



An LMI approach to exponential stability analysis for neutral Delay differential systems

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ABSTRACT

In this Article we investigate the problem of the neutral delay differential systems, the problem of determining the exponential stability is studied. By constructing new Lyapunov-Krasovskii (LKF) functional with new integral terms sufficient condition derived on the Lyapunov technique, we present several useful criteria of exponential stability for the systems are derived. The stability measures are formulated in terms of linear matrix inequality (LMI's) which can be certainly explained by using the MATLAB LMI toolbox. Numerical example is addressed to show the effectiveness of the proposed result.

KEYWORDS: Neutral system, Exponential stability, Lyapunov method, linear matrix inequality (LMI's).

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I. INTRODUCTION

Many scholars are functioned in problem of stability analysis of neutral delay differential systems has briefly in many articles from the past few decades due to its wide application in various practical problems like Automatic control, bioengineering, neural networks, automatic control Mechanics, Population propagation, turbojet engine, microwave oscillator, nuclear reactor and also instability in many control systems problems. Stability as an important index of control systems receives considerable attention.

In [5] mentioned stability analysis has got more attention and To analyze stability by using Lyapunov method or formation of Linear matrix inequality method because of the following three factors

- i. A variety of design specifications and constraints can be expressed as LMIs
- ii. Once formulated in terms of LMIs, a problem can be solved exactly by efficient convex optimization algorithms.
- iii. While most problems with multiple constraints or objectives lack analytical solutions in terms of matrix equations, they often remain tractable in the LMI framework. This makes LMI-based design a valuable alternative to classical analytical methods.

In [15] Tao.wu.et.al studies the problem of robust stability for uncertain neutral systems with distributed delay. By utilizing the incorporation of a new integral inequality technique and a novel Lyapunov-Krasovskii functional, some reduced conservative delay-dependent stability conditions for asymptotic stability are established.

In [16] Leping Sun study about the asymptotic stability is concerned for a class of neutral delay differential-algebraic equations. We will present two criteria by evaluating a corresponding harmonic function on the boundary of a torus region. Stability regions are also presented so as to locate all possible unstable characteristic roots of Neutral delay differential-algebraic equations.

In [10] Yajuan Liu.et.al worked on the problem of delay-dependent exponential stability criteria for neutral delay system with nonlinear uncertainties. By constructing a new class of Lyapunov-Krasovskii functional and using the free-weighting matrices methods, In [11] by using LMI techniques studies the admissibility

problem for a class of linear singular systems with time-varying delays. In order to highlight the relations between the delay and the state, the singular system is transformed into a neutral form.

In [12] to solve neutral delay differential system Lyapunov functional are introduced for sampled-data control in the presence of a constant input delay based on the vector extension of Wirtinger's inequality to show the improvement of the result.

In [13] the problems of the delay-dependent stability analysis of neutral systems with mixed interval time-varying delays with/without nonlinear perturbations are work with the help of Lyapunov functional to construct LMIs. Not only limited to these authors in [1],[2],[6] to [10] many researchers are worked on this field by inspiring of all these here we continued work on the neutral delay differential system and we shows the improvement results, we use MATLAB LMI Toolbox to illustrate our examples.

Notations and preliminaries

Let \mathcal{R}^n denote the n-dimensional real space

$\mathcal{R}^{n \times n}$ Denote the set of all real n by n matrices.

$\lambda_M(A)$ and $\lambda_m(A)$ denote the maximal and minimal eigenvalue of A respectively.

$\|x\|$ Denotes the Euclid norm of the vector of x

$\|A\|$ Denotes the induced norm of the matrix A, i.e. $\|A\| = \sqrt{\lambda_M(A^T A)}$

$d = \dot{h}(t)$

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a Neutral delay-differential system of the form

$$\dot{x}(t) = Ax(t) + Bx(t - h(t)) + C\dot{x}(t - h(t)) \quad (1)$$

With the initial condition function

$$x(t_0 + \theta) = \phi(\theta), \text{ for all } \theta \in (-h, 0) \quad (2)$$

$x(t) \in \mathcal{R}^n$ is the state vector, A, B and $C \in \mathcal{R}^{n \times n}$ are constant matrices, $h(t)$ is positive time-varying differentiable bounded delays satisfying,

$$0 < h(t) \leq \bar{h} < \infty \quad \dot{h}(t) \leq 1 \quad (3)$$

$\bar{h} = \max h(t)$ and $\phi(\cdot)$ is the given continuously differentiable function on $(-h, 0)$, and the system matrix A is assumed to be a Hurwitz matrix.

