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Research Paper

Positive Periodic Solutions for a Nonlinear Delay Difference Equation

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ABSTRACT: *This paper is concerned with a nonlinear delay difference equation. By using the Schauder's fixed point theorem, sufficient conditions for the existence of positive periodic solutions of the equation are obtained. An example is given to illustrate the main results.*
KEYWORDS: *Positive periodic solution; Difference equation; Delay.*

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I. INTRODUCTION

Since the behavior of discrete systems is sometimes sharply different from the behavior of the corresponding continuous systems and discrete analogs of continuous problems may yield interesting dynamical systems in their own right, many scholars have investigated difference equations independently; see, for example, [1-13].

In [10], Raffoul considered the equation

$$\Delta x(n) + a(n)x(n-\tau(n)) = 0.$$

In [11], Yankson considered the equation

$$\Delta x(n) + \sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) = 0.$$

Jin and Luo in [12] and Chen in [13] considered the equation

$$\Delta x(n) + a(n)f(x(n-\tau(n))) = 0.$$

In these works, the authors only studied the stability of zero solution of the equations. However, to the best of our knowledge, there are seldom results on the existence of positive periodic solutions of nonlinear delay difference equations.

Motivated by the above, in this paper, we present some existence results for the nonlinear delay difference equation

$$\Delta x(n) + \sum_{s=n-\tau}^{n-1} p(n,s)g(x(s)) = 0, n > T, \tag{1.1}$$

where $x : \square \to \square$, Δ denotes the forward difference operator, $\Delta x(n) = x(n+1) - x(n)$,

$p \in C(\square \times \square, \square)$, $g \in C((0,\infty),(0,\infty)), \tau, T \in \square$.

To reach our desired end we use Schauder's fixed point theorem to show the existence of positive periodic solutions of the equation (1.1).

Theorem 1.1 (Schauder's fixed point theorem [14]). Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S:\Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x \in \Omega$ such that $Sx = x$.

II. EXISTENCE OF PERIODIC SOLUTIONS

In this section we shall study the existence of positive ω-periodic solutions of (1.1).

Lemma 2.1 Suppose that there exists a positive continuous function $k(n,s), n-\tau \le s \le n$, such that

$$\sum_{u=n}^{n+\omega-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big) = 0, n > T. \tag{2.1}$$

Then the function

$$f(n) = \exp\Big(-\sum_{u=T}^{n-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big), n > T,$$

is ω -periodic.

Proof: For $n > T$, we obtain

$$\begin{aligned} f(n+\omega) &= \exp\Big(-\sum_{u=T}^{n+\omega-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big) \\ &= \exp\Big(-\sum_{u=T}^{n-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big) \times \exp\Big(-\sum_{u=n}^{n+\omega-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big) \\ &= f(n). \end{aligned}$$

Thus the function f is ω -periodic.

Theorem 2.1 Suppose that there exists a positive continuous function $k(n,s), n-\tau \le s \le n,$ such that (2.1) holds and

$$\begin{aligned} &\exp\Big(\sum_{u=T}^{n} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big) \\ &\times g\Big(\exp\Big(-\sum_{u=T}^{s-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big)\Big) = k(n,s), n > T. \end{aligned} \tag{2.2}$$

Then (1.1) has a positive ω -periodic solution.

Proof: Let $X = C([T-\tau,\infty),\mathbb{R})$ be the Banach space with the norm $\|x\| = \sup_{n \ge T-\tau} |x(n)|$. We set

$$f(n) = \exp\Big(-\sum_{u=T}^{n-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big), n > T.$$

With regard to Lemma 2.1 we have $m \le f(n) \le M$, where

$$\begin{aligned} m &= \inf_{n\in(T,\infty)} \Big\{\exp\Big(-\sum_{u=T}^{n-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big)\Big\}, \\ M &= \sup_{n\in(T,\infty)} \Big\{\exp\Big(-\sum_{u=T}^{n-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big)\Big\}. \end{aligned} \tag{2.3}$$

We now define a closed, bounded and convex subset Ω of X as follows

$$\Omega = \{x \in X : x(n+\omega) = x(n), n > T, m \le x(n) \le M, n > T,$$

$$k(n,s)x(n+1) = g(x(s)), n > T, n-\tau \le s \le n, x(n) = 1, T-\tau \le n \le T\}.$$

Define the operator $S:\Omega \to X$ as follows

$$(Sx)(n)=\begin{cases}\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big), n>T,\\ 1, T-\tau\le n\le T.\end{cases}$$

We shall show that for any $x\in\Omega$ we have $Sx\in\Omega$.

