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Research Paper

Linear Difference Equation Applications on Time Series Models

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ABSTRACT
This study was conducted to apply homogeneous linear difference equations on time series models. In this study, information on homogeneous linear difference equation was given. The time series methods addressed in studies in the field of agriculture were expressed openly with the back shift operators. These models were examined with the solution methods of homogeneous linear difference equations in different conditions. Generally, the homogeneous difference equation of the ARIMA (0,1,1) models were solved.
KEYWORDS: *Linear difference equations, time series, homogeneous.*

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I. INTRODUCTION

Difference equations are used in many applied sciences. They are commonly used in biology, chemistry, physics, engineering sciences, and mathematics [1]. Solutions are made with difference equations in autoregressive time series, moving average time series and autoregressive moving average time series [2].

The simplest expression of difference equations was seen in 2000 BC. Finding the root of an equation using difference equations was first used by the Babylonians [3]. Fibonacci, Nasir Al-Tusi, Yang Hui, Al-Banna, Al-Farisi and Shih-Chieh made significant contributions to the difference equations and sequence recurrence relations between 1200 and 1600 [4].

Difference equation theory is quite similar to the theory of differential equations. The analysis of difference equations is a newer concept than the differential equations. The developments in different branches of science such as genetic and quantum in radiation in the 20th century showed the need of all natural events for statements other than continuity statements. The states of discontinuity encountered in differential equations are intended to be eliminated with difference equations [5].

Difference equations is the algebraic relation between the finite differences and independent variables of a function with one or more variables. Difference equations, which are similar to differential equations and are new in the analysis process, are also called functional equations [6].

The difference equation is the discrete similar to the differential equation [7]. The difference equation theory has a more substantial content than its equivalent differential equation theory [8].

The aim of this study was to solve time series model using linear difference equations.

II. METHODOLOGY

Time Series Models

A general nth order linear difference equation with constant coefficients is given by

$$C_0X_t + C_1X_{t-1} + C_2X_{t-2} + \cdots + C_nX_{t-n} = e_t$$

where C_i, i=0,1,2,…,n are constants. This equation is said to be nonhomogeneous if $e_t \neq 0$. Back shift operator and letting

$$C(B) = (1 + C_1B + C_2B^2 + \cdots + C_nB^n$$

it can be written follow as [9].

$$C(B)X_t = e_t$$

Time series consists of autoregressive (AR), moving average (MA) and autoregressive moving average (ARMA) models.

A pth-order autoregressive model AR(p) model is denoted as [10].

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + e_t$$

Closed form of AR model

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)X_t = e_t$$

in the form [11].
AR(p) model uses a linear combination of past values of the target to make forecasts.
A qth-order moving average process, expressed MA(q), is characterized by [12].

$$X_t = -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} + e_t$$

Closed form of MA model

$$X_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_p B^q)e_t$$

in the form [11].
ARMA(p,q) model composed of a pth-order autoregressive and qth-order moving average process and it is characterized by [13].

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

Linear Difference Equations

Linear difference equations play an important role in time series. They are especially more important in autoregressive time series. Consider that $\{X_t : t\epsilon T\}$ is a time series. B lag (backshift) operator is defined as $B^k X_t = X_{t-k}$. B lag operator transforms the value of X_t random variable at any t moment to t-1 value. ∇ difference operator is defined with B support (∇$= 1 - $B). The first order difference series of X_t series is calculated as follows to show ∇ difference operator:

$$\nabla X_t = (1 - B)X_t = X_t - X_{t-1}$$

Similarly, the second-order difference series is as follows [14]:

$$\nabla^2 X_t = (1 - B)^2 X_t = X_t - 2X_{t-1} - X_{t-2}$$

The following equations can be written (Akdi, 2010):

$$(1 - B)^{-1} = \frac{1}{1 - B} = 1 + B + B^2 + B^3 + \cdots$$

$$(1 - mB)^{-1} = 1 + mB + m^2 B^2 + m^3 B^3 + \cdots$$

These statements can be expanded with the Taylor series expansion and the following calculations can be made:

$$(1 + B)^{-1} = \frac{1}{1 + B} = 1 - B + B^2 - B^3 + \cdots$$

$$(1 - B)^{-2} = 1 + 2B + 3B^2 + 4B^3 + \cdots$$

Linear homogeneous difference equation

The linear homogeneous difference equation at the p degree can be written as follows:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + e_t$$

or

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} = e_t$$

The solution of the linear difference equation must be found. If this equation is

$$C(B) = 1 + \phi_1 B + \phi_2 B^2 + \cdots + \phi_p B^p$$

then $C(B)X_t = e_t$ can be written. To solve the equation system, the $C(B)X_t = 0$ homogeneous equation system must first be solved [14]. The following Lemmas are givin for the solution of equation systems [9].