The system given in (1) often appears in the theory of automatic control or population dynamics. First, we establish a delay-independent criterion, for the asymptotic stability of the delay-differential system (1) using Lyapunov method in terms of LMI.

Definition 1.1: [2] System (1) is said to be globally exponentially stable with a convergence rate α if there are two positive constants α and λ such that

$$\|x(t)\| \leq \lambda e^{-\alpha t} \quad t \geq 0 \quad (4)$$

Lemma 1.2. [3] For any constant matrix $M \in \mathcal{R}^{n \times n}$, $M = M^T > 0$, scalar $\eta > 0$, vector function $w: [0, \eta] \rightarrow \mathcal{R}^n$ such that the integrations concerned are well defined, then

$$\left[\int_0^\eta w(s) ds \right]^T M \left[\int_0^\eta w(s) ds \right] \leq \eta \int_0^\eta w^T(s) M w(s) ds \quad (5)$$

Lemma 1.3. [4] The following matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0 \quad (6)$$

Where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ and $S(x)$ depend affinely on x , is equivalent to $R(x) < 0$, $Q(x) < 0$ and $Q(x) - S(x)R^{-1}(x)S^T(x) < 0$ (7)

This Lemma also called as Schur-complement Lemma.

Lemma 1.4. [1] For any positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0 \quad (8)$$

The following integral inequality holds

$$-\int_{t-h}^t \dot{x}^T(s) X_{33} \dot{x}(s) \leq \int_{t-h}^t [x^T(t) \quad x^T(t-h) \quad \dot{x}^T(s)] \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(s) \end{bmatrix} ds \quad (9)$$

III. MAIN RESULT

Theorem 3. 1: System (1) is globally exponentially stable with the convergence rate of $\alpha > 0$ With given Scalar $h > 0$, if there exist positive-definite symmetric matrices $P, Q, R, S \in \mathcal{R}^{n \times n}$ are of appropriate dimensions and a

positive semi-definite Matrix $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0$ such that the following LMIs hold:

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} < 0$$

where

$$\begin{aligned} \phi_{11} &= A^T P + PA + 2\alpha P + Q + A^T AR + h^2 A^T AS + e^{-2\alpha h} (hX_{11} + X_{13} + X_{13}^T + X_{33} - 2S) \\ \phi_{12} &= PB + A^T RB + h^2 A^T BS + e^{-2\alpha h} (2S + hX_{12} - X_{23} - X_{23}^T - X_{33}) \\ \phi_{13} &= PC + A^T CR + h^2 A^T CS \\ \phi_{21} &= B^T P + ARB^T + h^2 B^T AS + e^{-2\alpha h} (2S + hX_{12}^T + X_{23} - X_{13}^T - X_{33}) \\ \phi_{22} &= B^T BR - (1-d)e^{-2\alpha h} Q + h^2 B^T BS + e^{-2\alpha h} (hX_{22} - X_{23} - X_{23}^T + X_{33} - 2S) \\ \phi_{23} &= B^T CR + h^2 SB^T C \\ \phi_{31} &= C^T P + C^T AR + h^2 C^T AS \\ \phi_{32} &= C^T BR + h^2 C^T BS \\ \phi_{33} &= C^T CR - (1-d)e^{-2\alpha h} R + h^2 C^T CS \end{aligned}$$

Proof: Consider the Delay differential system (1), using the Lyapunov- Krasovskii functional candidate in the following form, we can write

$$V(t, x(t)) = v_1(t, x(t)) + v_2(t, x(t)) + v_3(t, x(t)) + v_4(t, x(t)) \quad (10)$$

