]. Jukimovski,A., Ageneralization of the lotosky method of summability ,Michigan Math,J.,6,271-290(1959).
[7]. Mohapatra,R.N. and P.Chandra,Holder continous functions and their Eular,Borel and Taylor means,Math,Chronicle,11,81-96(1982).
[8]. Prossdorf,S., Zur Knoverginz der Fourierrihen Holdersteliger funktionen,Math.Nachr.,69,506-526(1965).
[9]. Smith,G.,on the (F, d_n) method of summability,Canand,Math,17,506-526(1965).
[10]. Vuckoric,v., The Mutual inclusion of Kammata – Strikling method of summation,Michigan,Math,J.,6. 291-291(1959).
[11]. Rathore,C.J. and Srivastava, U.K.,On degree of approximation ,Acta Ciencia Indica,Vol. XXX M,No.2,375(2004).
[12]. Chui, C.K. and Holland, A.S.B., On the order of approximation by Euler and Taylor means, Journal of Approximation Theory, 39,24-38(1983).
[13]. Hardy, G.H.,Divergent Series, Oxford Uni. Press, (Clarendon) Oxford.
[14]. Hardy, G.H., Littleword, J.E. and Polya, G. Inequalities, Cambridge University Press, London/New York,(1967).
[15]. Jakimovski, A.,A Generalisation of Lototsky method of Summability, Michigan Math. J.6,277-290(1959).
[16]. Meir,A.and Sharma,A.,A generalization of $S_\propto$Summation method, proc. Camb. Phil.Soc.61-67(1970).
[17]. MEYER-Konig, Untersuchungen Uber einie Verwandte,Limitierungs verfahren Math.
[18]. Zygmund,A., Trigonometric series, Vol.I & II, Cambridge University Press London/New York(1959)

A J Khan. "On Degree of Approximation of the Functions by Product of Summability Means of Lipschitz Class." *Quest Journals Journal of Research in Applied Mathematics* , vol. 06, no. 05, 2020, pp. 01-04.