Lemma 1. If $X_t^{(1)}$ and $X_t^{(2)}$ are solutions of the homogeneous equation, then $b_1 X_t^{(1)} + b_2 X_t^{(2)}$ is also a solution for any arbitrary constants b_1 and b_2.

Lemma 2. If $X_t^{(H)}$ is a solution to the homogeneous equation and $X_t^{(P)}$ is a particular solution of the nonhomogeneous equation, then $X_t^{(H)} + X_t^{(P)}$ is the general solution of the complete equation.

Lemma 3. Let $(1 - B)^m X_t = 0$. Then a solution is given by $X_t = bt^j$ where b is any constant and j is a nonnegative integer less than m.

Lemma 4. Let $(1 - RB)^m X_t = 0$. Then a solution is given by $X_t = t^j R^t$, where j is any nonnegative integer less than *m*, the general solution is given by $X_t = \left(\sum_{j=0}^{m-1} b_j t^j\right) R^t$, where b_j is a constant.

When the second-order linear homogeneous difference equation is considered as follows;

$$C(B) = (1 - \phi_1 B - \phi_2 B^2) = 0$$

$(1 - \phi_1 B - \phi_2 B^2)$ statement can be written as

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + B$$

or

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = B$$

The last equation is also the AR(2) model [2]. In the following special condition where the equation equals to zero,

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = 0$$

And a transformation like $X_t = \lambda^t$ is made and put in place on the equation above,

$$\lambda^t - \phi_1 \lambda^{t-1} - \phi_2 \lambda^{t-2} = 0$$

is obtained. When B is bracketed in λ^{t-2} bracket, it becomes

$$\lambda^{t-2}(\lambda^2 - \phi_1 \lambda - \phi_2) = 0$$

Both sides of the equation are multiplied with $1/\lambda^{t-2}$ to find a value of λ.

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

The equation above is obtained. Two characteristic roots are obtained with the solution set of this quadric (2nd order) equation [15]. The following formula is used to find two characteristic roots.

$$\lambda_{1,2} = \frac{-\phi_1 \mp \sqrt{\phi_1^2 - 4 * 1 * \phi_2}}{2 * 1}$$

or

$$\lambda_{1,2} = \frac{-\phi_1 \mp \sqrt{\Delta}}{2}$$

are calculated. It is found as $\Delta = \phi_1^2 - 4\phi_2$ in here [15].

If it is $\phi_1^2 - 4\phi_2 > 0$, and there are two different reel roots like λ_1 and λ_2, the general solution for λ_1^t and λ_2^t base solutions are as follows;

$$X_t = c_1 \lambda_1^t + c_2 \lambda_2^t$$

and c_1 and c_2 are absolute values [15].

If it is $\phi_1^2 - 4\phi_2 = 0$, then $(\Delta = 0)$ λ_1 and λ_2 are equivelant to one another and there is one reel root. The solution of the homogeneous equation is $\lambda_{1,2} = \lambda = -\frac{\phi_1}{2}$. Therefore, there is a homogeneous solution with

$$X_t = c_1 \lambda_1^t + c_2 \lambda_2^t t$$

[16].

If $\phi_1^2 + 4\phi_2 < 0$, it follows that Δ is negative so that the characteristic roots are imaginary. Because $\phi_1^2 \geq 0$, imaginary roots occur only if $\phi_2 < 0$. Although this might be hard to interpret directly, if we switch to polar coordinates it is possible to transform the roots into more easly understood trigonometric functions.

$$\lambda_1 = (\phi_1 + i\sqrt{\Delta})/2$$
$$\lambda_2 = (\phi_1 - i\sqrt{\Delta})/2$$

where $i = \sqrt{-1}$

It can be used de Moivre's theorem to write the homogeneous solution is

$$X_t^h = c_1 r^t \cos(\theta t + c_2)$$

where c_1 and c_2 are arbitrary constants, $r = (-\phi_2)^{1/2}$, and the value of θ is chosen so as to satisfy

$$\cos(\theta) = \phi_1/[(-\phi_2)^{\frac{1}{2}}]$$

The trigonometric functions impart a wavelike pattern to the time path of the homogeneous solution; note that the frequency of the oscillations is determined by θ.

Because $\cos(\theta t) = \cos(2\pi + \theta t)$, the stability condition is determined solely by the magnitude of $r = (-\phi_2)^{1/2}$. If $|\phi_2| = 1$, the oscillations are of unchanging amplitude; the homogeneous solution is periodic. The oscillations will dampen if $|\phi_2| < 1$ and explode if $|\phi_2| > 1$ [17].