Where

$$v_1(t, x(t)) = e^{2\alpha t} x^T(t) P x(t) \quad (11)$$

$$v_2(t, x(t)) = \int_{-h(t)}^0 e^{2\alpha(t+s)} x^T(t+s) Q x(t+s) ds \quad (12)$$

$$v_3(t, x(t)) = \int_{-h(t)}^0 e^{2\alpha(t+s)} \dot{x}^T(t+s) R \dot{x}(t+s) ds \quad (13)$$

$$v_4(t, x(t)) = h \int_{-h}^0 \int_{t+\theta}^t e^{2\alpha(t+s)} \dot{x}^T(s) S \dot{x}(s) ds d\theta \quad (14)$$

Then the time derivative of $V(t, x(t))$ with respect to t along with the system is

$$\dot{V}(t, x(t)) = \dot{v}_1(t, x(t)) + \dot{v}_2(t, x(t)) + \dot{v}_3(t, x(t)) + \dot{v}_4(t, x(t)) \quad (15)$$

$$\begin{aligned} \dot{v}_1(t, x(t)) &= 2\alpha e^{2\alpha t} x^T(t) P x(t) + e^{2\alpha t} x^T(t) P \dot{x}(t) + e^{2\alpha t} \dot{x}^T(t) P x(t) \\ &= e^{2\alpha t} (x^T(t) (A^T P + PA + 2\alpha P) x(t) + x^T(t-h) (B^T P) x(t-h) + \dot{x}^T(t-h) (C^T P) x(t) \\ &\quad + x^T(t) P B x^T(t-h) + x^T(t) P C \dot{x}^T(t-h)) \end{aligned} \quad (16)$$

$$\dot{v}_2(t, x(t)) = e^{2\alpha t} (x^T(t) Q x(t) - (1-d)e^{-2\alpha h} x^T(t-h) Q x(t-h)) \quad (17)$$

$$\begin{aligned} \dot{v}_3(t, x(t)) &= e^{2\alpha t} (\dot{x}^T(t) R \dot{x}(t) - (1-d)e^{-2\alpha h} \dot{x}^T(t-h) R \dot{x}(t-h)) \\ &= e^{2\alpha t} (x^T(t) A^T AR x(t) + x^T(t) A^T BR x(t-h) + x^T(t) A^T CR \dot{x}(t-h) \\ &\quad + x^T(t-h) B^T BR x(t-h) + x^T(t-h) B^T AR x(t) + x^T(t-h) B^T CR \dot{x}(t-h) \\ &\quad + \dot{x}^T(t-h) C^T AR x(t) + \dot{x}^T(t-h) C^T BR x(t-h) + \dot{x}^T(t-h) C^T CR \dot{x}(t-h) \\ &\quad - (1-d)e^{-2\alpha h} \dot{x}^T(t-h) R \dot{x}(t-h)) \end{aligned} \quad (18)$$

$$\dot{v}_4(t, x(t)) = e^{2\alpha t} (\dot{x}^T(t) h^2 S \dot{x}(t) - h \int_{t-h}^t e^{2\alpha(s-t)} \dot{x}^T(s) S \dot{x}(s) ds) \quad (19)$$

$$\begin{aligned} &= e^{2\alpha t} ((Ax(t) + Bx(t-h(t)) + C\dot{x}(t-h(t)))^T h^2 S (Ax(t) + Bx(t-h(t)) + C\dot{x}(t-h(t))) \\ &- h \int_{t-h}^t e^{2\alpha(s-t)} \dot{x}^T(s) S \dot{x}(s) ds) \end{aligned}$$

Obviously $s \in [t - h, t]$ we have $e^{-2ah} \leq e^{2a(s-t)}$

And

$$-h \int_{t-h}^t e^{2a(s-t)} \dot{x}^T(s) S \dot{x}(s) ds \leq -he^{-2ah} \int_{t-h}^t \dot{x}^T(s) (S - X_{33}) \dot{x}(s) ds - he^{-2ah} \int_{t-h}^t \dot{x}^T(s) X_{33} \dot{x}(s) ds$$