For every $x\in\Omega$ and $n>T$ we get

$$(Sx)(n)=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)$$

$$=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big)\le M,$$

and $(Sx)(n)\ge m$.

For $n\in[T-\tau,T]$ we have $(Sx)(n)=1$, that is $(Sx)(n)\in\Omega$.

Further for every $x\in\Omega$ and $n>T$, $n-\tau\le s\le n$, according to (2.2) it follows

$$g((Sx)(s))=g\Big(\exp\Big(-\sum_{u=T}^{s-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)$$

$$=\exp\Big(-\sum_{u=T}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)\times\exp\Big(\sum_{u=T}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)$$

$$\times g\Big(\exp\Big(-\sum_{u=T}^{s-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)$$

$$=\exp\Big(\sum_{u=T}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big)\times g\Big(\exp\Big(-\sum_{u=T}^{s-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big)\Big)(Sx)(n+1)$$

$$=k(n,s)(Sx)(n+1).$$

Finally we shall show that for $x\in\Omega, n>T$ the function Sx is ω -periodic. For $x\in\Omega$, $n>T$ and with regard to (2.1) we get

$$(Sx)(n+\omega)=\exp\Big(-\sum_{u=T}^{n+\omega-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)$$

$$=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)\times\exp\Big(-\sum_{u=n}^{n+\omega-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big)$$

$$=(Sx)(n)\exp\Big(-\sum_{u=T}^{n+\omega-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big)=(Sx)(n).$$

So Sx is ω -periodic on $[T,\infty)$. Thus we have proved that $Sx\in\omega$ for any $x\in\Omega$.

We now show that S is completely continuous. First we shall show that S is continuous. Let $x_i\in\Omega$ be such that $x_i\to x\in\Omega$ as $i\to\infty$. For $n>T$ we have

$$|(Sx_i)(n)-(Sx)(n)|$$

$$=\Big|\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x_i(v))}{x_i(u+1)}\Big)-\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\frac{g(x(v))}{x(u+1)}\Big)\Big|.$$

By applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{i\to\infty}\|Sx_i-Sx\|=0.$$

For $n\in[T-\tau,T]$, the relation above is also valid. This means that S is continuous.

We now show that $S\Omega$ is relatively compact. Similar to the proof of Lemma 2.2 in [15], we omit the rest of the proof.

By Theorem 1.1 there exists an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that x_0 is a positive ω-periodic solution of (1.1). The proof is complete.

Corollary 2.1 Suppose that there exists a positive continuous function $k(n,s), n-\tau \le s \le n$, such that (2.1) holds and

$$\exp\Big(\sum_{u=s}^{n} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big) = k(n,s), n > T. \qquad (2.4)$$

Then the equation

$$\Delta x(n) + \sum_{s=n-\tau}^{n-1} p(n,s)x(s) = 0, n > T, \qquad (2.5)$$

has a positive ω-periodic solution

$$x(n) = \exp\Big(-\sum_{u=T}^{n-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big), n > T.$$

III. AN EXAMPLE

In this section, we give an example to illustrate our main results.

Example 1. Let $p(n) = \sin \pi n, \tau \in \square, \tau > 0$, then $\omega = 2$. We choose

$$k(n,s) = g(1), n-\tau \le s \le n.$$

For the condition (2.1), we have

$$\sum_{u=n}^{n+\omega-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big) = 0, n > T.$$

For the condition (2.2), we have

$$\exp\Big(\sum_{u=T}^{n} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big) \times g\Big(\exp\Big(-\sum_{u=T}^{s-1} \ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big)\Big)$$

$$= g(1) = k(n,s), n > T.$$

Therefore, all conditions of Theorem 2.1 are satisfied, then equation (1.1) has a positive 2-periodic solution.

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