III. RESULTS AND DISCUSSION

Applications were made on the results obtained from the studies conducted with time series in the field of agriculture. When the production of almonds is modelled with a time series, the ARIMA(0,1,1) model was obtained [18]. The coefficient of parameters of this model was found as $\theta = 0.417$.

$$X_t = X_{t-1} - 0.417\varepsilon_{t-1} + \varepsilon_t$$

To find the linear difference equation of the model above,

$$X_t(1-B) = \varepsilon_t(1-\theta B)$$

the ARIMA(0,1,1) model, which can be written as given above, can also be written as follows:

$$\varepsilon_t = X_t(1-B)(1-\theta B)^{-1}$$

When $(1-\theta B)^{-1}$ statement was placed instead of the calculation value as $1 + \theta B + (\theta B)^2 + (\theta B)^3 + \cdots$ with the help of the back shift operator,

$$\varepsilon_t = X_t(1-B)(1+\theta B + \theta^2 B^2 + \theta^3 B^3 + \cdots)$$

$$\varepsilon_t = X_t(1+\theta B + \theta^2 B^2 + \theta^3 B^3 - B - \theta B^2 - \theta^2 B^3 - \theta^3 B^4 - \cdots)$$

$$\varepsilon_t = X_t[(1 + B(\theta - 1) + B^2(\theta^2 - \theta) + B^3(\theta^3 - \theta^2) + \cdots)]$$

$$\varepsilon_t = X_t + (\theta - 1)X_{t-1} + (\theta^2 - \theta)X_{t-2} + \cdots$$

it became solvable as a second-order homogeneous linear difference equation above.

$$X_t + (\theta - 1)X_{t-1} + (\theta^2 - \theta)X_{t-2} = 0$$

It is taken as above. When the $\theta = 0.417$ value was placed in the equation,

$$X_t + (0.417 - 1)X_{t-1} + (0.417^2 - 0.417)X_{t-2} = 0$$

$$X_t - 0.583X_{t-1} - 0.243X_{t-2} = 0$$

$$X_t = \lambda^t$$

and the above transformation was applied to the equation,

$$\lambda^t - 0.583\lambda^{t-1} - 0.243\lambda^{t-2} = 0$$

$$\lambda^{t-2}(\lambda^2 - 0.583\lambda - 0.243) = 0$$

$$\lambda^2 - 0.583\lambda - 0.243 = 0$$

the second-order equation above was formed. The discriminant (Δ) of this equation is calculated as follows:

$$\Delta = b^2 - 4ac = (-0.583)^2 - 4 * 1 * (-0.243) = 1.312$$

$$\Delta = 1.312 > 0$$

$$\lambda_{1,2} = \frac{0.583 \mp \sqrt{1.312}}{2}$$

$$\lambda_1 = -0.281$$

$$\lambda_2 = 0.864$$

If the values obtained are placed on the $X_t = c_1\lambda_1^t + c_2\lambda_2^t$ equation,

$$X_t = c_1(-0.281)^t + c_2 0.864^t$$

is obtained. When arbitrary different values are given to t=0 as X_0 and t=1 as X_1, then different values of c_1 and c_2 coefficients can be obtained. For X_0=2 and X_1=5, c_1=-2.858 and c_2=4.858 were found. The new equation obtained is as follows:

$$X_t = -2.858\,(-0.281)^t + 4.858\,(0.864)^t$$

In another study, the production of peanuts was modelled as ARIMA(0,1,1) with the time series [19]. The coefficient of the parameter of the model is $\theta = 0.472$. Since it is a similar model to the abovementioned almond production model, the same operations are made, and as a result of the

$$X_t - 0.528X_{t-1} - 0.249X_{t-2} = 0$$

and

$$\lambda^t - 0.528\lambda^{t-1} - 0.249\lambda^{t-2} = 0$$

statements,

$$\lambda^2 - 0.528\lambda - 0.249 = 0$$

the equation above is obtained.

$$\Delta = 1.276 > 0$$

The reel roots of the equation are found as follows:

$$\lambda_1 = 0.829$$

$$\lambda_2 = -0.301$$

The following equation is obtained.

$$X_t = c_1 0.829^t + c_2(-0.301)^t$$

When arbitrary different values are given to t=0 as X_0 and t=1 as X_1, consider using X_0=-1 and X_1=4 to find the c_1 and c_2 coefficients. The following equation set is obtained.