Using Lemma 1.2

$$\begin{aligned} & -he^{-2ah} \int_{t-h}^t \dot{x}^T(s) (S - X_{33}) \dot{x}(s) ds - he^{-2ah} \int_{t-h}^t \dot{x}^T(s) X_{33} \dot{x}(s) ds \leq \\ & -e^{-2ah} \left(\int_{t-h}^t \dot{x}(s) ds \right)^T (S - X_{33}) \left(\int_{t-h}^t \dot{x}(s) ds \right) - he^{-2ah} \int_{t-h}^t \dot{x}^T(s) X_{33} \dot{x}(s) ds \\ & \leq -e^{-2ah} [x(t) - x(t-h)]^T (S - X_{33}) [x(t) - x(t-h)] - he^{-2ah} \int_{t-h}^t \dot{x}^T(s) X_{33} \dot{x}(s) ds \end{aligned}$$

Where

$$-he^{-2ah} \int_{t-h}^t \begin{bmatrix} x^T(t) & x^T(t-h) & \dot{x}^T(s) \end{bmatrix} X_{33} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(s) \end{bmatrix} ds$$

So

$$\begin{aligned} \dot{v}_4 = e^{2at} & (x^T(t) (e^{-2ah} (hX_{11} + X_{13} + X_{13}^T + X_{33} - 2S) + h^2 A^T A S) x(t) \\ & + x^T(t) (h^2 A^T B S + e^{-2ah} (2S + hX_{12} - X_{23} - X_{23}^T - X_{33})) x(t-h) + x^T(t-h) (h^2 B^T A S \\ & + e^{-2ah} (2S + hX_{12}^T + X_{23} - X_{23}^T - X_{33})) x(t) \\ & + x^T(t-h) (h^2 B^T B S + e^{-2ah} (hX_{22} - X_{23} - X_{23}^T + X_{33} - 2S)) x(t-h) \\ & + x^T(t) h^2 A^T C S \dot{x}(t-h) + x^T(t-h) (h^2 B^T C S) \dot{x}(t-h) \\ & + \dot{x}^T(t-h) h^2 C^T A S x(t) + \dot{x}^T(t-h) h^2 C^T B S x(t-h) + \dot{x}^T(t-h) h^2 C^T C S \dot{x}(t-h) \end{aligned} \quad (20)$$

Then the complete derivative

$$\dot{V}(t, x(t)) = \dot{v}_1(t, x(t)) + \dot{v}_2(t, x(t)) + \dot{v}_3(t, x(t)) + \dot{v}_4(t, x(t)) \quad (21)$$

Becomes

$$\begin{aligned} \dot{V}(t, x(t)) = e^{2at} & (x^T(t) (A^T P + PA + 2\alpha P + Q + A^T A R + h^2 A^T A S + e^{-2ah} (hX_{11} + X_{13} + X_{13}^T + X_{33} - \\ & 2S) x t + x^T t P B + A^T R B + h^2 A^T B S + e^{-2ah} (2S + hX_{12} - X_{23} - X_{23}^T - X_{33}) x t - h + x^T t P C + A^T R C + h^2 A^T C S x t - \\ & h + x^T t - h (B^T P + A R B^T + h^2 B^T A S + e^{-2ah} (2S + hX_{12}^T + X_{23} - X_{23}^T - X_{33})) x t + x^T t - h B^T B R - 1 - de - 2ah Q \\ & + h^2 B^T B S + e^{-2ah} (hX_{22} - X_{23} - X_{23}^T + X_{33} - 2S) x t - h + x^T t - h (B^T R C + h^2 B^T C S) x t - h + x^T t - h (C^T P + C^T \\ & A R + h^2 C^T A S) x t + x^T t - h (C^T R B + h^2 C^T B S) x t - h + x^T t - h (C^T C R - 1 - de - 2ah R + h^2 C^T C S) x t - h) \end{aligned} \quad (22)$$

Furthermore, using the Schur – complement Lemma, Re-writing in terms of LMI's of the form