$-1 = c_1 0.829^0 + c_2(-0.301)^0$
$4 = c_1 0.829^1 + c_2(-0.301)^1$

When this equation set is solved, c_1=3.274 and c_2=-4.274 were found. The new equation obtained is as follows:

$$X_t = 3.274\ (0.829)^t - 4.274\ (-0.301)^t$$

[20] modelled potato production as ARIMA(0,1,1) using time series. The coefficient of the parameter of the model is $\theta = 0.52$.

$$X_t - 0.48X_{t-1} - 0.25X_{t-2} = 0$$
$$\lambda^2 - 0.48\lambda - 0.25 = 0$$

The roots of the equation above were found as 0.795 and -0.315

$$X_t = c_1 0.795^t + c_2(-0.315)^t$$

is obtained. Considering X_0=6 for t=0 and X_1=3 for t=1, the equation system on this condition is as follows:

$6 = c_1 0.795^0 + c_2(-0.315)^0$
$3 = c_1 0.795^1 + c_2(-0.315)^1$

The following equation system was formed when the necessary modifications were made:

$6 = c_1 + c_2$
$3 = 0.795c_1 - 0.315c_2$

When this equation system was solved, c_1=4.405 and c_2=1.595 were found.

$$X_t = 4.405\ (0.795)^t + 1.595\ (-0.315)^t$$

The difference equation above was obtained.

[21] modelled the fig production as ARIMA(0,1,1) model and found the coefficient of the parameter of the model as $\theta = -0.461$. Model was first as follows:

$$X_t = X_{t-1} + 0.461\varepsilon_{t-1} + \varepsilon_t$$

After necessary modifications were made,

$$\varepsilon_t = X_t + (\theta - 1)X_{t-1} + (\theta^2 - \theta)X_{t-2} + \cdots$$

$$\varepsilon_t = X_t - 1.461X_{t-1} + 0.674X_{t-2} + \cdots$$

They were able to write the model, which was expressed in $\varepsilon_t$coefficients, in X_t coefficients as above.

$$X_t - 1.461X_{t-1} + 0.674X_{t-2} = 0$$

Considering it as above, the following occurs:

$$\lambda^t - 1.461\lambda^{t-1} + 0.674\lambda^{t-2} = 0$$
$$\lambda^{t-2}(\lambda^2 - 1.461\lambda + 0.674) = 0$$
$$\lambda^2 - 1.461\lambda + 0.674 = 0$$
$$\Delta = -0.56 < 0$$

There are no reel roots but complex roots. These roots are as follows:

$$\lambda_1 = 0.73 - 0.375i$$
$$\lambda_2 = 0.73 + 0.375i$$

$$r = \sqrt{0.73^2 + 0.375^2} = 0.82$$

$$tan\theta = \frac{0.375}{0.73} = 0.514$$
$$\theta = 0.475$$

It is 0.475 radian =27.22 degree.

$$X_t = c_1(0.73 + i\ 0.375)^t + c_2(0.73 - i\ 0.375)^t$$

$$X_t = 0.82^t(c_1 cos\ 0.475t - c_2 sin\ 0.475t)$$

$X_0 = 1$ The special solutions for $X_1 = 2$ are as follows (when t=0 and t=1):

$$1 = 0.82^0(c_1 cos\ 0.475\ (0) - c_2 sin\ 0.475\ (0))$$
$$c_1 = 1$$
$$2 = 0.82^1(1\ cos\ 0.475\ (1) - c_2 sin\ 0.475\ (1))$$
$$2 = 0.82(0.8893 - c_2\ 0.4573)$$
$$c_2\ = -3.39$$
$$X_t = 0.82^t(cos\ 0.475t + 3.39\ sin\ 0.475t)$$

IV. CONCLUSION

Some time series analysis methods applied are the first-order integrated moving average method as ARIMA(0,1,1). The open version of the model was expressed with the back shift operator with these models. When the linear difference equation was applied to the $X_t = X_{t-1} - 0.417\varepsilon_{t-1} + \varepsilon_t$ model, the second-order homogeneous linear difference equation was applied. Since the discriminant of the equation was higher than zero and 2 different reel roots were obtained, a linear difference equation where different special solutions can be found could be written. Linear difference equations were written for $X_t - 0.528X_{t-1} - 0.249X_{t-2}$ and $X_t - 0.48X_{t-1} - 0.25X_{t-2}$ models since the discriminants were similarly higher than zero. For $X_t = X_{t-1} + 0.461\varepsilon_{t-1} + \varepsilon_t$ model, the discriminant was negative (lower than zero); thus, it had complex roots and the linear difference equation was applied using the trigonometric feature. Shortly, it was seen that linear difference equations are significantly used in time series.

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