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} < 0 \quad (23)$$

Where

$$\begin{aligned} \phi_{11} & = A^T P + PA + 2\alpha P + Q + A^T A R + h^2 A^T A S + e^{-2ah} (hX_{11} + X_{13} + X_{13}^T + X_{33} - 2S) \\ \phi_{12} & = PB + A^T R B + h^2 A^T B S + e^{-2ah} (2S + hX_{12} - X_{23} - X_{23}^T - X_{33}) \\ \phi_{13} & = PC + A^T C R + h^2 A^T C S \\ \phi_{21} & = B^T P + A R B^T + h^2 B^T A S + e^{-2ah} (2S + hX_{12}^T + X_{23} - X_{23}^T - X_{33}) \\ \phi_{22} & = B^T B R - (1 - d) e^{-2ah} Q + h^2 B^T B S + e^{-2ah} (hX_{22} - X_{23} - X_{23}^T + X_{33} - 2S) \\ \phi_{23} & = B^T C R + h^2 S B^T C \\ \phi_{31} & = C^T P + C^T A R + h^2 C^T A S \\ \phi_{32} & = C^T B R + h^2 C^T B S \\ \phi_{33} & = C^T C R - (1 - d) e^{-2ah} R + h^2 C^T C S \end{aligned}$$

By applying Lemma 1.3 in ϕ with some effort, we get $\phi < 0$. Therefore, by Lyapunov–Krasovskii stability theorem $\dot{V}(t, x(t)) < 0$ we conclude the following result

$$\lambda_m(P) e^{2at} \|x(t)\|^2 \leq V(t) \leq V(0) \quad (24)$$

Where

$$\begin{aligned} V(0) & = x^T(0) P x(0) + \int_{-h(0)}^0 e^{2as} x^T(s) Q x(s) ds + \int_{-h(0)}^0 e^{2as} \dot{x}^T(s) R \dot{x}(s) ds + h \int_{-h}^0 \int_{\theta}^0 e^{2as} \dot{x}^T(s) S \dot{x}(s) ds d\theta \\ & = \lambda |\mu|_h^2 \end{aligned}$$

And $\lambda = \lambda_M(P) + h\lambda_M(Q) + h\lambda_M(R) + h^3\lambda_M(S)$

Hence

$$\|x(t)\| \leq \sqrt{\frac{(\lambda_M(P) + h\lambda_M(Q) + h\lambda_M(R) + h^3\lambda_M(S))}{\lambda_m(P)}} \cdot |\mu|_h \cdot e^{-\alpha t}, t \geq 0 \quad (25)$$

This implies that the system (1) is globally exponentially stable with convergence rate α

IV. NUMERICAL EXAMPLE:

To illustrate the improvement of our proposed method, we present the following simulation example.

Consider the following system

$$\dot{x}(t) = A x(t) + B x(t - h) + C \dot{x}(t - h) \quad (26)$$

With

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.2 & 0.5 \\ -0.3 & -0.4 \end{bmatrix}, C = \begin{bmatrix} 0.3 & 0.1 \\ -0.2 & 0.1 \end{bmatrix}$$

For $d = 0.5$ and $h = 0.1$ applying Theorem 3.1 the following feasible solution is obtained by using MATLAB LMI toolbox

$$\alpha = 2.6628e + 008,$$

$$P = 1.0e + 005 * \begin{bmatrix} 0.0402 & -0.4462 \\ -0.4462 & 4.9485 \end{bmatrix}, Q = 1.0e + 008 * \begin{bmatrix} 2.8739 & -0.0455 \\ -0.0455 & 2.8100 \end{bmatrix}$$

$$R = 1.0e + 006 * \begin{bmatrix} 0.0809 & -0.8969 \\ -0.8969 & 9.9467 \end{bmatrix}, S = 1.0e + 008 * \begin{bmatrix} 0.0098 & -0.1090 \\ -0.1090 & 1.2085 \end{bmatrix}$$

Then we conclude that system (1) is globally exponentially stable, these Type of Results are helpful in analyzing the stability in control system, Robust stability system, Delay differential equation of Electrical and Electronics related system of equations and also useful in Bio engineering stability problems.

V. Conclusion:

In this Article we studied about the sufficient conditions exponential stability of the Neutral delay differential systems. The derived conditions are expressed in terms of LMI using schur-complement lemma and Numerical examples are given to demonstrate of our result with help of MATLAB LMI tool box.

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